#1 Evaluate the following limits, if they exist:

a) \[ \lim_{x \to 0} \frac{e^{10x} - e^{-5x}}{25x} \]  
b) \[ \lim_{x \to 1} \frac{x^{15} - 1}{x^5 - 1} \]  
c) \[ \lim_{x \to 0} \frac{\sin 7x - 5x}{7x} \]  
d) \[ \lim_{x \to 0} \frac{e^{4x} - 1}{7 \sin x} \]

a) \[ \lim_{x \to 0} \frac{x^2 - x}{\sin 2x} \]  
f) \[ \lim_{x \to \pi} \frac{\cos x + 1}{3 \sin x} \]  
g) \[ \lim_{x \to 1} \frac{2x^{10} - 2}{3x^7 - 3} \]

#2 Find the derivatives of the following functions using the definition of derivative:

a) \[ f(x) = 2x^2 - 5x \]  
b) \[ f(x) = -2x^2 + 3x - 4 \]

#3 Find the derivatives of the following functions:

a) \[ f(x) = 3x^4 \sec (5x) \]  
b) \[ f(x) = \sqrt{x} \tan (3x) \]  
c) \[ f(x) = \frac{4x^2 - 5}{2x^2 - 1} \]  
d) \[ f(x) = \sin (7x) \cos (5x) \]  
e) \[ f(x) = \frac{2x^2}{x^2 - 16} \]

#4 Find the derivative \( y' = \frac{dy}{dx} \) of the following functions, using logarithmic differentiation:

a) \[ y = (3x + 1)^5 \left( x^4 - 2 \right)^7 \left( 5x^5 + 3 \right)^9 \]  
b) \[ y = x^x \]

#5 Find the equation of the tangent line, in slope-intercept form, to the curve:

a) \[ f(x) = 2x^3 + 5x^2 + 6 \] at \((-1,9)\)  
b) \[ f(x) = 4x - x^2 \] at \((1,3)\)

#6 Using implicit differentiation, find the equation of the tangent line to the given curve at the given point:

a) \[ 3x^2y^2 - 3y - 17 = 5x + 14 \] at \((-1,-3)\)  
b) \[ y^2 - 7xy + x^3 - 2x = 9 \] at \((0,3)\)

#7 If a snowball melts so its surface area decreases at a rate of 1 cm²/min, find the rate at which the diameter decreases when the diameter is 6 cm.

#8 The radius of a sphere increases at a rate of 3 in/sec. How fast is the volume increasing when the diameter is 24in?

#9 The radius of a cone is increasing at a rate of 3 inches/sec, and the height of the cone is 3 times the radius. Find the rate of change for the volume of that cone when the radius is 7 inches.

#10 Sand pours from a chute and forms a conical pile whose height is always equal to its base diameter. The height of the pile increases at a rate of 5 feet/hour. Find the rate of change of the volume of the sand in the conical pile, when the height of the pile is 4 feet.

#11 A cylindrical tank with radius 8 m is being filled with water at a rate of 2 m³/min. What is the rate of change of the water height in this tank?

#12 A box with a square base and an open top must have a volume of 256 cubic inches. Find the dimensions of the box that will minimize the amount of material used (the surface area).

#13 A farmer wishes to enclose a rectangular plot using 200 m of fencing material. One side of the land borders does not need fencing. What is the largest area that can be enclosed?
#14 Sketch the graph of the given polynomial. Calculate and label all local maxima, minima, and points of inflection:

\[ f(x) = x^3 - 3x^2 - 1 \]

#15 Sketch the graphs of the following functions. Check for symmetry, find and label any intercepts and asymptotes:

a) \( f(x) = \frac{-2}{4 - x^2} \)  

b) \( f(x) = \frac{x}{x^2 - 1} \)  

c) \( f(x) = \frac{2x^2}{x^2 - 4} \)

#16 Evaluate each of the following definite integrals:

a) \( \int \frac{2 - 3x^4 + 4}{x^3} \, dx \)  

b) \( \int \frac{4e^x + 3}{-3} \, dx \)  

c) \( \int \frac{3x^6 - 2x^2 + 2e^x + 2}{x^4} \, dx \)

Answers to questions:

One can solve all problems in #1 just using l'Hospital's rule.

#1  

a) \(-\frac{1}{5}\)  

b) 3  

c) \(\frac{2}{7}\)  

d) \(\frac{4}{7}\)  

e) \(-\frac{1}{2}\)  

f) 0  

g) \(\frac{20}{21}\)

#2 a) 

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h)^2 - 5(x+h) - (2x^2 - 5x)}{h} = \lim_{h \to 0} \frac{4xh + 2h^2 - 5h}{h} = \lim_{h \to 0} \frac{4x + 2h - 5}{h} = \lim_{h \to 0} (4x + 2h - 5) = 4x - 5 \]

b) \(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-2(x+h)^2 + 3(x+h) - 4 - (-2x^2 + 3x - 4)}{h} = \lim_{h \to 0} \frac{-4xh - 2h^2 + 3h}{h} = \lim_{h \to 0} (-4x - 2h + 3) = -4x + 3 \)

#3 a) \(12x^3 \sec(5x) + 15x^4 \sec(5x) \tan(5x)\)  

b) \(\frac{1}{2} \tan(3x) + \sqrt{x} (1 + \tan^2(3x))\)  

c) \(\frac{12x}{(2x^2 - 1)^2}\)

d) \(7 \cos(7x) \cos(5x) - 5 \sin(7x) \sin(5x)\)  

e) \(\frac{-64x}{(x^2 - 16)^2}\)

#4 a) Take ln of both sides: \(\ln y = 5 \ln(3x + 1) + 7 \ln(x^4 - 2) + 9 \ln(5x^5 + 3)\) then take the derivative of both sides w.r.t \(x\):  

\[ \frac{y'}{y} = \frac{15}{3x + 1} + \frac{28x^3}{x^4 - 2} + \frac{225x^4}{5x^5 + 3} \]

and after multiplying both sides by \(y\), expressing \(y\) and simplifying, we get:

\[ y' = 15(3x + 1)^4(x^4 - 2)^7(5x^5 + 3)^9 + 28x^3(3x + 1)^5(x^4 - 2)^6(5x^5 + 3)^9 + 225x^4(3x + 1)^5(x^4 - 2)^7(5x^5 + 3)^8 \]

b) Follow the steps in part a) to get \(y' = (1 + \ln x) x^x\)

#5 a) The equation of the tangent line at \(x = -1\) is given by \(y = 5 - 4x\).

b) The equation of the tangent line at \(x = 1\) is given by \(y = 1 + 2x\).

#6 a) The derivative as a function \(y' = \frac{5 - 6xy^2}{6x^2y - 3}\) one computes by implicit differentiation. The slope of the tangent line at the given point is the derivative evaluated at \((1, -3)\), that is \(y'(1) = \frac{7}{3}\). The equation of the tangent line is given by:

\(y = -3 + \frac{7}{3}(x - 1) = \frac{7}{3}x - \frac{16}{3}\)  

\(b) \quad y' = \frac{7y - 3x^2 + 2}{2y - 7x}, \quad y'(0) = \frac{23}{6}, \quad y = 3 + \frac{23}{6}(x - 0) = 3 + \frac{23}{6}x\)
#7  The rate of change of the diameter is \( \frac{dD}{dt} = -\frac{1}{12\pi} = -0.0265 \text{ cm/min} \)

#8  The rate of change of the volume is \( \frac{dV}{dt} = 1728\pi \text{ in}^3/\text{sec} \)

#9  \( \frac{dV}{dt} = 441\pi = 1385.4 \text{ in}^3/\text{sec} \) (Note: \( V = \frac{1}{3}\pi r^2 h = \pi r^3 \), since \( h = 3r \))

#10 \( \frac{dV}{dt} = 20\pi = 62.8 \text{ ft}^3/\text{hour} \) (Note: \( V = \frac{1}{12}\pi h^3 \), since \( r = \frac{h}{2} \))

#11 \( V = \pi r^2 h = 64\pi h \rightarrow \frac{dV}{dt} = 64\pi \frac{d}{dt} \frac{dh}{dt} + \frac{dV}{dt} = \frac{1}{64\pi} \frac{dV}{dt} = \frac{2}{64\pi} = \frac{1}{32\pi} \text{ m/min} \)

#12  The base is 8 and height is 4, thus the dimensions are: 8×8×4

#13  The area is maximized when one side of the rectangle is 50m and the other is 100m, which gives an area of 5000m².

#14  The relative minimum is at the point (2,-5), the relative maximum is at (0,-1). The second derivative changes sign at the point on the curve (1,-3), which is thus an inflection point.

#15  a) Even function \( f(-x) = f(x) \), symmetric w.r.t. the y-axis
    Vertical asymptotes at \( x = \pm 2 \)
    Horizontal asymptote at \( y = 0 \)

   b) Odd function \( f(-x) = -f(x) \), symmetric w.r.t. zero
    Vertical asymptotes at \( x = \pm 1 \)
    Horizontal asymptote at \( y = 0 \)

   c) Even function \( f(-x) = f(x) \), symmetric w.r.t. the y-axis
    Vertical asymptotes at \( x = \pm 2 \)
    Horizontal asymptote at \( y = 2 \)

#16  a) \( 2 \int_{1}^{3} x^4 - 3 \int_{1}^{3} x^4 dx + 4 \int_{1}^{3} x^3 dx = 2(3-1) - 3\left( \frac{3^5}{5} - \frac{1^5}{5} \right) + 4\left( \frac{3^2}{2} - \frac{1^2}{2} \right) = \frac{6274}{45} = -139.422 \)

b) \( 4e^3 e^{2x} \left[ e^{2x} \right]_3^4 = 4e^3 e^2 - 4e^3 e^3 = 4e^5 - 4 \approx 589.65 \)

c) \( 3 \int_{1}^{4} x^2 dx - 4 \int_{1}^{4} x^2 dx + 2e^4 \int_{1}^{4} e^x dx = 3 \left( \frac{4^3}{3} - \frac{1^3}{3} \right) - 2\left( \frac{4^{-1}}{-1} - \frac{1^{-1}}{-1} \right) + 2e^2 (e^4 - e^1) = \frac{123}{2} + 2e^6 - 2e^3 \)