Lecture 7 - Separable Equations

Separable equations is a very special type of differential equations where you can “separate” the terms involving only y on one side of the equation and terms involving only t on the other side. This separation will allow us to integrate both sides of the equation. We will see that the logistic equation we encountered in the previous lecture is separable and use the method we learn here to solve it.

The key to solving separable differential equations will be the chain rule of derivatives. The chain rule tells us how to find the derivative of a function when it is a composition of two functions and we already know the derivatives of both of them. Let us recall the chain rule.

Chain Rule of Derivatives:
If \( f(t) = h(g(t)) \), then \( f'(t) = h'(g(t))g'(t) \).

Slogan: “derivative of outer function with the inner function plugged in · derivative of inner function”

Example: Find \( \frac{dz}{dt} \) if \( z(t) = y(t) - \frac{1}{3}y(t)^3 \).

\[
\frac{dz}{dt} = \frac{dy}{dt} - \frac{1}{3} \cdot 3y(t)^2 \frac{dy}{dt} = \frac{dy}{dt} - y(t)^2 \frac{dy}{dt} = (1 - y(t)^2) \frac{dy}{dt}
\]

Problem: Solve the differential equation \( \frac{dy}{dt} = \frac{t^2}{1 - y^2} \).

Solution: First we rewrite the equation separating the y and t terms: \( (1 - y^2) \frac{dy}{dt} = t^2 \).

The left side is an expression involving only y and the entire expression is multiplied by \( \frac{dy}{dt} \). The right side is an expression involving only t. This is precisely what it means to be separable. It is absolutely key that all y terms be multiplied by \( \frac{dy}{dt} \)!

Now that we separated the y and t terms, we can integrate both sides.

Right side: \( \int t^2 \, dt = \frac{t^3}{3} + C \)

Left side: \( \int (1 - y^2) \frac{dy}{dt} \, dt = y - \frac{1}{3}y^3 + D \)

We know this immediately because we just computed above that the derivative of \( (1 - y^2) \frac{dy}{dt} \) is \( y - \frac{1}{3}y^3 \). However there is an easy trick to computing such antiderivatives that we could have otherwise used. Notice that \( \int (1 - y^2) \frac{dy}{dt} \, dt \) becomes \( \int 1 - y^2 \, dy \) if we cancel the dt. Since the integral \( \int 1 - y^2 \, dy \) ends in \( dy \), this tells us that we need to integrate with respect to \( y \), that is \( y \) now becomes the independent variable and we FORGET that it was a function of \( t \). Therefore:

\( \int 1 - y^2 \, dy = y - \frac{1}{3}y^3 / 3 + C \) is a standard calculus integration.

As we discussed earlier, it is NOT a legal mathematical operation to cancel the dt, but because the notation was chosen very cleverly, we can use the chain rule as a justification that this fake cancelation actually works to produce the correct result.

Getting back to our problem, we have \( y - \frac{1}{3}y^3 / 3 + D = \frac{1}{3}t^3 + C \).

\[
y - \frac{1}{3}y^3 = \frac{1}{3}t^3 + E
\]

It is very difficult to solve this equation explicitly for \( y \)! In most cases, the equation for \( y \) will obtain out of the separable equations algorithm will be only approximable numerically by a computer. This is a major downside of this algorithm. An equation that is not solved explicitly for the dependent variable is called implicit.

Problem: Solve the differential equation \( \frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)} \), \( y(0) = -1 \).

Solution:

Step 1: Check if the equation is separable.

The equation is separable: \( 2(y - 1) \frac{dy}{dx} = 3x^2 + 4x + 2 \)

Step 2: Integrate both sides!

Left side: \( \int 2(y - 1) \frac{dy}{dx} \, dx = \int 2(y - 1) \, dy = \int 2y - 2 \, dy = y^2 - 2y + C \).

Right side: \( \int 3x^2 + 4x + 2 \, dx = x^3 + 2x^2 + 2x + D \).

So \( y^2 - 2y + C = x^3 + 2x^2 + 2x + D \).
\[ y^2 - 2y = x^3 + 2x^2 + 2x + E \]

**Step 3:** Solve for \( E \).
\[
(-1)^2 - 2(-1) = 0^3 + 2 \cdot 0^2 + 2 \cdot 0 + E
\]
\[ 3 = E \]
\[ y^2 - 2y = x^3 + 2x^2 + 2x + 3 \]

**Step 4:** See if it is possible to solve explicitly for \( y \).
In this case, it is possible to solve the equation explicitly for \( y \) by using the quadratic formula!

First we move all terms over to the left side:
\[ y^2 - 2y - (x^3 + 2x^2 + 2x + 3) = 0 \]
Next we use the quadratic formula: \( a = 1, b = -2, c = -(x^3 + 2x^2 + 2x + 3) \)
\[
y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 + 4(x^3 + 2x^2 + 2x + 3)}}{2} = \frac{2 \pm \sqrt{4(1 + x^3 + 2x^2 + 2x + 3)}}{2} = \frac{2 \pm \sqrt{x^3 + 2x^2 + 2x + 4}}{2}
\]
So we have:
\[ y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} \]

But this equation gives us two functions! Equations that result from applying the separation technique are often satisfied by more than one function. A simple example of an equation that is satisfied by more than one function is \( x^2 + y^2 = 4 \). The graph of this equation is a circle with radius 2.

The upper blue part of the circle is a function \( f(x) = \sqrt{4 - x^2} \). The lower red part of the circle is a function \( g(x) = -\sqrt{4 - x^2} \). In these cases, we use the initial condition again to choose the correct function.

First we consider: \( y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4} \) and calculate \( y(0) = 1 + \sqrt{4} = 3 \)
So this cannot be the correct solution!

Next we consider: \( y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \) and calculate \( y(0) = 1 - \sqrt{4} = 1 - 2 = -1! \)
So the correct solution is:
\[ y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \]

Can it happen that more than one function satisfies both the differential equation and the initial condition? Yes, we will see an example of this shortly.

**Problem:** Solve the differential equation \( \frac{dy}{dx} = \frac{4x - x^3}{4 + y^3} \) so that the point \((0, 1)\) lies on the solution curve.

**Solution:**
**Step 1:** Check if the equation is separable.
The equation is separable: \( (4 + y^3) \frac{dy}{dx} = 4x - x^3 \).

**Step 2:** Integrate both sides!
Left side: \( \int 4 + y^3 \, dy = 4y + y^4/4 + C \)
Right side: \[ \int 4x - x^3 \, dx = 2x^2 - x^4/4 + D \]
So \[ 4y + y^4/4 + C = 2x^2 - x^4/4 + D. \]

\[ 4y + y^4/4 = 2x^2 - x^4/4 + E \]

**Step 3:** Solve for \( E \).
\[ 4 \cdot 1 + 1^4/4 = 2 \cdot 0^2 - 0^4/4 + E \]
\[ 4 + 1/4 = 17/4 = E \]

\[ 4y + y^4/4 = 2x^2 - x^4/4 + 17/4 \]

We can make the expression simpler by eliminating the denominators:

\[ 16y + y^4 = 8x^2 - 4x^4 + 17 \]

It is very difficult to solve this equation explicitly for \( y \). Here is a graph of the equation.

Can we separate this graph into two functions in the same way we separated the circle above? Yes! We need to find the exact location where the graph starts to loop.

Notice that these are exactly the place where the tangent line to the graph is VERTICAL, meaning that the derivative does NOT exist!
\[ \frac{dy}{dx} \text{ does not exist when } 4 + y^3 = 0. \]
\[ y^3 = -4 \]
\[ y \approx -1.59 \]

In the previous lecture, we discussed the logistic equation as a means of modeling population dynamics.

\[ \frac{dy}{dt} = ry(1 - \frac{1}{K}y) \]

Next, we will solve a specific example of a logistic equation.

**Problem:** Solve \( \frac{dy}{dt} = y(1 - \frac{1}{2}y) \) \((r = 1, K = 2)\).

**Solution:**

**Step 1:** Check if the equation is separable.

The logistic equation is clearly separable: \[ \frac{1}{y(1 - \frac{1}{2}y)} \frac{dy}{dt} = 1. \]

**Step 2:** Integrate both sides!

Left side: \[ \int \frac{1}{y(1 - \frac{1}{2}y)} dy \]

To integrate this expression requires a technique from Calculus called **Integration by Partial Fractions.**

This technique is applied to integrals whose numerator and denominator are polynomials. The basic idea of the method is to factor the denominator polynomial and decompose the entire fraction into a sum of fractions with the factors as denominators.

We decompose \[ \frac{1}{y(1 - \frac{1}{2}y)} = \frac{A}{y} + \frac{B}{1 - \frac{1}{2}y} \]

We solve for \( A \) and \( B \):

\[ \frac{A}{y} + \frac{B}{1 - \frac{1}{2}y} = \frac{A(1 - \frac{1}{2}y) + By}{y(1 - \frac{1}{2}y)} \]

We conclude that \( A(1 - \frac{1}{2}y) + By = 1 \).

\[ A - \frac{1}{2}Ay + By = 1 \]
\[ (B - \frac{1}{2}A)y + A = 1 = 0y + 1 \]
\[ B - \frac{1}{2}A = 0 \text{ and } A = 1 \]
\[ A = 1 \text{ and } B = \frac{1}{2} \]
So \( \frac{1}{y(1 - \frac{1}{2}y)} = \frac{1}{y} + \frac{1}{1 - \frac{1}{2}y} \).

We integrate

\[
\int \frac{1}{y(1 - \frac{1}{2}y)} \, dy = \int \frac{1}{y} \, dy + \frac{1}{2} \int \frac{1}{1 - \frac{1}{2}y} \, dy = \ln |y| + \frac{1}{2} (-2) \ln |1 - \frac{1}{2}y| + C = \ln |y| - \ln |1 - \frac{1}{2}y| = \ln \left| \frac{y}{1 - \frac{1}{2}y} \right| + C
\]

Right side: \( \int 1 \, dt = t + D. \)

So \( \ln \left| \frac{y}{1 - \frac{1}{2}y} \right| + C = t + D. \)

\[
\ln \left| \frac{y}{1 - \frac{1}{2}y} \right| = t + E
\]

**Step 3:** See if it is possible to solve explicitly for \( y \).

Applying \( \ln \) to both sides, we obtain:

\[
y(1 - \frac{1}{2}y) = F e^t \quad (F = \pm e^E)
\]

\[
y = F e^t (1 - \frac{1}{2}y) = F e^t - \frac{F}{2} e^t y
\]

\[
y + \frac{F}{2} e^t y = F e^t
\]

\[
y(1 + \frac{F}{2} e^t) = F e^t
\]

\[
y = \frac{F e^t}{1 + \frac{F}{2} e^t} = \frac{F}{e^{-t} + \frac{F}{2}}
\]

Since \( K = 2 \), we know that \( \lim_{t \to \infty} y = 2 \). Let us check this.

\[
\lim_{t \to \infty} \frac{F}{e^{-t} + \frac{F}{2}} = \frac{F}{2} = 2.
\]

**Problem:** Solve \( \frac{dy}{dt} = y^{1/3}, \ y(0) = 0 \)

**Solution:**

**Step 1:** Check if the equation is separable.

This is equation is clearly separable: \( y^{-1/3} \frac{dy}{dt} = 1 \).

**Step 2:** Integrate both sides!

Left side: \( \int y^{-1/3} \, dy = \frac{3}{2} y^{2/3} + C \)

Right side: \( \int 1 \, dt = t + D. \)

So \( \frac{3}{2} y^{2/3} + C = t + D. \)

\[
\frac{3}{2} y^{2/3} = t + E
\]

**Step 3:** Solve for \( E. \)

\[
\frac{3}{2} \cdot 0^{2/3} = 0 + E
\]

\[
E = 0
\]

\[
\frac{3}{2} y^{2/3} = t
\]

**Step 4:** See if it is possible to solve explicitly for \( y. \)

\[
y^{2/3} = \frac{2}{3} t
\]

\[
y^2 = \left( \frac{2}{3} t \right)^3
\]

\[
y = \pm \left( \frac{2}{3} t \right)^3
\]
Notice that we first raised both sides to the third power and only then took the square root of both sides. The reason is that taking square root produces two solutions! Now as before, we use the initial condition again to choose the correct function. First we consider \( y = (\frac{2}{3}t)^3 \) and see that \( y(0) = 0 \). Next we consider \( y = - (\frac{2}{3}t)^3 \) and again see that \( y(0) = 0! \) In this case both functions satisfy the differential equation and the initial condition! This is our first example of a differential equation with an initial condition having more than one solution! How could this have happened? In the first step of the solution we divided both sides by \( y^{1/3} \) but we never stated that \( y \neq 0 \). Later we used precisely the value \( y = 0 \) to find the constant. Note that we were still able to obtain a valid solution. The only problem that arose was the existence of another solution!

**General Formula:** Solve \( g(y) \frac{dy}{dt} = h(t), \ y(a) = b \).

**Step 1:** Rewrite the equation: \( g(y) \frac{dy}{dt} = h(t) \).

**Step 2:** Integrate both sides!
Left side: \( \int g(y) \, dy = G(y) + C \)
Right side: \( \int h(t) \, dt = H(t) + D \)
So \( G(y) + C = H(t) + D \).

\[
G(y) = H(t) + E
\]

**Step 3:** Solve for \( E \).
\( G(b) = H(a) + E \)
\( E = G(b) - H(a) \)

\( G(y) = H(t) + G(b) - H(a) \)

**Step 4:** We see if it is possible to solve explicitly for \( y \).

What happens if we cannot integrate one or both of the sides? Can we still obtain the solution numerically?

We rewrite the solution as: \( G(y) - G(b) = H(t) - H(a) \)
This can, in turn, be rewritten as:
\[
\int_b^y g(z) \, dz = \int_a^t h(s) \, ds
\]

For example, say \( y(0) = 1 \) and need to compute \( y(3) \).
1) We compute the area \( \int_0^3 h(s) \, ds = N \).
2) We compute a value of \( y \) such that \( \int_1^y g(z) \, dz = N \).

Finally, note that the separable equations method works on differential equations of the form \( \frac{dy}{dt} = ay - b \) as well because these are clearly separable equations. The first step which factors out \( a \) is in fact separating the variables.