Lecture 15 - Existence and uniqueness of solutions II

Recall that in the first part of the course, we learned a theorem giving the conditions which a first order linear differential equation $\frac{dy}{dt} = p(t)y + g(t)$ must satisfy to guarantee that a solution exists and is unique:

**Theorem:** If the functions $p(t)$ and $g(t)$ are continuous on an interval $(a, b)$ containing the point $t = t_0$, then there exists a unique function $y$ that satisfies the differential equation

$$\frac{dy}{dt} = p(t)y + g(t), \quad y(t_0) = y_0$$

on the interval $(a, b)$.

In this lecture we will learn the conditions a second order linear differential equation must satisfy to guarantee that a solution exists and is unique.

**Theorem:** If the functions $q(t)$, $r(t)$, and $g(t)$ are continuous on an interval $(a, b)$ containing the point $t = t_0$, then there exists a unique function $y$ that satisfies the differential equation

$$y'' = q(t)y' + r(t)y + g(t), \quad y(t_0) = m, \quad y'(t_0) = n$$

on the interval $(a, b)$.

**Problem:** Find the largest interval on which the differential equation $(t^2 - 3t)y'' + ty' - (t + 3)y = 0$, $y(1) = 2$, $y'(1) = 1$ is guaranteed to have a unique solution.

**Solution:**
First, we transform the equation into the form: $y'' = q(t)y' + r(t)y + g(t)$.

$$y'' = -\frac{t}{t^2 - 3t}y' + \frac{t + 3}{t^2 - 3t}y$$

Now we see that:

$q(t) = -\frac{t}{t^2 - 3t}$
$r(t) = \frac{t + 3}{t^2 - 3t}$
$g(t) = 0$

The function $q(t) = -\frac{t}{t^2 - 3t}$ is continuous for all $t \neq 0$ and $t \neq 3$.
The function $r(t) = \frac{t + 3}{t^2 - 3t}$ is continuous for all $t \neq 0$ and $t \neq 3$.
The function $g(t) = 0$ is continuous for all $t$.

Therefore the largest interval containing $t = 1$ on which all three functions are continuous is $(0, 3)$.
Therefore the interval $(0, 3)$ is the largest interval on which the differential equation is guaranteed to have a unique solution.

**Problem:** Find the largest interval on which the differential equation $(t - 5)y'' + ty' + 3\ln |t|y = 0$, $y(2) = 3$, $y'(2) = 1$ is guaranteed to have a unique solution.

**Solution:**
First, we transform the equation into the form: $y'' = q(t)y' + r(t)y + g(t)$.

$$y'' = -\frac{t}{t - 5}y' - \frac{3\ln |t|}{t - 5}y$$

Now we see that:

$q(t) = -\frac{t}{t - 5}$
$r(t) = -\frac{3\ln |t|}{t - 5}$
$g(t) = 0$

The function $q(t) = -\frac{t}{t - 5}$ is continuous for all $t \neq 5$. 

The function \( r(t) = -\frac{3\ln|t|}{t-5} \) is continuous for all \( t \neq 0 \) and \( t \neq 5 \) (since \( \ln(0) \) is undefined).

The function \( g(t) = 0 \) is continuous for all \( t \).

Therefore the largest interval containing \( t = 2 \) on which all three functions are continuous is \((0, 5)\).

Therefore the interval \((0, 5)\) is the largest interval on which the differential equation is guaranteed to have a unique solution.