MAT 1575 Final Exam Review Problems

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1. Evaluate the following integrals:
   a. \( \int \frac{x}{\sqrt{x^2 + 9}} \, dx \)
   b. \( \int \frac{3x^2}{\sqrt[3]{x^3 - 5}} \, dx \)

2. Evaluate the following integrals:
   a. \( \int 5xe^{-5x} \, dx \)
   b. \( \int x^2 e^{-x^3} \, dx \)
   c. \( \int x \cos(3x) \, dx \)

3. Find the area of the region enclosed by the graphs of:
   a. \( y = 3 - x^2 \) and \( y = -2x \)
   b. \( y = x^2 - 2x \) and \( y = x + 4 \)

4. Find the volume of the solid found by revolving about the x-axis. The region is bounded by the graphs of:
   a. \( y = x^2 - 9 \) and \( y = 0 \)
   b. \( y = -x^2 + 9 \) and \( y = 0 \)

5. Evaluate the following integrals:
   a. \( \int \frac{1}{x^2 \sqrt{36 - x^2}} \, dx \)
   b. \( \int \frac{\sqrt{x^2 - 9}}{x^4} \, dx \)
   c. \( \int \frac{9}{x^2 \sqrt{x^2+9}} \, dx \)
   d. \( \int \frac{6}{x^2 \sqrt{x^2 - 36}} \, dx \)

6. Evaluate the following integrals:
   a. \( \int \frac{3 - 5x^2}{x^3 + 6x} \, dx \)
   b. \( \int \frac{5x + 6}{x^2 - 36} \, dx \)
   c. \( \int \frac{3x + 2}{x^2 + 2x - 8} \, dx \)

7. Determine whether the integral is convergent or divergent. Evaluate the integral if convergent:
   a. \( \int_0^\infty \frac{2}{(x+2)^3} \, dx \)
   b. \( \int_0^\infty \frac{5}{\sqrt[3]{x}+5} \, dx \)
   c. \( \int_4^\infty \frac{3}{\sqrt[4]{(x-3)^4}} \, dx \)
8. Decide if the following series converges or not. Justify your answer using an appropriate test:
   a. \( \sum_{n=1}^{\infty} \frac{9n^2}{3n^5 + 5} \)  
   b. \( \sum_{n=1}^{\infty} \frac{5}{10^n} \)  
   c. \( \sum_{n=1}^{\infty} \frac{5n}{10^n} \)  
   d. \( \sum_{n=1}^{\infty} 5n(0.1)^n \)

9. Determine whether the series is absolutely or conditionally convergent or divergent:
   a. \( \sum_{n=1}^{\infty} (-1)^n \frac{10}{7n + 2} \)  
   b. \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^5}} \)  
   c. \( \sum_{n=0}^{\infty} (-1)^n 5^{-n} \)  
   d. \( \sum_{n=1}^{\infty} (-1)^n \frac{n^2 - n - 1}{2n^2 + n + 1} \)

10. Find the radius and the interval of convergence of the following power series:
   a. \( \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{5^n} x^n \)  
   b. \( \sum_{n=0}^{\infty} (-7)^{n+2} x^{n+1} \)

11. Find the Taylor polynomial of degree 2 for the functions:
   a. \( f(x) = x^2 e^{-x} \) at \( a = -1 \)  
   b. \( f(x) = 2x \cos(5x) \) at \( a = 2\pi \).

12. Find the Taylor polynomial of degree 3 for the given function, centered at the given number \( a \).
    Graph the function and the Taylor polynomial on the same screen:
   a. \( f(x) = 1 + xe^{-x} \) at \( a = -1 \)  
   b. \( f(x) = \sin(x) \) at \( a = \frac{\pi}{2} \)

Answers:

(1a). \( \sqrt{x^2 + 9} + C \)  
(1b). \( \frac{3}{2} \sqrt{(x^3 - 5)^2} + C \)

(2a). \( -xe^{-5x} - \frac{1}{5} e^{-5x} + C \)  
(2b). \( -(x^2 + 2x + 2)e^{-x} + C \)  
(2c). \( \frac{1}{3} x \sin(3x) + \frac{1}{9} \cos(3x) + C \)

(3a). The area of the region between the two curves is:

\[ \text{Area} = \int_{-1}^{3} (3 - x^2 - (-2x)) \, dx = \frac{32}{3} \]
(3b). The area of the region between the two curves is:

\[ \text{Area} = \int_{-1}^{4} \left( x + 4 - (x^2 - 2x) \right) dx = \frac{125}{6} \]

(4a). Approximate the volume of the solid by vertical disks with radius \( y = x^2 - 9 \) between \( x = -3 \) and \( x = 3 \);

in the limit, the volume is \( V = \int_{-3}^{3} \pi (x^2 - 9)^2 \, dx = \frac{1296}{5} \pi \)

(4b). The solid of revolution is the same by symmetry, therefore the volume must be the same as well:

\[ V = \int_{-3}^{3} \pi (-x^2 + 9)^2 \, dx = \frac{1296}{5} \pi \]

(5a). \(-\frac{\sqrt{36 - x^2}}{36} + C \)  
(5b). \( \frac{(x^2 - 9)^{3/2}}{27x^3} + C \)  
(5c). \(-\frac{\sqrt{x^2 + 9}}{x} + C \)  
(5d). \( \frac{\sqrt{x^2 - 36}}{6x} + C \)

(6a). \( \frac{\ln|x|}{2} - \frac{11}{4} \ln(x^2 + 6) + C \)  
(6b). \( 3 \ln|x - 6| + 2 \ln|x + 6| + C \)  
(6c). \( \frac{5}{3} \ln|x + 4| + \frac{4}{3} \ln|x - 2| + C \)

(7a). \( \frac{1}{4} \)  
(7b). The integral does not converge  
(7c). 9

(8a). The sum converges by the Comparison (or Limit Comparison) Test and the p-Test.

(8b). This is a geometric series, which converges to 5/9:

\[ \sum_{n=1}^{\infty} \frac{5}{10^n} = \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \cdots = \frac{5}{1 - r} = \frac{5/10}{1 - 1/10} = \frac{5/10}{9/10} = \frac{5}{9} \]

(8c). The series converges by the Ratio Test. (8d). This is the same series as in (8c).

(9a). Conditionally convergent: convergent by the Alternating Series Test but not absolutely convergent since \( \sum_{n=1}^{\infty} \frac{10}{7n + 2} \) diverges (like the Harmonic series, by the Limit Comparison Test).
(9b). Absolutely convergent: convergent by the Alternating Series Test and absolutely convergent since \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \) converges by the p-Test with \( p = 2.5 \).

(9c). Absolutely convergent: convergent by the Alternating Series Test and absolutely convergent since \( \sum_{n=0}^{\infty} 5^{-n} = \frac{1}{1 - 1/5} = \frac{5}{4} \) converges as a geometric series.

(9d). Divergent: by the Test for Divergence. The limit of the general term does not exist. Note that the Alternating Series Test does not apply.

(10a). The radius of convergence is 5 and the interval of convergence is \( |x|<5 \) or \(-5<x<5\). This follows from either using the Ratio Test or from recognizing that the power series is a geometric series \( \sum_{n=0}^{\infty} \left( \frac{-x}{5} \right)^n \) which converges when \( \left| \frac{-x}{5} \right|<1 \) thus giving the interval of convergence \( |x|<5 \). Note that the series does not converge at the end points \( x = \pm 5 \). Inside the interval of convergence, one can sum up the geometric series to get the function: \( -\sum_{n=0}^{\infty} \left( \frac{-x}{5} \right)^n = - \frac{1}{1 - (-x/5)} = - \frac{1}{1 + x/5} = - \frac{5}{x+5} \)

(10b). The radius of convergence is 1/7 and the interval of convergence is \( |x|<\frac{1}{7} \) or \(-\frac{1}{7}<x<\frac{1}{7}\). This follows from either using the Ratio Test or from recognizing that the power series is a geometric series \( (-7)^2 x \sum_{n=0}^{\infty} (-7x)^n \) which converges when \( |-7x|<1 \) thus giving the interval of convergence \( |x|<\frac{1}{7} \). Note that the series does not converge at the end points \( x = \pm 1/7 \). Inside the interval of convergence, one can sum up the geometric series to get the function: \( (-7)^2 x \sum_{n=0}^{\infty} (-7x)^n = 49x \frac{1}{1 - (-7x)} = \frac{49x}{1 + 7x} \)

(11a). \( T_2(x) = e^2 - 4e^2(x+1) + 7e^2(x+1)^2 \)

(11b). \( T_2(x) = 4\pi + 2(x-2\pi) - 50\pi(x-2\pi)^2 \)

(12a). \( T_3(x) = 1 - e + 2e(x+1) - \frac{3}{2}e(x+1)^2 + \frac{2}{3}e(x+1)^3 \)

(12b). \( T_3(x) = 1 - \frac{1}{2} \left( x - \frac{\pi}{2} \right)^2, \) only even powers exist