Lecture 16 - Second order nonhomogeneous linear differential equations: Method of Undetermined Coefficients

In this lecture we learn to solve nonhomogeneous equations of the form \( ay'' + by' + cy = g(t) \) using the "method of undetermined coefficients".

In previous lectures we learned to how to solve any equation of the form \( ay'' + by' + cy = 0 \). We will apply these techniques to help us solve the nonhomogeneous equation \( ay'' + by' + cy = g(t) \) by understanding the connection between solutions to homogeneous and nonhomogeneous equations.

**Question:** Is there any relationship between solutions to a nonhomogeneous equation \( p(t)y'' + q(t)y' + r(t)y = g(t) \) and the solutions to its homogeneous version \( p(t)y'' + q(t)y' + r(t) = 0 \)?

**Answer:** YES!

**Fact:** Suppose \( u(t) \) is a solution to the differential equation \( p(t)y'' + q(t)y' + r(t)y = g(t) \). Then ALL solutions are given by functions of the form \( a(t) + u(t) \) where \( a(t) \) is SOME solution to \( p(t)y'' + q(t)y' + r(t)y = 0 \).

In other words, to build ANY solution to a nonhomogeneous equation, we need the following building blocks:

1. ONE solution to that nonhomogeneous equation,
2. ALL solutions to the homogeneous version of that equation.

**Proof:**

Fix a solution \( u(t) \) to the equation \( p(t)y'' + q(t)y' + r(t)y = g(t) \).

We have to prove the following two statements:

1. If \( a(t) \) is a solution to \( p(t)y'' + q(t)y' + r(t)y = 0 \), then the sum \( y(t) = a(t) + u(t) \) is a solution to \( p(t)y'' + q(t)y' + r(t)y = g(t) \).
2. If \( y(t) \) is a solution to \( p(t)y'' + q(t)y' + r(t)y = g(t) \), then it is a sum \( y(t) = a(t) + u(t) \) for some solution \( a(t) \) to \( p(t)y'' + q(t)y' + r(t)y = 0 \).

Proof of (1):

Start with \( y(t) = a(t) + u(t) \) where \( a(t) \) is a solution to \( p(t)y'' + q(t)y' + r(t)y = 0 \).

\[
y'(t) = a'(t) + u'(t)
\]

\[
y''(t) = a''(t) + u''(t)
\]

We substitute:

\[
p(t)(a''(t) + u''(t)) + q(t)(a'(t) + u'(t)) + r(t)(a(t) + u(t)) = 0
\]

\[
p(t)a''(t) + p(t)u''(t) + q(t)a'(t) + q(t)u'(t) + r(t)a(t) = 0 + r(t)u(t)
\]

\[
p(t)a''(t) + q(t)a'(t) + r(t)a(t) + q(t)u'(t) + r(t)u(t) = 0 + g(t) = g(t)
\]

Therefore \( y(t) = a(t) + u(t) \) is a solution to \( p(t)y'' + q(t)y' + r(t)y = g(t) \! \)

Proof of (2):

Start with a solution \( y(t) \) for \( p(t)y'' + q(t)y' + r(t)y = g(t) \).

We need to show that \( y(t) = a(t) + u(t) \) where \( a(t) \) is a solution to \( p(t)y'' + q(t)y' + r(t)y = 0 \).

Notice that this is the same as showing that \( a(t) = y(t) - u(t) \) is a solution to \( p(t)y'' + q(t)y' + r(t)y = 0 \! \)

\[
a'(t) = y'(t) - u'(t)
\]

\[
a''(t) = y''(t) - u''(t)
\]

We substitute:

\[
p(t)(y''(t) - u''(t)) + q(t)(y'(t) - u'(t)) + r(t)(y(t) - u(t)) = 0
\]

\[
p(t)y''(t) - p(t)u''(t) + q(t)y'(t) - q(t)u'(t) + r(t)y(t) = 0
\]

\[
p(t)y''(t) + q(t)y'(t) + r(t)y(t) - p(t)u''(t) + q(t)u'(t) + r(t)u(t) = 0
\]

\[
g(t) - g(t) = 0
\]

Therefore \( a(t) = y(t) - u(t) \) is a solution to \( p(t)y'' + q(t)y' + r(t)y = 0 \).

This analysis gives us a strategy of how to find solutions to nonhomogeneous equations of the form \( ay'' + by' + cy = g(t) \). We already know that every solution to \( ay'' + by' + cy = 0 \) has the form \( c_1a(t) + c_2b(t) \) where \( a(t) \) and \( b(t) \) is a fundamental set of solutions. Therefore we know how to find ALL
solutions to \( ay'' + by' + cy = 0 \). All that remains to be learned is how to find a solution \( u(t) \) for the nonhomogeneous version \( ay'' + by' + cy = g(t) \).

**Strategy:**

**Step 1:** Find a fundamental set of solutions \( a(t) \) and \( b(t) \) to the homogeneous version \( ay'' + by' + cy = 0 \).

**Step 2:** Find a solution \( u(t) \) to the nonhomogeneous version \( ay'' + by' + cy = g(t) \): use the method of undetermined coefficients.

**Step 3:** General solution: \( y(t) = c_1 a(t) + c_2 b(t) + u(t) \)

**Problem:** Solve the differential equation \( y'' - 3y' - 4y = 3e^{2t} \).

**Solution:**

**Step 1:** Find a fundamental set of solutions to the homogeneous version \( y'' - 3y' - 4y = 0 \).

The characteristic equation is \( r^2 - 3r - 4 = 0 \).

So \( r = 4 \) and \( r = -1 \).

\[ a(t) = e^{4t} \text{ and } b(t) = e^{-t} \]

is a fundamental set of solutions to the homogeneous version.

**Step 2:** Find a solution to the nonhomogeneous version \( y'' - 3y' - 4y = 3e^{2t} \).

Guess for a solution:

\[ u(t) = Ae^{2t} \]

where \( A \) is the “undetermined coefficient”.

The guess we make will always be “very similar” to \( g(t) \) and the pattern will be summarized at the end of the lecture.

\[ u(t) = Ae^{2t} \]
\[ u'(t) = 2Ae^{2t} \]
\[ u''(t) = 4Ae^{2t} \]

We substitute:

\[ 4Ae^{2t} - 3(2Ae^{2t}) - 4Ae^{2t} = 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = -6Ae^{2t} = 3e^{2t} \]

\[ A = -\frac{1}{2} \]

So we found a solution to the nonhomogeneous version:

\[ u(t) = -\frac{1}{2}e^{2t} \]

**Step 3:**

General solution:

\[ y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t} \]

**Problem:** Solve the differential equation \( y'' - y' - 2y = e^{-t} \).

**Solution:**

**Step 1:** Find a fundamental set of solutions to the homogeneous version \( y'' - y' - 2y = 0 \).

The characteristic equation is \( r^2 - r - 2 = 0 \).

So \( r = 2 \) and \( r = -1 \).

\[ a(t) = e^{2t} \text{ and } b(t) = e^{-t} \]

is a fundamental set of solutions to the homogeneous version.

**Step 2:** Find a solution to the nonhomogeneous version \( y'' - y' - 2y = e^{-t} \).

Guess for a solution: \( u(t) = Ae^{-t} \)

\[ u'(t) = -Ae^{-t} \]
\[ u''(t) = Ae^{-t} \]

We substitute:

\[ Ae^{-t} + Ae^{-t} - 2Ae^{-t} = 0 = e^{-t} \]

But this is impossible!

Let us analyze what went wrong:
Notice that $Ae^{-t}$ is a solution to the homogeneous version! It is impossible to be simultaneously a solution to the homogeneous and the nonhomogeneous version. So this is an obviously wrong guess to begin with. In the future, if we encounter the situation that the guess we want to make is already a solution to the homogeneous version, our algorithm will be to **multiply the guess by $t$**.

**Second guess:** $u(t) = Ae^{-t}$

$u'(t) = Ae^{-t} - Ae^{-t}$

$u''(t) = -2Ae^{-t} - Ae^{-t} = -2Ae^{-t} + Ae^{-t}$

We substitute:

$-2Ae^{-t} + Ae^{-t} - (Ae^{-t} - Ae^{-t}) - 2Ae^{-t} = -2Ae^{-t} + Ae^{-t} - Ae^{-t} + Ae^{-t} - 2Ae^{-t} = -3Ae^{-t} = e^{-t}$

$A = -1/3$.

So we found a solution to the nonhomogeneous version:

$$u(t) = -1/3te^{-t}$$

**Step 3:**

General solution:

$$y(t) = c_1e^{2t} + c_2e^{-t} - 1/3te^{-t}$$

**Problem:** Solve the differential equation $y'' - 4y' + 4y = 7e^{2t}$.

**Solution:**

**Step 1:** Find a fundamental set of solutions to the homogeneous version $y'' - 4y' + 4y = 0$.

The characteristic equation is $r^2 - 4r + 4 = 0$.

$r^2 - 4r + 4 = (r - 2)^2 = 0$

So $r = 2$.

$$a(t) = e^{2t} \text{ and } b(t) = te^{2t}$$

is a fundamental set of solutions to the homogeneous version.

**Step 2:** Find a solution to the nonhomogeneous version $y'' - 4y' + 4 = 7e^{2t}$.

**Guess** for a solution: $u(t) = Ae^{2t}$

We check and see that the guess is already a solution to the homogenous version! So using the algorithm given above, we make the **second** guess:

$$u(t) = Ae^{2t}$$

We check and see that this guess is a solution to the homogeneous version as well! In this case, we make the **third** guess:

$$u(t) = At^2e^{2t}$$

In the situation where the initial two guesses fail, the third guess will ALWAYS work!

$u(t) = At^2e^{2t}$

$u'(t) = 2Ate^{2t} + 2At^2e^{2t}$

$u''(t) = 2Ae^{2t} + 4Ate^{2t} + 4At^2e^{2t} + 4At^2e^{2t} = 2Ae^{2t} + 8At^2e^{2t} + 4At^2e^{2t}$

We substitute:

$2Ae^{2t} + 8At^2e^{2t} + 4At^2e^{2t} - 4(2Ate^{2t} + 2At^2e^{2t}) + 4At^2e^{2t} = 2Ae^{2t} + 8At^2e^{2t} + 4At^2e^{2t} - 8At^2e^{2t} + 4At^2e^{2t} = 2Ae^{2t} = 7e^{2t}$

Therefore $A = 7/2$.

So we found a solution to the nonhomogeneous version:

$$u(t) = \frac{7}{2}t^2e^{2t}$$

**Step 3:**

General solution:

$$y(t) = c_1e^{2t} + c_2te^{2t} + \frac{7}{2}t^2e^{2t}$$

**Problem:** Solve the differential equation $y'' - 2y' - 8 = 2\sin(t)$.

**Solution:**
**Step 1**: Find a fundamental set of solutions to the homogeneous version \( y'' - 2y' - 8y = 0 \).

The characteristic equation is \( r^2 - 2r - 8 = 0 \).

\[ r^2 - 2r - 8 = (r - 4)(r + 2) = 0 \]

So \( r = 4 \) and \( r = -2 \).

\[ a(t) = e^{4t} \text{ and } b(t) = e^{-2t} \]

is a fundamental set of solutions to the homogeneous version.

**Step 2**: Find a solution to the nonhomogeneous version \( y'' - 2y' - 8 = 2\sin(2t) \).

**Guess** for a solution:

\[ u(t) = A\sin(2t) + B\cos(2t) \]

This guess will be used whenever \( g(t) \) is a multiple of \( \sin(nt) \) or \( \cos(nt) \).

We check and see that the guess is not a solution to the homogeneous version. So we go ahead with trying for a solution:

\[ z(t) = A\sin(2t) + B\cos(2t) \]

\[ z'(t) = 2A\cos(2t) - 2B\sin(2t) \]

\[ z''(t) = -4A\sin(2t) - 4B\cos(2t) \]

We substitute:

\[ -4A\sin(2t) - 4B\cos(2t) - 2(2A\cos(2t) - 2B\sin(2t)) - 8(A\sin(2t) + B\cos(2t)) = \]

\[ -4A\sin(2t) - 4B\cos(2t) - 4A\cos(2t) + 4B\sin(2t) - 8A\sin(2t) - 8B\cos(2t) = \]

\[ (-4A + 4B - 8A)\sin(2t) + (-4B - 4A - 8B)\cos(2t) = \]

\[ (-12A + 4B)\sin(2t) + (-12B - 4A)\cos(2t) = 2\sin(2t) \]

We conclude that:

\[-12A + 4B = 2\]

\[-12B - 4A = 0 \]

Here is the reason we are allowed to draw this conclusion:

**Fact**: If \( a\sin(x) + b\cos(x) = c\sin(x) + d\cos(x) \) for EVERY \( x \), then \( a = c \) and \( b = d \).

**Proof**: \( a\sin(0) + b\cos(0) = c\sin(0) + d\cos(0) \)

So \( b = d \).

\[ a\sin(\pi/2) + b\cos(\pi/2) = c\sin(\pi/2) + d\cos(\pi/2) \]

So \( a = c \).

Back to our system of equations:

\[ B = 1/20 \]

\[ A = -3/20 \]

So we found a solution to the nonhomogeneous version:

\[ u(t) = -3/20\sin(2t) + 1/20\cos(2t) \]

**Step 3**: General solution:

\[ y(t) = c_1e^{4t} + c_2e^{-2t} - 3/20\sin(2t) + 1/20\cos(2t) \]

**Problem**: Solve the differential equation \( y'' + 9y = 3\cos(3t) \).

**Solution**:

**Step 1**: Find a fundamental set of solutions to the homogeneous version \( y'' + 9y = 0 \).

The characteristic equation is \( r^2 + 9 = 0 \).

So \( r = 3i \) and \( r = -3i \).

\[ a(t) = \cos(3t) \text{ and } b(t) = \sin(3t) \]

is a fundamental set of solutions to the homogeneous version.

**Step 2**: Find a solution to the nonhomogeneous version \( y'' + 9y = 3\cos(3t) \).

**Guess** for a solution: \( u(t) = A\sin(3t) + B\cos(3t) \)

We check and see that the guess is already a solution to the homogeneous version.

So using the algorithm given above, we make the **second** guess:

\[ u(t) = At\cos(3t) + Bt\sin(3t) \]
We substitute:
\[ u(t) = A \cos(3t) - 3At \sin(3t) + B \sin(3t) + 3Bt \cos(3t) \]
\[ u''(t) = -3A \sin(3t) - 3A \sin(3t) - 9At \cos(3t) + 3B \cos(3t) + 3B \cos(3t) - 9Bt \sin(3t) = -6A \sin(3t) - 9At \cos(3t) + 6B \cos(3t) - 9Bt \sin(3t) \]
We substitute:
\[ -6A \sin(3t) - 9At \cos(3t) + 6B \cos(3t) - 9Bt \sin(3t) + 9(At \cos(3t) + Bt \sin(3t)) = -6A \sin(3t) - 9At \cos(3t) + 6B \cos(3t) + 9Bt \sin(3t) + 9At \cos(3t) + 9Bt \sin(3t) = -6A \sin(3t) + 6B \cos(3t) = 3 \cos(3t) \]
Therefore:
\[ -6A = 0 \]
\[ 6B = 3 \]
\[ A = 0 \]
\[ B = 1/2 \]
So we found a solution to the nonhomogeneous version:
\[ u(t) = \frac{1}{2} t \cos(3t) \]

**Step 3:**
General solution:
\[ y(t) = c_1 \cos(3t) + c_2 \sin(3t) + \frac{1}{2} t \cos(3t) \]

**Problem:** Solve the differential equation \( y'' - 4y' - 12y = 2t^3 - t + 3 \).

**Solution:**

**Step 1:** Find a fundamental set of solutions to the homogeneous version \( y'' - 4y' - 12y = 0 \).
The characteristic equation is \( r^2 - 4r - 12 = 0 \).
\[ r^2 - 4r - 12 = (r - 6)(r + 2) = 0 \]
So \( r = 6 \) and \( r = -2 \).
\[ a(t) = e^{6t} \text{ and } b(t) = e^{-2t} \]
is a fundamental set of solutions to the homogeneous version.

**Step 2:** Find a solution to the nonhomogeneous version \( y'' - 4y' - 12y = 2t^3 - t + 3 \).

**Guess** for a solution:
\[ u(t) = At^3 + Bt^2 + Ct + D \]
In the case where \( g(t) \) is a polynomial function, our guess will always be a polynomial of the same degree with unknown coefficients.

We check and see that the guess is not a solution to the homogeneous version. So we go ahead with trying to determine the undetermined coefficients.
\[ u(t) = At^3 + Bt^2 + Ct + D \]
\[ u'(t) = 3At^2 + 2Bt + C \]
\[ u''(t) = 6At + 2B \]
We substitute:
\[ 6At + 2B - 4(3At^2 + 2Bt + C) - 12(At^3 + Bt^2 + Ct + D) = 0 \]
\[ 6At + 2B - 12At^2 - 8Bt - 4C - 12At^3 - 12Bt^2 - 12Ct - 12D = -12At^3 + (-12A - 12B)t^2 + (6A - 8B - 12C)t + (2B - 4C - 12D) = 2t^3 - t + 3 \]
Therefore:
\[ -12A = 2 \]
\[ -12A - 12B = 0 \]
\[ 6A - 8B - 12C = -1 \]
\[ 2B - 4C - 12D = 3 \]
We solve:
\[ A = -1/6 \]
\[ -12 \cdot -1/6 - 12B = 0 \rightarrow 2 - 12B = 0 \rightarrow 2 = 12B \rightarrow B = 1/6 \]
\[ 6 \cdot (1/6) - 8 \cdot 1/6 - 12C = -1 \rightarrow -1 - 4/3 - 12C = -1 \rightarrow -1 - 4/3 + 1 = 12C \rightarrow -4/3 = 12C \rightarrow C = -1/9 \]
\[2 \cdot \frac{1}{6} - 4 \cdot \left(-\frac{1}{9}\right) - 12D = 3 \quad \rightarrow \quad 1/3 + 4/9 - 12D = 3 \quad \rightarrow \quad 1/3 + 4/9 - 3 = 12D \quad \rightarrow \quad -20/9 = 12D \quad \rightarrow \quad D = -\frac{5}{27}\]

So we found a solution to the nonhomogeneous version:

\[u(t) = -\frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}\]

**Step 3:**

General solution:

\[y(t) = c_1e^{5t} + c_2e^{-2t} - \frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}\]

**Problem:** Solve the differential equation \(y'' - 3y' - 4y = -8e^t\cos(2t)\).

**Solution:**

**Step 1:** Find a fundamental set of solutions to the homogeneous version \(y'' - 3y' - 4y = 0\).

The characteristic equation is \(r^2 - 3r - 4 = 0\).

\[r^2 - 3r - 4 = (r - 4)(r + 1) = 0\]

So \(r = 4\) and \(r = -1\).

\[a(t) = e^{4t}\ and \ b(t) = e^{-t}\]

is a fundamental set of solutions to the homogeneous version.

**Step 2:** Find a solution to the nonhomogeneous version \(y'' - 3y' - 4y = -8e^t\cos(2t)\).

**Guess for a solution:**

\[u(t) = Ae^t\sin(2t) + Be^t\cos(2t)\]

This guess will be used whenever \(g(t)\) is a multiple of \(e^{kt}\sin(kt)\) or \(e^{kt}\cos(kt)\).

We check and see that the guess is not a solution to the homogeneous version. So we go ahead with trying to determine the undetermined coefficients.

\[u(t) = Ae^t\sin(2t) + Be^t\cos(2t)\]

\[u'(t) = Ae^t\sin(2t) + 2Ae^t\cos(2t) + Be^t\cos(2t) - 2Be^t\sin(2t) = (A - 2B)e^t\sin(2t) + (2A + B)e^t\cos(2t)\]

\[u''(t) = (A - 2B)e^t\sin(2t) + 2(A - 2B)e^t\cos(2t) + (2A + B)e^t\cos(2t) - 2(2A + B)e^t\sin(2t) = (A - 2B)e^t\sin(2t) + (2A - 4B)e^t\cos(2t) + (2A + B)e^t\cos(2t) + (-4A - 2B)e^t\sin(2t) = (A - 2B)e^t\sin(2t) + (2A - 4B + 2A + B)e^t\cos(2t) = (-3A - 4B)e^t\sin(2t) + (4A - 3B)e^t\cos(2t)\]

We substitute:

\[(-3A - 4B)e^t\sin(2t) + (4A - 3B)e^t\cos(2t) - 3((A - 2B)e^t\sin(2t) + (2A + B)e^t\cos(2t)) - 4(Ae^t\sin(2t) + Be^t\cos(2t)) = (A - 2B)e^t\sin(2t) + (2A - 4B)e^t\cos(2t) - 3(A - 2B)e^t\sin(2t) - 3(2A + B)e^t\cos(2t) - 4Ae^t\sin(2t) - 4Be^t\cos(2t) = (-3A - 4B)e^t\sin(2t) + (4A - 3B)e^t\cos(2t) + (-3A + 6B)e^t\sin(2t) + (-6A - 3B)e^t\cos(2t) - 4Ae^t\sin(2t) - 4Be^t\cos(2t) = (-10A + 2B)e^t\sin(2t) + (-2A - 10B)e^t\cos(2t) = -8e^t\sin(2t)\]

Therefore:

\[-10A + 2B = 0\]
\[-2A - 10B = -8\]

Here is the reason we are allowed to draw this conclusion:

**Fact:** If \(ae^{mx}\sin(x) + be^{mx}\cos(x) = ce^{mx}\sin(x) + de^{mx}\cos(x)\) for EVERY \(x\), then \(a = c\) and \(b = d\).

**Proof:** \(ae^0\sin(0) + be^0\cos(0) = ce^0\sin(0) + de^0\cos(0)\)

So \(b = d\).

\(ae^{\pi/2}\sin(\pi/2) + be^{\pi/2}\cos(\pi/2) = ce^{\pi/2}\sin(\pi/2) + de^{\pi/2}\cos(\pi/2)\)

So \(a = c\).

Back to our system of equations:

We solve:

\[-10A + 2B = 0 \quad \rightarrow \quad -10A = -2B \quad \rightarrow \quad A = 1/5B\]
\(-2 \cdot 1/5B - 10B = -8 \rightarrow -2/5B - 10B = -8 \rightarrow -52/5B = -8 \rightarrow B = 10/13\)

So we found a solution to the nonhomogeneous version:

\[ u(t) = \frac{2}{13}e^t \sin(2t) + \frac{10}{13}e^t \cos(2t) \]

**Step 3:**

**General solution:**

\[ y(t) = c_1e^{4t} + c_2e^{-t} + \frac{2}{13}e^t \sin(2t) + \frac{10}{13}e^t \cos(2t) \]

**Problem:** Solve the differential equation \(y'' - 3y' - 4y = 3e^{2t} - 8e^t \cos(2t)\).

**Solution:**

**Step 1:** Find a fundamental set of solutions to the homogeneous version \(y'' - 3y' - 4y = 0\).

The characteristic equation is \(r^2 - 3r - 4 = 0\).

\(r^2 - 3r - 4 = (r - 4)(r + 1) = 0\)

So \(r = 4\) and \(r = -1\).

\(a(t) = e^{4t}\) and \(b(t) = e^{-t}\)

is a fundamental set of solutions to the homogeneous version.

**Step 2:** Find a solution to the nonhomogeneous version \(y'' - 3y' - 4y = 3e^{2t} - 8e^t \cos(2t)\).

Looking back over the notes, we see that we already found solutions for:

1. \(y'' - 3y' - 4y = 3e^{2t}\):
   \[ u(t) = -\frac{1}{2}e^{2t} \]

2. \(y'' - 3y' - 4y = -8e^t \cos(2t)\):
   \[ u(t) = \frac{2}{13}e^t \sin(2t) + \frac{10}{13}e^t \cos(2t) \]

We conclude that for \(y'' - 3y' - 4y = 3e^{2t} - 8e^t \cos(2t)\):

\[ u(t) = -\frac{1}{2}e^{2t} + \frac{2}{13}e^t \sin(2t) + \frac{10}{13}e^t \cos(2t) \]

**Step 3:**

**General solution:**

\[ y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t} + \frac{2}{13}e^t \sin(2t) + \frac{10}{13}e^t \cos(2t) \]

Here is the reason we are allowed to add the individual solutions:

**Fact:** IF

\[ u_1(t) \] is a solution to \(ay'' + by' + cy = g_1(t)\)

\[ u_2(t) \] is a solution to \(ay'' + by' + cy = g_2(t)\)

\[ \vdots \]

\[ u_m(t) \] is a solution to \(ay'' + by' + cy = g_m(t)\)

THEN

\[ u(t) = u_1(t) + u_2(t) + \cdots + u_m(t) \] is a solution to \(ay'' + by' + cy = g_1(t) + g_2(t) + \cdots + g_m(t)\).

**Proof:**

\[ u'(t) = u'_1(t) + u'_2(t) + \cdots + u'_m(t) \]

\[ u''(t) = u''_1(t) + u''_2(t) + \cdots + u''_m(t) \]

\[ a(u''_1(t) + u''_2(t) + \cdots + u''_m(t)) + b(u'_1(t) + u'_2(t) + \cdots + u'_m(t)) + c(u_1(t) + u_2(t) + \cdots + u_m(t)) = g_1(t) + g_2(t) + \cdots + g_m(t) \]

Therefore \(u(t) = u_1(t) + u_2(t) + \cdots + u_m(t)\) is a solution to \(ay'' + by' + cy = g_1(t) + g_2(t) + \cdots + g_m(t)\).

**Problem:** Solve the differential equation \(y'' + 2y' = t^2\).

**Solution:**

**Step 1:** Find a fundamental set of solutions to the homogeneous version \(y'' + 2y' = 0\).

The characteristic equation is \(r^2 + 2r = 0\).

\(r^2 + 2r = r(r + 2) = 0\)
So $r = 0$ and $r = -2$. 

$$a(t) = 1 \text{ and } b(t) = e^{-2t}$$

is a fundamental set of solutions to the homogeneous version.

**Step 2:** Find a solution to the nonhomogeneous version $y'' + 2y = t^2$.

**Guess** for a solution:

$$u(t) = At^2 + Bt + C$$

We check and see that the guess is not a solution to the homogeneous version. So we go ahead with trying to determine the undetermined coefficients.

$$u(t) = At^2 + Bt + C$$
$$u'(t) = 2At + B$$
$$u''(t) = 2A$$

We substitute:

$$2A + 2(2At + B) = 2A + 4At + 2B = 4At + (2A + 2B) = t^2$$

This is clearly impossible since a polynomial of degree 1 cannot equal a polynomial of degree 2!

In this case, we did check the solution against the solutions to the homogeneous case but we still ran into a problem. This illustrates that it is possible to make a wrong guess where the reason for why the guess is wrong is not immediately clear!

It is however easy to fix the guess. We use the same algorithm as before:

**Second guess:** $u(t) = At^3 + Bt^2 + Ct$
$$u'(t) = 3At^2 + 2Bt + C$$
$$u''(t) = 6At + 2B$$

We substitute:

$$6At + 2B + 2(3At^2 + 2Bt + C) = 6At + 2B + 6At^2 + 4Bt + 2C = (6A)t^2 + (6A + 4B)t + (2B + 2C) = t^2$$

Therefore:

$$6A = 1$$
$$6A + 4B = 0$$
$$2B + 2C = 0$$

We solve:

$$A = 1/6$$
$$B = -1/4$$
$$C = 1/4$$

So we found a solution to the nonhomogeneous version:

$$u(t) = \frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{1}{4}t$$

**Step 3:**

General solution:

$$y(t) = c_1 + c_2e^{-2t} + \frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{1}{4}t$$

**General Strategy** for the method of undetermined coefficients: $ay'' + by' + cy = g(t)$

**Case 1:** $g(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$ ($g(t)$ is a polynomial of degree $n$)

First guess:

$$u(t) = A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0$$

where $A_i$ are the **undetermined coefficients**.

**Case 2:** $g(t) = (a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0)e^{mt}$

First guess:

$$u(t) = (A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0)e^{mt}$$

**Case 3:** $g(t) = (a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0)e^{mt} \sin(kt)$

First guess:

$$u(t) = (A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0)e^{mt} \sin(kt) + (B_nt^n + B_{n-1}t^{n-1} + \cdots + B_1t + B_0)e^{mt} \cos(kt)$$
Case 4: \( g(t) = (a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0)e^{mt} \sin(kt) \)
Same as case 3!

Case 5: \( g(t) = g_1(t) + g_2(t) + \cdots g_n(t) \) where each \( g_i(t) \) is a function from cases (1) – (4).
This case gets broken up into \( n \) problems for each function \( g_i(t) \) and the solutions are added to obtain a solution to the entire problem.

All cases:
If the first guess fails, modify the guess by multiplying with \( t \).
If the second guess fails, make a third (and final!) guess by multiplying again with \( t \).