Write $0.02303$ as a fraction.

**Solution:**

$$0.02303 = 0.02 + \frac{303}{10^5} + \frac{303}{10^8} + \ldots$$

the second part is a geometric series with initial term $a = \frac{303}{10^5}$ and $r = \frac{1}{1000}$, and so using the formula $a/(1 - r)$ for the geometric series, which is allowed since $r < 1$, we get

$$\frac{2}{100} + \frac{\frac{303}{100000}}{1 - \frac{1}{1000}} = \frac{2}{100} + \frac{\frac{303}{100000}}{\frac{100 - 1}{100}} = \frac{2 \times 999}{100 	imes 999} + \frac{303}{99900} = \frac{2301}{99900}$$

where in the second step, we multiplied top and bottom with $100,000$ to simplify the compounded fraction.

Write the following definite integral

$$\int_0^{1/2} \ln(1 + 4x^2) \, dx$$

as a series and show that it is convergent, using the following steps:

1. find a power series for $\frac{1}{1 + 4x}$ and determine its radius of convergence;
2. using anti-derivatives and the series from (1), find a power series for $\ln(1 + 4x)$ and determine its radius of convergence;
3. using substitution and the series from (2), find a power series for $\ln(1 + 4x^2)$ and determine its radius of convergence;
4. using anti-derivatives and the series from (3), find a power series for the indefinite integral $\int \ln(1 + 4x^2) \, dx$ and determine its radius of convergence;
5. use the fundamental theorem of calculus to find a series for $\int_0^{1/2} \ln(1 + 4x^2) \, dx$, and show that it converges.

**Solution:**

(1) Using the geometric series $\frac{1}{1-a} = \sum a^n$ for $a = -4x$, we get

$$\frac{1}{1 + 4x} = \sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-4)^n x^n.$$
(2) Taking anti-derivatives of both sides (and compensating with a factor $1/4$ caused by the chain rule), we get

$$\frac{1}{4} \ln(1 + 4x) = \sum_{n=0}^{\infty} \frac{(-4)^n x^{n+1}}{n+1}$$

which has the same radius of convergence $R = 1/4$;

(3) Bringing the factor to the other side and substituting $x^2$ for $x$ gives

$$\ln(1 + 4x^2) = 4 \sum_{n=0}^{\infty} \frac{(-4)^n (x^2)^{n+1}}{n+1} = 4 \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n+2}}{n+1}$$

and this converges for $|x^2| = |x|^2 < 1/4$, whence $|x| < 1/2$;

(4) Taking again the anti-derivative and then using the fundamental theorem of calculus gives

$$\int_{0}^{1/2} \ln(1 + 4x^2) \, dx = 4 \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n+3}}{n+1(2n+3)} \bigg|_{0}^{1/2}$$

$$= 4 \sum_{n=0}^{\infty} \frac{(-4)^n}{(n+1)(2n+3)} \left( \frac{1}{2} ight)^{2n+3} - 0$$

$$= \frac{4}{8} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(2n+3)}$$

where we simplified in the last line by writing

$$\left( \frac{1}{2} \right)^{2n+3} = \left( \frac{1}{2} \right)^{2n} \left( \frac{1}{2} \right)^{3} = \left( \left( \frac{1}{2} \right)^2 \right)^n \frac{1}{8} = \left( \frac{1}{4} \right)^n \frac{1}{8} = \frac{1}{8} \cdot 4^n.$$

The final series converges by the alternating series test since the limit of the general term (without sign) $\lim_{n \to \infty} \frac{1}{(n+1)(2n+3)} = 1/\infty = 0$. 