Lecture 19 - The Laplace transform and differential equations

In this lecture we will learn how to use the Laplace transform to solve a differential equation with initial conditions. Central to these techniques is the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

**Key Fact I:** Suppose that \( f \) is continuous and \( f' \) is piecewise continuous and eventually \( |f(t)| \leq Ke^{at} \) where \( K \) and \( a \) are some constants. Then \( \mathcal{L}\{f'(t)\} \) exists for \( s > a \) and
\[
\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \quad \text{for } s > a.
\]

We will only prove the statement in the case where \( f' \) is also continuous and not just piecewise continuous. The proof of the other case is essentially the same, only more messy.

**Proof:**
First note that if \( \int_0^\infty |g(t)| \, dt \) converges, then so does \( \int_0^\infty g(t) \, dt \).

The intuitive reason behind this being that the areas under the graph of the absolute value of a function are the same or larger than the areas of under graph of the function itself.

So if we can show that \( \int_0^\infty e^{-st}|f(t)| \, dt \) converges, then it will follow that \( \int_0^\infty e^{-st}f(t) \, dt \) converges as well.

We start off by showing that the domain of \( \mathcal{L}\{f(t)\} \) is at least \( s > a \):

Eventually: \( e^{-st}|f(t)| \leq e^{-st}Ke^{at} = Ke^{(a-s)t} \) (since \( |f(t)| \leq Ke^{at} \)).

So by the Comparison Method, if we can show \( \int_0^\infty Ke^{(a-s)t} \, dt \) converges, then we will know that \( \int_0^\infty e^{-st}f(t) \, dt \) converges as well!

\[
\int_0^\infty Ke^{(a-s)t} \, dt = K \lim_{z \to \infty} \int_0^z e^{(a-s)t} \, dt = K \lim_{z \to \infty} \frac{1}{a-s} e^{(a-s)z} = \frac{1}{a-s}
\]

This limit exists when \( a - s < 0 \).

Therefore \( \int_0^\infty e^{(a-s)t} \, dt \) converges when \( s > a \).

Therefore \( \int_0^\infty e^{-st}|f(t)| \, dt \) converges when \( s > a \) (by Comparison Method).

Therefore \( \int_0^\infty e^{-st}f(t) \, dt \) converges when \( s > a \).

Finally, therefore the domain of \( \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t) \, dt \) is at least \( s > a \).

Next we compute the formula for \( \mathcal{L}\{f'(t)\}: \)
\[
\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t) \, dt = \lim_{z \to \infty} \int_0^z e^{-st}f'(t) \, dt
\]

First we integrate \( \int_0^z e^{-st}f'(t) \, dt \) using integration by parts:
\[
u(t) = e^{-st} \quad w'(t) = f'(t) \\
u'(t) = -se^{-st} \quad w(t) = f(t)
\]

\[
\int e^{-st}f'(t) \, dt = e^{-st}f(t) + s \int e^{-st}f(t) \, dt
\]

\[
\int_0^z e^{-st}f'(t) \, dt = e^{-st}f(t) \bigg|_0^z + s \int_0^z e^{-st}f(t) \, dt = e^{-sz}f(z) - f(0) + s \int_0^z e^{-st}f(t) \, dt
\]

\[
\lim_{z \to \infty} \int_0^z e^{-st}f'(t) \, dt = \lim_{z \to \infty} e^{-sz}f(z) - f(0) + s \lim_{z \to \infty} \int_0^z e^{-st}f(t) \, dt = \lim_{z \to \infty} e^{-sz}f(z) + s \int_0^\infty e^{-st}f(t) \, dt - f(0) = \lim_{z \to \infty} e^{-sz}f(z) + s \mathcal{L}\{f(t)\} - f(0) = s\mathcal{L}\{f(t)\} - f(0) (\text{since } \lim_{z \to \infty} e^{-sz}f(z) = 0!)
\]

It should be clear that \( \lim_{z \to \infty} e^{-sz}f(z) = 0 \) for \( s > a \) since according to our argument above the integral \( \int_0^\infty e^{-st}f(t) \, dt \) converges for \( s > a \) and we learned in the previous lectures that for an improper integral \( \int_0^\infty f(x) \, dx \) to converge it must be that \( \lim_{x \to \infty} f(x) = 0! \)

**Example:** We showed in the previous lecture that the Laplace transform of \( f(t) = \sin(t) \) is \( \frac{1}{s^2 + 1} \). Use this fact to find the Laplace transform of \( g(t) = \cos(t) \).
(sin(t))' = cos(t)
L{\cos(t)} = L\{(\sin(t))'\} = sL\{\sin(t)\} - \sin(0) = \frac{s}{s^2 + 1} - 0 = \frac{s}{s^2 + 1}

**Key Fact II:** Suppose that \( f(t), f'(t), \ldots, f^{(n-1)}(t) \) are continuous and \( f^{(n)}(t) \) is piecewise continuous and eventually \( |f(t)| \leq Ke^{at}, |f'(t)| \leq Ke^{at}, \ldots, |f^{(n-1)}(t)| \leq Ke^{at} \) where \( K \) and \( a \) are some constants. Then \( L\{f^{(n)}(t)\} \) exists for \( s > a \) and

\[
L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0) \quad \text{for} \quad s > a.
\]

**Example:**

\[
L\{f^{(3)}(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)
\]

Armed with Key Fact II, we will use the Laplace transform to solve a differential equation with initial conditions.

**Problem:** Solve the differential equation \( y'' - y' - 2y = 0, \ y(0) = 1, \ y'(0) = 0. \)

**Solution:** Of course, we already know how to solve this problem using the characteristic equation. But here we will see how to solve it using the Laplace transform.

**Step 1:** Apply the Laplace transform to both sides of the differential equation.

\[
L\{y'' - y' - 2y\} = L\{0\}
\]

Right side: \( L\{0\} = 0 \)

Left side:

\[
L\{y''\} - L\{y'\} - 2L\{y\} = s^2 L\{y\} - sy(0) - y'(0) - (sL\{y\} - f(0)) - 2L\{y\} = \]

\[
s^2 L\{y\} - sy(0) - y'(0) - sL\{y\} + y(0) - 2L\{y\} = \]

\[
(s^2 - s - 2)L\{y\} + (1-s)y(0) + y'(0) = \]

\[
(s^2 - s - 2)L\{y\} + (1-s) \cdot 1 + 0 = \]

\[
(s^2 - s - 2)L\{y\} + (1-s) = \]

Therefore:

\[
(s^2 - s - 2)L\{y\} + (1-s) = 0
\]

**Step 2:** Solve for \( L\{y\} \).

\[
L\{y\} = \frac{s-1}{s^2 - s - 2}
\]

**Step 3:** Invert the Laplace transform to find the function \( y \).

Here is how we do it:

First, we use the method of partial fractions to decompose the expression:

\[
\frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}
\]

\[
\frac{(s+1)A + (s-2)B}{(s-2)(s+1)} = \frac{As + A + Bs - 2B}{(s+1)(s-2)}
\]

\[
(\frac{A + B}{s+1} + \frac{A - 2B}{s-2} = s-1)
\]

So \( (A + B)s + (A - 2B) = s - 1 \).

\[
A + B = 1
\]

\[
A - 2B = -1
\]

\[
A = 1/3
\]

\[
B = 2/3
\]

\[
\frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}
\]

\[
L\{y\} = \frac{1/3}{s-2} + \frac{2/3}{s+1}
\]

Now we make a guess at \( y \):

1. \( L\{e^{2t}\} = 1/(s-2) \)
2. \( \mathcal{L}\{e^{-t}\} = 1/(s + 1) \)

3. \( \mathcal{L}\{1/3e^{2t} + 2/3e^{-t}\} = \frac{1/3}{s-2} + \frac{2/3}{s+1}! \)

\[ y = 1/3e^{2t} + 2/3e^{-t} \]

Applying the Laplace transform and solving the resulting equation for \( \mathcal{L}\{y\} \) is a matter of solving algebra problem. Inverting the Laplace transform to find \( y \) is a matter of solving another algebra problem together with having a table of the Laplace transforms of common functions. Actually, there is a formula to invert the Laplace transform but it requires knowledge of complex analysis and is therefore too advanced for this course. The key point is that using the Laplace transform “transforms” a differential equation problem into an algebra problem!

**Problem:** Solve the differential equation \( y'' + y = \sin(2t) \), \( y(0) = 2 \), \( y'(0) = 1 \).

**Solution:** As in the previous problem, we already know how to solve this problem using the method of undetermined coefficients. But here we will see how to solve it using the Laplace transform.

**Step 1:** Apply the Laplace transform to both sides of the differential equation.
\[
\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin(2t)\}
\]
Right side: \( \mathcal{L}\{\sin(2t)\} = 2/(s^2 + 4) \)
Left side:
\[
\mathcal{L}\{y'' + y\} = \\
\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \\
s^2 \mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \\
(s^2 + 1) \mathcal{L}\{y\} - sy(0) - y'(0) = \\
(s^2 + 1) \mathcal{L}\{y\} - s \cdot 2 - 1 = \\
(s^2 + 1) \mathcal{L}\{y\} - 2s - 1
\]
Therefore:
\[
(s^2 + 1) \mathcal{L}\{y\} = 2/(s^2 + 4) + 2s + 1 = 2 + (2s + 1)(s^2 + 4)/s^2 + 4 = 2 + 2s^3 + 8s + s^2 + 4 = 2s^3 + s^2 + 8s + 6 \\
\]

**Step 2:** Solve for \( \mathcal{L}\{y\} \).
\[
\mathcal{L}\{y\} = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 4)(s^2 + 1)}
\]

**Step 3:** Invert the Laplace transform to find the function \( y \).
Again, we use the method of partial fractions to decompose the expression:
\[
\frac{2s^3 + s^2 + 8s + 6}{(s^2 + 4)(s^2 + 1)}
\]
into a sum of simpler fractions.
\[
\frac{2s^3 + s^2 + 8s + 6}{(s^2 + 4)(s^2 + 1)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 1} = \frac{(s^2 + 1)(As + B) + (s^2 + 4)(Cs + D)}{(s^2 + 4)(s^2 + 1)} = \\
\frac{As^3 + Bs^2 + As + B + Cs^3 +Ds^2 + 4Cs + 4D}{(s^2 + 4)(s^2 + 1)} = \\
\frac{(A + C)s^3 + (B + D)s^2 + (A + 4C)s + (B + 4D)}{(s^2 + 4)(s^2 + 1)}
\]
So \((A + C)s^3 + (B + D)s^2 + (A + 4C)s + (B + 4D) = 2s^3 + s^2 + 8s + 6.
A + C = 2 \\
B + D = 1 \\
A + 4C = 8 \\
B + 4D = 6 \\
A = 2 - C \rightarrow 2 - C + 4C = 8 \rightarrow 2 + 3C = 8 \rightarrow 3C = 6 \rightarrow C = 2 \\
A = 2 - 2 \rightarrow A = 0 \\
B = 1 - D \rightarrow 1 - D + 4D = 6 \rightarrow 1 + 3D = 6 \rightarrow 3D = 5 \rightarrow D = 5/3 \\
B = 1 - 5/3 = -2/3
\[
\frac{2s^3 + s^2 + 8s + 6}{(s^2 + 4)(s^2 + 1)} = \frac{-2/3}{s^2 + 4} + \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1}
\]
\[
\mathcal{L}\{y\} = \frac{-2/3}{s^2 + 4} + \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1}
\]

Now we make a guess at \(y\):
1. \(\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2 + 4}\)
2. \(\mathcal{L}\{\cos(t)\} = \frac{s}{s^2 + 1}\)
3. \(\mathcal{L}\{\sin(t)\} = \frac{1}{s^2 + 1}\)
4. \(\mathcal{L}\{-1/3 \sin(2t) + 2 \cos(t) + 5/3 \sin(t)\} = \frac{-2/3}{s^2 + 4} + \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1}\)
\[
y = -1/3 \sin(2t) + 2 \cos(t) + 5/3 \sin(t)
\]