Lecture 14 - Reduction of Order Method

In the previous lectures we learned how to solve ANY second order homogeneous linear differential equation with constant coefficients:

\[ ay'' + by' + cy = 0. \]

In this lecture, we learn our first technique for solving second order homogeneous linear equations with nonconstant coefficients.

\[ p(t)y'' + q(t)y' + r(t)y = 0 \]

The main difficulty encountered in solving these equations is finding the two families of solutions. The simple idea of guessing that the solution must have the form \( e^{rt} \) does not work for the more general case of nonconstant coefficients. The technique we learn here will involve finding a second family of solutions if you are already KNOW one of the families. You are provided with one family of solutions and based on this information you are asked to find a second family. To find the second family we will use exactly the same method as we did for equations \( ay'' + by' + cy = 0 \) with characteristic equation having only one solution. That is, if \( y_1(t) = c_1 a(t) \) is a family of solutions to the differential equation at hand, we will guess that there is another solution of the form \( v(t) a(t) \) for some nonconstant function \( v(t) \).

Problem: Solve the differential equation \( 2 t^2 y'' + 3 t y' - y = 0 \), \( t > 0 \) given that \( y_1(t) = c_1 t^{-1} \) is one family of solutions.

Solution:

Guess for a second family: \( z(t) = v(t) t^{-1} \).

\[
\begin{align*}
z''(t) &= v''(t) t^{-1} - v'(t) t^{-2} - v(t) t^{-2} + 2 v(t) t^{-3} = v''(t) t^{-1} - 2v'(t) t^{-2} + 2v(t) t^{-3} \\
\end{align*}
\]

We substitute \( z(t) \) and its derivatives into the differential equation:

\[
2t^2 v''(t) t^{-1} - 2 v'(t) t^{-2} + 2 v(t) t^{-3} + 3 t (v'(t) t^{-1} - v(t) t^{-2}) - v(t) t^{-1} = 0
\]

\[
2tv''(t) - 4tv'(t) t^{-1} + 3v'(t) t^{-1} - 3v(t) t^{-1} - v(t) t^{-1} = 0
\]

\[
2tv''(t) - v(t) = 0
\]

At first glance, it may look like we now have to solve another second order differential equation to obtain \( v(t) \), but this is not the case!

Notice that the terms of the equation involve only \( v''(t) \) and \( v'(t) \). There is no term involving \( v(t) \)!

This means that the equation we obtained is a first order linear differential equation for \( w(t) = v'(t) \).

\[
2tw'(t) - w(t) = 0
\]

Thus, we reduced the problem from solving a second order differential equation to solving a first order differential equation. This is precisely the reason the method is named “Reduction of Order”.

We solve the equation

\[
2tw'(t) - w(t) = 0
\]

\[
u(t) = e^{-\int \frac{1}{2t} dt} = e^{-1/2 \ln(t)} = e^{\ln(t^{-1/2})} = t^{-1/2} \]

(We can remove absolute value since \( t > 0 \).)

\[
w(t) t^{-1/2} + C = \int 0 \, dt = D
\]

\[
w(t) = E t^{1/2}
\]

The simplest such \( w(t) \):

\[
w(t) = t^{1/2} \quad (E = 1)
\]

So now we have \( w(t) = v'(t) = t^{1/2} \).

To find \( v(t) \), we take the antiderivative of \( w(t) \).

\[
v(t) = \int t^{1/2} \, dt = 2/3 t^{3/2} + C
\]

Again, to obtain a second family of solutions, we only need ONE \( v(t) \), so we take the simplest one with \( C = 0 \).

\[
v(t) = 2/3 t^{3/2}
\]

Finally, we have a second family of solutions:

\[
y_2(t) = c_2 t^{3/2} t^{-1} = c_2 t^{1/2}
\]
You might wonder where the coefficient $2/3$ in front of $v(t)$ disappeared. Since $2/3c_2$ is just another constant, we rename the entire term $c_2$.

General solution:  
$$y(t) = c_1t^{-1} + c_2t^{1/2}$$

Question: Are the two solutions independent?

Solution:  
$$a(t) = t^{-1}$$  
$$b(t) = t^{1/2}$$  
$$a'(t) = -t^{-2}$$  
$$b'(t) = 1/2t^{-1/2}$$  
$$W(t) = a(b) - a'(b)b(t) = 1/2t^{-1}t^{-1/2} + t^{-2}t^{1/2} = 1/2t^{-3/2} + t^{-3/2} = 3/2t^{-3/2}$$

It clear that $W(t) = 0$ only if $t = 0$.

But the assumption in the problem is that $t > 0$!

Therefore for $t > 0$, the two solutions are independent (form a fundamental set).

Problem: Solve the differential equation $ty'' - y' + 4t^3y = 0$, $t > 0$ given that $y_1(t) = c_1 \sin(t^2)$ is one family of solutions.

Solution:  
**Guess for a second family:** $z(t) = v(t)\sin(t^2)$.  
$$z'(t) = v'(t)\sin(t^2) + 2tv(t)\cos(t^2)$$  
$$z''(t) = v''(t)\sin(t^2) + 2tv'(t)\cos(t^2) + (2v(t) + 2tv'(t))\cos(t^2) - 4t^2v(t)\sin(t^2) = v''(t)\sin(t^2) + 2tv'(t)\cos(t^2) + 2v(t)\cos(t^2) + 2tv'(t)\cos(t^2) - 4t^2v(t)\sin(t^2) = v''(t)\sin(t^2) + 4tv'(t)\cos(t^2) + 2v(t)\cos(t^2) - 4t^2v(t)\sin(t^2)$$

We substitute $z(t)$ and its derivatives into the differential equation:  
$$t(v''(t)\sin(t^2) + 4tv'(t)\cos(t^2) + 2v(t)\cos(t^2) - 4t^2v(t)\sin(t^2)) - (v'(t)\sin(t^2) + 2v(t)\cos(t^2)) + 4t^3v(t)\sin(t^2) = tv''(t)\sin(t^2) + 4t^2v'(t)\cos(t^2) + 2v(t)\cos(t^2) - 4t^2v(t)\sin(t^2) = tv''(t)\sin(t^2) + 4t^2v'(t)\cos(t^2) - v'(t)\sin(t^2) = 0$$

Notice that again all the $v(t)$ terms canceled out allowing us to “reduce the order”.  
Let $w(t) = v'(t)$.

Rewriting the equation in terms of $w(t)$, we obtain a first order linear differential equation:  
$$t\sin(t^2)w'(t) + (4t^2\cos(t^2) - \sin(t^2))w(t) = 0$$

$$w'(t) + \left(\frac{4t^2\cos(t^2) - \sin(t^2)}{t\sin(t^2)}\right)w(t) = 0$$

$$w'(t) + \left(\frac{4t^2\cos(t^2)}{\sin(t^2)} - 1\right)\frac{1}{t}w(t) = 0$$

$$\mu(t) = e^{\int\left(\frac{4t^2\cos(t^2)}{\sin(t^2)} - 1\right)\frac{1}{t}dt}$$

We integrate $\int 4t\cos(t^2) dt$.

$$\mu(t) = \sin(t^2)$$

$$\frac{du}{dt} = 2t\cos(t^2)$$

$$\int 4t\cos(t^2) \frac{du}{dt} dt = 2\int \frac{du}{u} dt = 2\int \frac{1}{u} du = 2 \ln|u| + C = 2 \ln|\sin(t^2)| = \ln((\sin(t^2))^2)$$

(Remember that we ignore constants in the calculation of $\mu(t)$.)

So $\mu(t) = e^{\ln((\sin(t^2))^2)} = e^{\frac{\ln((\sin(t^2))^2)}{t}}$.
To find \( v \), we take the antiderivative of \( w(t) \).
\[
v(t) = \int \frac{t}{\sin(t^2)^2} dt = 1/2 \int \frac{du}{\sin(u)^2} dt = 1/2 \int \frac{1}{\sin(u)^2} du = -1/2 \cot(u) + C = -1/2 \cot(t^2) + C
\]

Again, to obtain a second family of solutions, we only need ONE \( v(t) \), so we take the simplest one with \( C = 0 \).

\[
v(t) = -1/2 \cot(t^2)
\]

Finally, we have a second family of solutions:
\[
y_2(t) = c_2 \cot(t^2) \sin(t^2) = c_2 \cos(t^2)
\]

**General solution:**
\[
y(t) = c_1 \sin(t^2) + c_2 \cos(t^2)
\]

**Question:** Do we always get the “reduction of order”? In other words, do the \( v(t) \) terms always cancel out?

To answer the question suppose \( p(t) y'' + r(t) y' + q(t) y = 0 \) is a differential equation and we know one family of solutions \( y_1(t) = c_1 a(t) \).

Guess a second family: \( z(t) = v(t) a(t) \)
\[
z(t) = v(t) a(t) + v'(t) a'(t) + v''(t) a''(t)
\]

We substitute \( z(t) \) and its derivatives into the differential equation:
\[
p(t) v''(t) a(t) + 2p(t) a'(t) v'(t) a'(t) + p(t) a''(t) v(t) + q(t) a'(t) v'(t) + q(t) a''(t) v(t) + r(t) a(t) v(t) = 0
\]

Since the function \( a(t) \) is a solution to the differential equation \( p(t) y'' + q(t) y' + r(t) y = 0 \), we have
\[
p(t) a''(t) + q(t) a'(t) + r(t) a(t) = 0.
\]

Thus, the coefficient of \( v(t) \) is 0!

So the answer is YES we always get the reduction of order.