Lecture 11 - Functions on Complex Numbers: computing $e^t$

There is a subject in mathematics called **Complex Analysis** which deals with extending Calculus concepts from functions on the real numbers to functions on the complex numbers. Calculus concepts such as limits, derivatives, integrals, and infinite series are extended in appropriate ways to work with complex numbers. Many of the same rules continue to operate.

To handle differential equations $ay'' + by' + cy = 0$ with characteristic equation $ar^2 + br + c = 0$ that has complex solutions, we will need to study how to extend the function $f(t) = e^t$ to the complex numbers. For instance how do you compute $e^i$ or $e^{3i}$ or $e^{1 + 2i}$?

At first exponentiation made sense only for whole number exponents.

**Example 1:** If $n$ is a positive whole number, then $e^n = e \cdot e \cdot \ldots \cdot e$ 

\[
e^3 = e \cdot e \cdot e \approx 20.09\]

Next, exponentiation was extended to 0, in such a way that all usual rules of exponentiation held.

**Example 2:** If $n = 0$, then $e^n = 1$.

Next, exponentiation was extended to negative numbers in such a way that all usual rules of exponentiation held.

**Example 3:** If $n = -m$ is a negative whole number, then $e^n = 1/e^{-n}$.

\[
e^{-3} = 1/e^3 \approx 0.05\]

Next, exponentiation was extended to rational numbers in such a way that all rules of exponentiation held.

**Example 4:** If $a/b$ is a fraction, then $e^{a/b} = \sqrt[b]{e^a}$.

\[
e^{2/3} = \sqrt[3]{e^2} \approx 1.95\]

Next, exponentiation was extended to irrational numbers in such a way that all rules of exponentiation held.

**Example 5:** If $r = s.s_1s_2s_3s_4 \ldots$ is an irrational number, then $e^r = \lim_{n \to \infty} e^{s.s_1 \ldots s_n}$.

\[
e^\pi = \lim\{e^3, e^{3.1}, e^{3.14}, e^{3.141}, e^{3.14159}, \ldots\} \approx 23.14\]

**Question:** How was exponentiation extended to complex numbers?

There are various formulas to approximate $f(t) = e^t$.

**Formula 1:** $e^t = \lim_{n \to \infty} (1 + \frac{1}{n})^nt$

**Formula 2:** $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$

Let us make some calculations using the second formula:

\[
e^1 = 2.71828 \ldots\]
\[
1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} = 1 + 1 + 1/2 + 1/6 + 1/24 = 2.70833 \ldots\]
\[
1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} = 1 + 1 + 1/2 + 1/6 + 1/24 + 1/120 = 2.71666 \ldots\]
\[
1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \frac{1^6}{6!} = 1 + 1 + 1/2 + 1/6 + 1/24 + 1/120 + 1/720 = 2.71805 \ldots\]
\[
1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \frac{1^6}{6!} + \frac{1^7}{7!} = 1 + 1 + 1/2 + 1/6 + 1/24 + 1/120 + 1/720 + 1/5040 = 2.71825 \ldots\]

\[
e^{1/2} = 1.64871 \ldots\]
\[
1 + 1/2 + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \frac{(1/2)^4}{4!} = 1.64843 \ldots\]
\[
1 + 1/2 + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \frac{(1/2)^4}{4!} + \frac{(1/2)^5}{5!} = 1.64869 \ldots\]
\[
1 + 1/2 + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \frac{(1/2)^4}{4!} + \frac{(1/2)^5}{5!} + \frac{(1/2)^6}{6!} = 1.64871 \ldots\]
Formally, we call \(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \) an **infinite series** and write it mathematically as:

\[
e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}
\]

The symbol \(\sum\) means sum! We calculate \(\sum_{n=0}^{\infty} \frac{t^n}{n!}\) by finding \(\lim_{n \to \infty} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!}\)

The functions \(f(t) = \sin(t)\) and \(g(t) = \cos(t)\) can also be approximated by infinite series!

Here is the infinite series for \(\sin(t)\):

\[
\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}
\]

Notice that the infinite series for \(\sin(t)\) looks very similar to the infinite series for \(e^t\). The difference is that the \(\sin(t)\) series skips all the **even** powers of \(t\) and alternates + with −.

Let us make some calculations using the infinite series for \(\sin(t)\):

\[
\sin(1) = 0.84147 \cdots
\]

\[
1 - \frac{1^3}{3!} + \frac{1^5}{5!} - \frac{1^7}{7!} + \frac{1^9}{9!} = 1 - 1/6 + 1/120 - 1/5040 = 0.84146 \cdots
\]

\[
\sin(2) = 0.90929 \cdots
\]

\[
2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} = 2 - 8/6 + 32/120 - 128/5040 = 0.90793 \cdots
\]

Here is the infinite series for \(\cos(t)\):

\[
\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}
\]

Notice that the infinite series for \(\cos(t)\) also looks very similar to the infinite series for \(e^t\). The difference is that the \(\cos(t)\) series skips the **odd** powers of \(t\) and alternates + and −.

Let us make some calculations using the infinite series for \(\cos(t)\):

\[
\cos(1) = 0.54030 \cdots
\]

\[
1 - \frac{1^2}{2!} + \frac{1^4}{4!} - \frac{1^6}{6!} = 1 - 1/2 + 1/24 - 1/720 = 0.54027 \cdots
\]

We use the formula

\[
e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \cdots
\]

to extend the function \(e^t\) to complex numbers. Using this formula preserves all the rules of exponentiation.

**Example:**

\[
e^i = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \frac{i^4}{4!} + \frac{i^5}{5!} + \frac{i^6}{6!} + \frac{i^7}{7!} + \cdots + \cdots
\]

How do we compute what complex number this infinite sum is equal to?

We use the table below which simplifies powers of \(i\) and infinite series for \(\sin(t)\) and \(\cos(t)\).

\[
i^1 = i
\]
\[
i^2 = -1
\]
\[
i^3 = -i
\]
\[
i^4 = 1
\]
\[
i^5 = i
\]
\[
i^6 = -1
\]
\[
i^7 = -i
\]
\[
i^8 = 1
\]
\[
\vdots
\]
\[ e^i = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \frac{i^4}{4!} + \frac{i^5}{5!} + \frac{i^6}{6!} + \frac{i^7}{7!} + \cdots = \]
\[ 1 + i - \frac{1}{2!} - \frac{i}{3!} + \frac{1}{4!} - \frac{i}{5!} - \frac{1}{6!} + \frac{i}{7!} + \cdots = \]
\[ (1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots) + i(1 + \frac{1}{3!} - \frac{1}{5!} + \frac{1}{7!} + \cdots) = \]
\[ \cos(1) + i \sin(1) \approx 0.54 + 0.84i \]

A general calculation gives us the formula:
\[ e^{it} = \cos(t) + i \sin(t) \]

This equation was first discovered by Euler and it is known as **Euler’s Formula**.

**Examples:**

1. \( e^{2i} = \cos(2) + i \sin(2) \approx -0.42 + 0.91i \)
2. \( e^{-1+i} = e^{-1}e^i = e^{-1}(\cos(1) + i \sin(1)) = e^{-1} \cos(1) + ie^{-1} \sin(1) \approx 0.20 + 0.31i \)
   (We are assuming here that using the infinite series for \( e^t \) with complex numbers preserves all the rules of exponentiation!)
3. \( e^{2-3i} = e^2e^{-3i} = e^2(\cos(-3) + i \sin(-3)) = e^2 \cos(-3) + ie^2 \sin(-3) \approx -7.32i - 1.04i \)