PARAMETERIZED STRATIFICATION AND PIECE NUMBER OF D-SEMIANALYTIC SETS

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Abstract. We obtain results on the geometry of $D$-semianalytic and subanalytic sets over a complete, non-trivially valued non-Archimedean field $K$, which is not necessarily algebraically closed. Among the results are the Parameterized Smooth Stratification Theorem and several results concerning the dimension theory of $D$-semianalytic and subanalytic sets. Also, an extension of Bartenwerfer’s definition of piece number for analytic $K$-varieties is provided for the $D$-semianalytic sets and the existence of a uniform bound for the piece number of the fibers of a $D$-semianalytic set is proved. There is a connection between the piece number and the complexity of a $D$-semianalytic set which is a subset of the affinoid line and therefore a simpler proof of the Complexity Theorem of Lipshitz and Robinson is made possible by these results. Finally we prove an analogue of a theorem by van den Dries, Haskell and Macpherson, which states that for each $D$-semianalytic $X$, there is a semialgebraic $Y$ such that one dimensional fibers of $X$ are among the one dimensional fibers of $Y$ through an easy application of our earlier results.

1. Introduction

In this paper we continue the investigation which started in [3] of basic geometric properties of $D$-semianalytic subsets of $(K^o)^m$, where $K$ is an arbitrary (i.e. not necessarily algebraically closed) non-trivially valued, complete non-Archimedean field and $K^o$ is its valuation ring. For most of our results we impose the additional condition that $\text{Char } K = 0$.

Two of our main results appear int the title of this paper. The first of these guarantees the existence of a decomposition of a $D$-semianalytic set into finitely many manifolds which remain manifolds when specialized at parameters (Theorem 4.3) and the second one guarantees the existence of a uniform bound on the piece numbers of fibers of a $D$-semianalytic set (Theorem 5.5). Moreover we establish a connection between the piece number and the complexity of one dimensional $D$-semianalytic sets to give a simpler proof of the existence of a uniform bound for the complexity of the one-dimensional fibers of $D$-semianalytic sets (Theorem 6.7) which was first proved by Lipshitz and Robinson in [12]. Along the way we also prove new results on the dimension theory of $D$-semianalytic and subanalytic sets. These are mostly the properties of a dimension function which are expected to hold for a reasonably well-behaved class of sets. Moreover we also prove that several notions of dimension agree for the subanalytic sets (this was proved for the $D$-semianalytic sets earlier in [3] by the author) and investigate the question of how the dimension of these sets changes if they are embedded in $(K'^o)^m$, where $K'$ is a

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complete extension of $K$. We finish by showing that for each $D$-semianalytic set $X$ there is a semialgebraic set $Y$ such that the one dimensional fibers of $X$ are among the one-dimensional fibers of $Y$ (Theorem 6.8). This is an analogue of Theorem A of [6] by van den Dries, Haskell and Macpherson.

We will introduce our main objects of study thoroughly in Section 2, but here we are going to briefly go over what they are and how they were developed. The $D$-semianalytic subsets of $(K^\circ)^m$ are finite unions of sets of points which satisfy finitely many norm inequalities (both strict and non-strict) between $D$-functions. These functions are obtained from members of rings of separated power series $S_{m,n}$ through the use of the restricted division operators $D_0$ and $D_1$, and substitution (see Definitions 2.2 and 2.3). The rings of separated power series $S_{m,n}(E,K)$ are superrings of the Tate algebras $T_{m+n}(K)$ and were first introduced by Lipshitz in [8]. This came out of the need to treat the case where $K$ is non-discretely valued, while proving that given a coordinate projection $\pi$, for each affinoid analytic $K$-variety $X$, there is a bound $\Gamma$ such that the fibers of $X$ under $\pi$ have at most $\Gamma$ isolated points, when $\text{Char} K = 0$ or $\text{Char} K = p > 0$ and $[K : K^p] < \infty$. In [1], Bartenwerfer introduced the notion of the piece number of an analytic $K$-variety and proved that under the same conditions on the characteristic of $K$ as above, an analytic $K$-variety $X$ which is the zero set of an ideal of $S_{m,n}$, there is a bound $\Gamma$ such that the fibers of $X$ under a coordinate projection have piece numbers less than $\Gamma$. As the piece number dominates the number of isolated points, this was a strengthening of the main theorem of [8].

In [9] Lipshitz showed that the class of subanalytic sets (i.e. the projections of $D$-semianalytic sets) coincides with the class of the $D$-semianalytic sets when $K$ is algebraically closed thus obtaining a quantifier elimination theorem for the analytic theory of such fields and later this result was generalized to a more general class of definable sets by Lipshitz and Robinson in [14]. Meanwhile in [13], the same authors obtained results on the geometry of subanalytic subsets of $(K^\circ)^m$ defined by $D$-functions with coefficients from $K$ where $K$ is an arbitrary complete non-Archimedean field and $\bar{K}$ is an algebraically closed complete extension of $K$. Among those results were the Smooth Stratification Theorem and results on the dimension of subanalytic sets. It was a natural question to ask how much of these results could be carried over to $D$-semianalytic subsets of $(K^\circ)^m$. Results similar to those in [13] and further results on the dimension theory of $D$-semianalytic sets in this more general context were obtained in [3]. One of the main tools in obtaining these results, the Parameterized Normalization Lemma for $D$-semianalytic Sets of [3], plays an essential role in proving many of the results in this paper too. The proof of this lemma relies on the ideas and results of [13], where much of the commutative algebra of the rings $S_{m,n}$ and quasi-affinoid algebras are worked out.

On the other hand a special type of $D$-semianalytic sets over an algebraically closed field $K$, the $R$-domains, stands out in the study of the quasi-affinoid geometry. These sets generalize the rational domains of affinoid geometry as in Definition 5 of §7.2.3 of [2] and some of their geometric properties were established by Lipshitz and Robinson in [11] and [12]. One of the main results of [11] is that such subsets of $K^\circ$ can be written as boolean combinations of discs (see Definition 6.1). In [12] the complexity of an $R$-subdomain of $K^\circ$ was defined to be the number of discs appearing in such a combination and it was shown that given a $D$-semianalytic subset $X$ of $(K^\circ)^{m+1}$ there is a bound $\Gamma$ such that for each parameter $\bar{p} \in (K^\circ)^m$, the fiber
$X(p)$ of $p$ in $X$ differs from an $R$-domain of complexity less than $\Gamma$ by at most $\Gamma$ points. This result is analogous to the Theorem A of [6] by van den Dries, Haskell and Macpherson which states that the one-dimensional fibers of a subanalytic (in the language of [5]) subset of $\mathbb{Z}_p^m+1$ can be obtained as one-dimensional fibers of a semialgebraic (in Macintyre’s Language) subset of some $\mathbb{Z}_p^M+1$. As explained in [6], the bound on “complexity” of one dimensional fibers follows immediately from this theorem.

The outline of this paper is follows. After the preliminary definitions in Section 2, we revisit the dimension theory of $D$-semianalytic sets in Section 3. In this section there are also refinements of the Parameterized Normalization Theorem of [3], in preparation to prove the Parameterized Smooth Stratification Theorem of Section 4. The methods developed for proving this theorem also have important consequences in the dimension theory of $D$-semianalytic and subanalytic sets and we spend the rest of the Section 4 discussing these. At the same time, the Parameterized Smooth Stratification Theorem lays out the groundwork for the results of Section 5. We start Section 5 by extending the definition of piece number for analytic $K$-varieties due to Bartenwerfer, to the $D$-semianalytic sets and prove the Piece Number Theorem for the fibers of those sets. In Section 6, we turn our attention to the fibers of $D$-semianalytic sets which are subsets of the affinoid line and establish a relation between the piece number and the complexity of such fibers resulting in a new, simpler proof of the Complexity Theorem of [12]. This theorem is analogous to the Theorem A of [6] in the $p$-adic setting, and we will finish by providing another, more readily recognizable, analogue of Theorem A in our setting. We would like to note that one could obtain the results guaranteeing the existence of uniform bounds like the Complexity Theorem by using non-standard models approach of [6].

2. Preliminaries

In this section we give the definitions and establish basic properties of our objects of study. In doing so, we follow [15] by Lipshitz and Robinson, where these objects were studied extensively. These objects are mainly the rings of separated power series and geometric and algebraic objects related to them.

In what follows, $K$ denotes an arbitrary non-trivially valued non-Archimedean complete field unless stated otherwise.

**Definition 2.1.** Let $x = (x_1,...,x_m)$ and $\rho = (\rho_1,...,\rho_n)$ denote variables, fix a complete, quasi-Noetherian subring $E$ of $K^\circ$ (which also has to be a discrete valuation ring in case $K$ is of positive characteristic) and let $\{a_i\}_{i \in \mathbb{N}}$ be a zero sequence in $K^\circ$, and $B$ be the local quasi-Noetherian ring

$$(E[a_0, a_1, ... \{a \in E|a_0,a_1,...:|a|=1\})^\wedge,$$

where $^\wedge$ denotes the completion in $|\cdot|$. Let $\mathcal{B}$ be the family of all such rings. Define the separated power series ring over $(E,K)$ of $(m,n)$ variables to be

$$S_{m,n}(E,K) := K \otimes_{K^\circ} \left( \varprojlim_{B \in \mathcal{B}} B \langle x \rangle \langle \rho \rangle \right).$$

For $f = \sum_{\alpha,\beta} a_{\alpha,\beta} x^\alpha \rho^\beta \in S_{m,n}(E,K)$, the Gauss norm of $f$ is defined as

$$||f|| := \sup_{\alpha,\beta} |a_{\alpha,\beta}|.$$
We will write \( S_{m,n} \) instead of \( S_{m,n}(E,K) \) when the ring \( E \) and the field \( K \) are clear from the context. We will also make use of two key results on algebra of these rings from [15] throughout. Specifically, these results are that the rings \( S_{m,n} \) are Noetherian and we have suitable Weierstrass Preparation and Division Theorems over these rings.

The ring \( S_{m,n} \) contains our “global” analytic functions. In other words, members of \( S_{m,n}(E,K) \) are convergent and have global power series expansions over the set \((K^\circ)^m \times (K^\circ)^n\) where \( K^\circ \) denotes the maximal ideal of \( K \). Nevertheless we are also interested in a more general class of analytic functions which may have different power series expansions at different localities. More precisely we wish to be able to take quotients of members of rings of separated power series as well as substitute them in other members. For this to work we need two restricted division operators as defined by Lipshitz in [9].

**Definition 2.2.** The restricted division operator \( D_0 : (K^\circ)^2 \to K^\circ \) is defined as

\[
D_0(x,y) := \begin{cases} 
  x/y & \text{if } |x| \leq |y| \neq 0 \\
  0 & \text{otherwise},
\end{cases}
\]

while the restricted division operator \( D_1 : (K^\circ)^2 \to K^\circ \) is defined as

\[
D_1(x,y) := \begin{cases} 
  x/y & \text{if } |x| < |y| \\
  0 & \text{otherwise}.
\end{cases}
\]

Now we can define the \( D \)-functions which were used in defining \( D \)-semianalytic sets.

**Definition 2.3.** \( D \)-functions over \((K^\circ)^m \times (K^\circ)^n\) are inductively defined as follows.

i) Any member of \( S_{m,n} \) is a \( D \)-function over \((K^\circ)^m \times (K^\circ)^n\).

ii) If \( f, g \) are \( D \)-functions over \((K^\circ)^m \times (K^\circ)^n\) and \( h \) is a \( D \)-function over \((K^\circ)^{m+1} \times (K^\circ)^n \) or \((K^\circ)^m \times (K^\circ)^{n+1}\) then \( h(x,D_0(f,g),\rho) \) (or \( h(x,\rho,D_1(f,g)) \)) is also a \( D \)-function over \((K^\circ)^m \times (K^\circ)^n\).

Note that what we call a \( D \)-function coincides with what was called an \( \mathcal{L}^0_{\text{aff}} \)-term in [9] and in [14]. However we will rarely use these terms in the rest of this paper and instead work with the “generalized rings of fractions over \( S_{m,n} \)”. Before we define such rings we need some more notation.

A ring \( B \) which is of the form \( S_{m,n}/I \) for some ring of separated power series \( S_{m,n} \) and ideal \( I \subset S_{m,n} \) is called a quasi-affinoid algebra. Let \( f, \bar{g} \in B \), \( z \) be a variable not appearing in \( S_{m,n} \) and \( f, g \in S_{m,n} \) be two elements whose canonical images in \( B \) are \( f \) and \( \bar{g} \) respectively, then we will write \( B(f/\bar{g}) \) (or \( B(\bar{z}/(\bar{g}z-f)) \) and \( B[[f/\bar{g}]] \) (or \( B[[z]]/(\bar{g}z-f) \)) for the rings \( S_{m+1,n}/(I \cdot S_{m+1,n} \cup \{gx_{m+1} - f \}) \) and \( S_{m,n+1}/(I \cdot S_{m,n+1} \cup \{g\rho_{n+1} - f \}) \) respectively. More generally if \( A \) is a quasi-affinoid algebra, \( y \) and \( \lambda \) are multi-variables not appearing in the presentation of \( A \) and

\[
\begin{align*}
B_1 &= A(y_1, \ldots, y_{M_1})[[\lambda_1, \ldots, \lambda_{N_1}]]/I_1 \\
B_2 &= A(y_{M_1+1}, \ldots, y_{M_2})[[\lambda_{N_1+1}, \ldots, \lambda_{N_2}]]/I_2
\end{align*}
\]

we will follow Definition 5.4.2 of [15] and define \( B_1 \otimes_A^s B_2 \), the separated tensor product of \( B_1 \) and \( B_2 \) over \( A \), to be the ring

\[
B_1 \otimes_A^s B_2 := A(y_1, \ldots, y_{M_2})[[\lambda_1, \ldots, \lambda_{N_2}]]/(I_1 \cup I_2).
\]
By Theorem 5.2.6 of [15], this product is independent of the presentations of $B_1$ and $B_2$.

Now following the Definition 5.3.1 of [15]:

**Definition 2.4.** A generalized ring of fractions over $S_{m,n}$ is inductively defined as follows:

i) $S_{m,n}$ is a generalized ring of fractions over $S_{m,n}$.

ii) If $B$ is a generalized ring of fractions over $S_{m,n}$, and $g,f_1,...,f_s,f'_1,...,f'_{t} \in B$, then

$$B \langle f_1/g,...,f_s/g \rangle [[f'_1/g,...,f'_{t}/g]]_s$$

is also a generalized ring of fractions over $S_{m,n}$.

If $B$ is a generalized ring of fractions over $S_{m,n}(E,K)$ then the members of $B$ can be treated as functions over the domain of $B$, which is defined below.

**Definition 2.5.** Let $K'$ be a complete extension of $K$ and let

$$B = S_{m+n+s,t}/\{\{g_i x_i - f_{i} \}_{i=1}^{m+s} \cup \{g'_j \rho_j - f'_{j} \}_{j=n+1}^{n+t}\}$$

be a generalized ring of fractions over $S_{m,n}$. Let $\bar{g}_i$, $\bar{f}_i$, $\bar{g}'_j$ and $\bar{f}'_j$ denote the images of $g_i$, $f_i$, $g'_j$ and $f'_j$ in $B$ respectively. Then we define $K'$-Dom$_{m,n}B$, the $K'$-rational points in the domain of $B$ to be the set

$$\{\bar{p} \in (K'^{\infty})^{m} \times (K'^{\infty})^{n} : |\bar{f}_i(\bar{p})| \leq |\bar{g}_i(\bar{p})| \neq 0, |\bar{f}'_j(\bar{p})| < |\bar{g}'_j(\bar{p})| \text{ for all } i,j\}.$$

When $K' = K$ we will simply write Dom$_{m,n}B$ instead of $K$-Dom$_{m,n}B$.

Note that if $K$ is the completion of algebraic closure of $\bar{K}$, then our definition for $K$-rational points of the domain of $B$ coincides with what is called the domain of $B$ in [14].

For a generalized ring of fractions $B$ over $S_{m,n}$ as above and, an ideal $I \subset B$ we will use the customary notation Dom$_{m,n}B \cap V(I)_K$ to denote the set

$$\text{Dom}_{m,n}B \cap V(I)_K := \{\bar{p} \in \text{Dom}_{m,n}B : f(\bar{p}) = 0 \text{ for all } f \in I\},$$

and omit the subscript $K$ when $K$ is algebraically closed.

With these notations established, we can give an alternate definition of the $D$-semianalytic sets. This alternate definition is useful because it connects algebraic objects with geometric objects in a customary way.

**Definition 2.6.** A $D$-semianalytic subset of $(K^{\infty})^{m+n}$ is a finite union of sets of the form Dom$_{m,n}B \cap V(I)_K$ where $B$ is a generalized ring of fractions over $S_{m,n}$ and $I$ is an ideal of $B$.

Next, we are going to make observations which will be helpful in the proofs of Theorems 4.3 and 4.5. Let $B$ be as in Definition 2.5 and let $J \subset S_{m+n+s,t}$ be the ideal $(\{g_i x_i - f_i \}_{i=1}^{m+s} \cup \{g'_j \rho_j - f'_j \}_{j=n+1}^{n+t})$. With this ideal we can associate a $K$-rational variety $V(J)_K$ of $(K^{\infty})^{m+s} \times (K^{\infty})^{n+t}$ and a semianalytic set

$$X := \{\bar{p} \in (K^{\infty})^{m+s} \times (K^{\infty})^{n+t} : \bar{p} \in V(J)_K \text{ and } \prod_{i=n+1}^{m+s} \bar{g}_i(\bar{p}) \cdot \prod_{j=n+1}^{n+t} \bar{g}'_j(\bar{p}) \neq 0\}.$$

Then the coordinate projection $\pi : (K^{\infty})^{m+s} \times (K^{\infty})^{n+t} \to (K^{\infty})^{m} \times (K^{\infty})^{n}$ maps $X$ bijectively onto Dom$_{m,n}B$. Furthermore by the Implicit Function Theorem, in fact if $X$ is non-empty then it is an $(m+n)$-dimensional $K$-analytic manifold (see Definition 4.2).
In order to understand the properties of the set $\text{Dom}_{m,n} B$, we will often look at the set $X$. However, to avoid complications coming from working over a field which may not be algebraically closed and to avoid irreducible components of $X$ that are contained in the excluded set

$$Z := \{ \bar{p} \in (K^\circ)^{m+s} \times (K^{\circ\circ})^{n+t} : \prod_{i=m+1}^{m+s} g_i(\bar{p}) \cdot \prod_{j=n+1}^{n+t} g'_j(\bar{p}) = 0 \}$$

we will prefer to work with the largest ideal that vanishes on $\text{Dom}_{m,n} B \cap V(I)_K$. That is, the ideal

$$I(\text{Dom}_{m,n} B \cap V(I)_K) := \{ f \in B : f(\bar{p}) = 0 \text{ for all } \bar{p} \in \text{Dom}_{m,n} B \cap V(I)_K \}$$

whose corresponding ideal $\bar{I}$ in $S_{m+s,n+t}$ does not have a minimal prime divisor $\mathfrak{p}$ such that $V(\mathfrak{p})_K \subset Z$, will show up often in our arguments.

There is a special type of $D$-semianalytic sets that we will come back to.

**Definition 2.7.** Let $B$ be a generalized ring of fractions over $S_{m,n}$ such that at each inductive step of the construction of $B$ as described in Definition 2.4, the ideal $(g,f_1,\ldots,f_s,f'_1,\ldots,f'_t)$ is the unit ideal. Let $\bar{K}$ be an algebraically closed complete extension of $K$, then $\bar{K}$-$\text{Dom}_{m,n} B$ is called an $R$-subdomain of $(\bar{K}^\circ)^m \times (\bar{K}^{\circ\circ})^n$.

$R$-domains generalize the rational domains of the affinoid geometry (as given in Definition 7.2.3.5 of [2]). They are also examples of quasi-affinoid subdomains and have the universal property described in Definition 5.3.4 of [15]. This guarantees that if $A$ and $B$ are two generalized rings of fractions over $S_{m,n}$ such that $X := \bar{K}$-$\text{Dom}_{m,n} A = \bar{K}$-$\text{Dom}_{m,n} B$ is an $R$-domain, then $A \simeq B$. Therefore it is possible to define a ring of analytic functions on $X$ with coefficients from $K$, which we will denote by $\mathcal{O}(X)_K := A \simeq B$.

### 3. Normalization and Dimension Theory

In this section we briefly discuss the dimension theory of $D$-semianalytic sets and the Parameterized Normalization Lemma. These are two of our most important tools in the proofs of the main results of this paper. Establishing these was the main purpose of [3] and we start by the basic definitions and results from that source, finishing up with improvements on the previous results. Those improvements will enable us to prove main theorems later in this paper.

For an arbitrary subset of $K^m$ we can define several notions of dimension as follows.

**Definition 3.1.** We define the **geometric dimension**, $\text{g-dim } X$, of a non-empty set $X \subset K^m$ to be the greatest integer $d$ such that the image of $X$ under a coordinate projection onto a $d$-dimensional coordinate hyper-plane has an interior point.

The **weak dimension**, $\text{w-dim } X$, of a nonempty set $X \subset K^m$ is defined to be the greatest integer $d$ such that the image of $X$ under a coordinate projection onto a $d$-dimensional coordinate hyper-plane is somewhere dense.

The **manifold dimension**, $\text{m-dim } X$, of a nonempty set $X \subset K^m$ is the greatest integer $d$ such that $X$ contains a $d$-dimensional analytic manifold (see Definition 4.2).

Define also

$$\text{m-dim } \emptyset = \text{g-dim } \emptyset = \text{w-dim } \emptyset = -1$$
For a generalized ring of fractions $B$ and an ideal $I$ of $B$, we will write $k$-dim $B/I$ to denote the Krull dimension of the algebra $B/I$.

Note that the reference to a $d$-dimensional coordinate hyperplane in the above definition is superfluous as it is easy to prove that maximizing over all $d$-dimensional hyperplanes will yield the same result for the dimensions above. One of the main theorems (Theorem 6.2) of [3] was:

**Theorem 3.2.** Assume $\text{Char} K = 0$. For a generalized ring of fractions $B$ over $S_{m,n}$ and ideal $I$ of $B$ satisfying $I = \mathcal{I}(\text{Dom}_{m,n}B \cap V(I)_K)$,

$$w\text{-dim} \text{Dom}_{m,n}B \cap V(I)_K = g\text{-dim} \text{Dom}_{m,n}B \cap V(I)_K = k\text{-dim} B/I.$$  

The main tool for proving Theorem 3.2, as well as for obtaining the results in this paper is the Parameterized Normalization Lemma (Lemma 5.3 of [3]). It is well known that given a quasi-affinoid algebra $B$, it is not always possible to find a ring of separated power series $S_{m,n}$ such that there is a finite injection $\phi: S_{m,n} \to A$ (see Example 2.3.5 of [15]). Nevertheless we can break up the $D$-semianalytic set associated with $B$ into finitely many smaller $D$-semianalytic sets whose associated quasi-affinoid algebras can be normalized in the sense above. Furthermore in this process of breaking-up, several key properties are preserved including the parameter structure of those algebras. Note that given a quasi-affinoid algebra there may be more than one way of considering it as a generalized ring of fractions depending on which variables we choose as representing the coordinates of the space in which the $D$-semianalytic set lives and which variables correspond to the fractions which appear in the construction of the $D$-semianalytic set. However, this fact plays no role in our discussions.

Let us write $S_{m+M,n+N}$ for the ring of separated power series in the variables $(x_1, \ldots, x_m), (\rho_1, \ldots, \rho_n), (y_1, \ldots, y_M)$ and $(\lambda_1, \ldots, \lambda_N)$. Now suppose $B$ is a generalized ring of fractions over $S_{m+M,n+N}$, then assigning appropriate $x$ or $\rho$ variables to fractions involving terms built up inductively from only the variables $x_1, \ldots, x_m$ and $\rho_1, \ldots, \rho_n$, $B$ can be written in the form

$$B = (S_{m,n}(x_{m+1}, \ldots, x_{m+s})[[\rho_{n+1}, \ldots, \rho_{n+t}]]_S/J_1)
(y_{1}, \ldots, y_{M+S})[[\lambda_{1}, \ldots, \lambda_{N+S}]]_S/J_2.$$  

We will call the $x$ and $\rho$ variables appearing above the parameter variables.

For a ring of separated power series $S_{m+M,n+N}$, and a fixed $1 \leq j \leq M$, we will call an automorphism $\phi$ of $S_{m+M,n+N}$, which is defined by $\phi(y_i) := y_i + y_i^j$ for some $r_i \in \mathbb{N}^+$ for $i < j$ and $\phi(y_i) = y_i$ for $i \geq j$, a Weierstrass change of variables among $y$ variables. We define the Weierstrass change of variables among $x$, $\rho$ or $\lambda$ variables similarly. Inductively, we will apply the same term for a composition of Weierstrass change of variables, so that a Weierstrass change of variables among a single type of variable (i.e. among either $x$, $\rho$, $y$ or $\lambda$ variables) respects the sort of the variables as well as the parameter structure.

In the following discussions we will often come across a situation where, for a quasi-affinoid algebra $B = S_{m+M,n+N}/J$, and an ideal $I \subset B$, the corresponding ideal $I$ of $S_{m+M,n+N}$ is such that after a Weierstrass variable change $\phi$, we can find another quasi-affinoid algebra $A$ such that the natural map $A \to S_{m+M,n+N}/\phi(I)$ is a finite inclusion. In such cases we will abuse the notation and say $A \to B/\phi(I)$ is a finite inclusion. Combining Lemmas 5.3 and 5.5 of [3] we have:
Lemma 3.3 (Parameterized Normalization Lemma). Let $B$ be as in Equation (1) and $I$ be an ideal of $B$, then there exist finitely many generalized rings of fractions

$$B_i = (S_{m,n} \langle x_{m+1}, \ldots, x_{m+s_i} \rangle [\langle \rho_{n+1}, \ldots, \rho_{n+s_i} \rangle] / J_i, 1)$$

with parameter rings $A_i := S_{m,n} \langle x_{m+1}, \ldots, x_{m+s_i} \rangle [\langle \rho_{n+1}, \ldots, \rho_{n+s_i} \rangle] / J_i, 1$, ideals $I_i \subset B_i$, and integers $M_i, N_i$ such that

i) $V(I) \cap \text{Dom}_{m,n+M+N} B = \bigcup_j (V(I_i) \cap \text{Dom}_{m,n+M+N} B_i)$,

ii) $I_i = I(V(I_i) \cap \text{Dom}_{m,n+M+N} B_i)$, $k\text{-dim } B/I \geq k\text{-dim } B_i/I_i$,

iii) after a Weierstrass change of variables $\phi_i$ among $y$ and $\lambda$ variables separately, the natural map

$$A_i/(A_i \cap I_i) \langle y_1, \ldots, y_{M_i} \rangle [\langle \lambda_1, \ldots, \lambda_{N_i} \rangle] \rightarrow B_i/\phi_i(I_i)$$

is a finite inclusion.

Although the previous statement of the Parameterized Normalization Lemma is quite useful for working with the geometric properties of projections of D-semianalytic sets, we need some improvements for our later applications. For us, the most important improvement is to carry the process of normalization one more step to have it look more like other well known normalization results from algebra.

Lemma 3.4. Let $B$ be as in (1) and $I$ be an ideal of $B$, then there exist finitely many generalized rings of fractions

$$B_i = S_{m+s_i, M+n+t_i+N+T_i} / J_i,$$

in the variables $x_1, \ldots, x_{m+s_i}$, $\rho_1, \ldots, \rho_{n+s_i}$, $y_1, \ldots, y_{M+n}$, $\lambda_1, \ldots, \lambda_{N+T}$; ideals $I_i$, and integers $m_i, n_i, M_i, N_i$ such that

i) $V(I) \cap \text{Dom}_{m,n+M+N} B = \bigcup_j (V(I_i) \cap \text{Dom}_{m,n+M+N} B_i)$,

ii) $I_i = I(V(I_i) \cap \text{Dom}_{m,n+M+N} B_i)$,

iii) $M_i + N_i \leq M + N$, $m_i + n_i \leq m + n$ and after a Weierstrass change of variables $\psi_i$ among $x, \rho, y$ and $\lambda$ variables separately the natural map

$$S_{m+n, M+n_i} \rightarrow B_i/\psi_i(I_i)$$

is a finite inclusion.

Proof. After applying Lemma 3.3 we may assume that there are integers $M'$ and $N'$, a generalized ring of fractions $A$ over $S_{m,n}$ and a Weierstrass change of variables $\phi$ among $y$ and $\lambda$ variables separately such that

$$A/(I \cap A) \langle y_1, \ldots, y_{M'} \rangle [\langle \lambda_1, \ldots, \lambda_{N'} \rangle] \rightarrow B/\phi(I)$$

is a finite inclusion. Apply Lemma 3.3 once more to $A/(I \cap A)$ to get generalized rings of fractions $A_i$ over $S_{m,n}$, ideals $J_i' \subset A_i$, integers $m_i, n_i$ and Weierstrass changes of variables $\phi_i$ among $x, \rho$ variables separately such that

$$V(I \cap A) \cap \text{Dom}_{m,n} A = \bigcup_j (\text{Dom}_{m,n} A_i \cap V(J_i') K),$$

$$J_i' = I(\text{Dom}_{m,n} A_i \cap V(J_i') K), k\text{-dim } A/(I \cap A) \geq k\text{-dim } A_i/J_i',$$

for all $i$, and $S_{m+n_i} \rightarrow A_i/\phi_i(J_i')$ is a finite inclusion. Then by Lemmas 2.4 and 2.5 of [3] we have

$$A_i/J_i' \langle y_1, \ldots, y_{M'} \rangle [\langle \lambda_1, \ldots, \lambda_{N'} \rangle] \rightarrow B_i/\phi(I \cup J_i' \cdot B_i),$$
is a finite inclusion where $B_i$ is the generalized ring of fractions over $S_{m+n,M,n+N}$. Which is the separated tensor product $A_i \otimes_A B$. Again by the same lemma

$$S_{m+M',n_i+N'} \rightarrow B_i/\phi_i \circ \phi(I \cup J'_i \cdot B_i).$$

is also a finite inclusion.

Let $I_i = T(V((I \cup J'_i) \cdot B_i)_K \cap \text{Dom}_{m+M,n+N}B_i)$, and notice that if $k\text{-dim } B_i/I_i$ is equal to $m_i + n_i + M' + N'$, then the natural map

$$S_{m+M',n_i+N'} \rightarrow B_i/\phi_i \circ \phi(I_i)$$

is also a finite inclusion. If $k\text{-dim } B_i/I_i$ is less than $m_i + n_i + M' + N'$, then we proceed inductively, applying the process described above to the generalized ring of fractions $B_i$ and ideal $I_i$. Note that, by Lemma 5.5 of [3], we have $m_i + n_i \leq m + n$ and $M' + N' \leq M + N$, so the termination of the described process is guaranteed. □

A couple of observations about the process of normalization described above is in order. They will be useful later.

**Remark 3.5.** i) Let us write $B_i = S_{m+s_i+M+S_i,n+t_i+N+T_i}/J_i$, for the generalized rings of fractions we found in the previous lemma. One important property in the above setting is that $x$, $\rho$, $\gamma$ and $\lambda$ variables are not mixed under $\phi_i \circ \phi$, indeed the image under the change of variables (automorphism) $\phi_i \circ \phi$ of $S_{m+s_i,n+t_i}$ is again $S_{m+s_i,n+t_i}$. We will call such an automorphism a parameter respecting automorphism of quasi-affinoid algebras with a parameter structure.

ii) This next observation will be used in the proof of Theorem 4.5. Suppose that the projection of $\text{Dom}_{m+M,n+N}B \cap V(I)_K$ onto the hyperplane $(K^\circ)^m \times (K^\circ)^n$ is somewhere dense and observe that this projection is contained in $\bigcup (\text{Dom}_{m,n}A_i \cap V(J'_i)_K)$ where each $A_i$ is a generalized ring of fractions over $S_{m,n}$ and each $J'_i \subset A_i$ is an ideal as in the proof above. Then for some index $i_0$ the projection of the $D$-semianalytic set $\text{Dom}_{m+M,n+N}B_{i_0} \cap V(I_{i_0})_K$ onto $(K^\circ)^m \times (K^\circ)^n$ contains a somewhere dense subset and is contained in $\text{Dom}_{m,n}A_{i_0} \cap V(J'_{i_0})_K$. Hence by Lemma 4.9 of [3] we have $k\text{-dim } A_{i_0}/J'_{i_0} = m + n$, and therefore $m_{i_0} + n_{i_0} = m + n$.

We finish this section with one last observation about a “localization” property of the normalization process we have been discussing.

**Lemma 3.6.** Let $B$ be a generalized ring of fractions over $S_{m,n}$ and let $I$ be an ideal of $B$. Suppose that $S_{m',n'} \rightarrow B/I$ is a finite inclusion, the origin $\mathbf{0}$ is in $\text{Dom}_{m,n}B \cap V(I)_K$ and that the maximal ideal $\mathfrak{m}$ of $B$ corresponding to the origin $\mathbf{0}$ is such that $k\text{-dim } B_\mathbf{0}/I \cdot B_\mathbf{m} = m' + n'$. Then for all $\varepsilon \in K^\circ$, $\varepsilon \not= 0$ such that the open ball $B_K(\mathbf{0},|\varepsilon|)$ of center $\mathbf{0}$ and radius $|\varepsilon|$ is contained in $\text{Dom}_{m,n}B$, there is a $\delta \in K^\circ$, $\delta \not= 0$ such that

$$T_{m'+n'}(\delta) \rightarrow T_{m'+n'}(\delta) \langle x_{m'+1}/\varepsilon, \ldots, x_m/\varepsilon, \rho_{n'+1}/\varepsilon, \ldots, \rho_n/\varepsilon \rangle / I$$

is a finite inclusion, where $T_{m'+n'}(\delta)$ stands for the Tate ring

$$K \langle x_1/\delta, \ldots, x_{m'}/\delta, \rho_1/\delta, \ldots, \rho_{n'}/\delta \rangle.$$

**Proof.** For simplicity in notation we will assume $n' = n = 0$, but the arguments below also work in the general case. Let

$$f_{m'+i} = x_{m'+i}^{s_{m'+i}} + b_{m'+i,s_{m'+i}-1} x_{m'+i}^{s_{m'+i}-1} + \ldots + b_{m'+i,0} x_{m'}$$
be the lowest degree monic polynomial in $I \cap K \langle x_1, \ldots, x_{m'} \rangle \langle x_{m'+i} \rangle$ for $i = 1, \ldots, m - m'$. For each such $i$, and $j = 1, \ldots, s_{m'+i}$, write

$$b_{m'+i,s_{m'+i}+j} = c_{m'+i,s_{m'+i}+j} + d_{m'+i,s_{m'+i}+j},$$

where $c_{m'+i,s_{m'+i}+j} \in K$ and $d_{m'+i,s_{m'+i}+j} \in (x_1, \ldots, x_{m'})K \langle x_1, \ldots, x_{m'} \rangle$. By continuity there is a $\delta \in K^\circ$, $\delta \neq 0$ such that whenever $\bar{p} \in B_K(0, |\delta|) \cap (K^\circ)^m$ we have $|d_{m'+i,s_{m'+i}+j}(\bar{p})| < |\varepsilon|^l$ for all $i = 1, \ldots, m - m'$ and $j = 1, \ldots, s_{m'+i}$.

Notice that in this case each $f_{m'+i}$ is regular in $x_{m'+i}/\varepsilon$ of some degree $r_{m'+i}$, $1 \leq r_{m'+i} \leq s_{m'+i}$, in

$$B \langle x_{m'+1}/\varepsilon, \ldots, x_{m'}/\varepsilon \rangle \simeq T_{m'}(\delta) \langle x_{m'+1}/\varepsilon, \ldots, x_{m'}/\varepsilon \rangle.$$

This shows that $T_{m'}(\delta) \langle x_{m'+1}/\varepsilon, \ldots, x_{m'}/\varepsilon \rangle/I$ is finite over $T_{m'}(\delta)$. On the other hand by Lemma 3.6 of [3] $k$-dim $T_{m'}(\delta) \langle x_{m'+1}/\varepsilon, \ldots, x_{m'}/\varepsilon \rangle/I = m'$ and we see that

$$T_{m'}(\delta) \rightarrow T_{m'}(\delta) \langle x_{m'+1}/\varepsilon, \ldots, x_{m'}/\varepsilon \rangle/I$$

is a finite inclusion. \qed

4. Parameterized Stratification

In this section, we start by sharpening the Smooth Stratification Theorem of [3] and [13] for $D$-semianalytic sets using the normalization results of the previous section. Our goal is to prove that a $D$-semianalytic set is in fact a finite union of $D$-semianalytic manifolds which remain manifolds when specialized at points of the parameter space. Moreover, the tools we develop in this process also enable us to prove many properties of the dimension theory of $D$-semianalytic and subanalytic sets.

Next lemma handles the main step in proving the Parameterized Smooth Stratification Theorem by establishing that a properly normalized quasi-affinoid variety can be written as a union of an analytic manifold and a smaller dimensional variety where the manifold is locally the graph of some analytic functions, with the normalized parameters being functions of only the free parameters of the normalization.

**Lemma 4.1.** Let $\text{Char } K = 0$ and let $I$ be a prime ideal of $S_{m+M,n+N}$ such that $I = I(V(I)_K)$. Suppose $S_{m'+M',n'+N'} \rightarrow S_{m+M,n+N}/I$ is a finite inclusion respecting parameters (see Remark 3.5 (i)), then we have

$$V(I)_K = Z \cup Y,$$

where $Z$ is a $(m' + M' + n' + N')$-dimensional analytic manifold whose charts are given by projections onto the space $(K^\circ)^{m'+M'} \times (K^\circ)^{n'+N'}$ and where $Y = V(I')_K$ for some ideal $I' \supseteq I$.

More precisely, for each $\bar{p} \in Z$ there is an open neighborhood $U$ of $\bar{p}$ and analytic functions $\alpha_{m'+i}(x', \rho')$, $\beta_{n'+j}(x', \rho')$, $\gamma_{M'+k}(x', \rho', y', X')$, $\delta_{N'+l}(x', \rho', y', X')$ over $U$ for $i = 1, \ldots, m - m'$, $j = 1, \ldots, n - n'$, $k = 1, \ldots, M - M'$, $l = 1, \ldots, N - N'$ and multi variables $x' = (x_1, \ldots, x_{m'})$, $\rho' = (\rho_1, \ldots, \rho_{m'})$, $y' = (y_1, \ldots, y_{n'})$, $X' = (\lambda_1, \ldots, \lambda_{N'})$ such that

$$Z \cap U = V\left(\{x_{m'+i} - \alpha_{m'+i}\}^{m-m'}_{i=1} \cup \{\rho_{n'+j} - \beta_{n'+j}\}^{n-n'}_{j=1} \cup \{y_{M'+k} - \gamma_{M'+k}\}^{M-M'}_{k=1} \cup \{\lambda_{N'+l} - \delta_{N'+l}\}^{N-N'}_{l=1}\right)_K \cap U.$$
Proof. Again for simplicity in notation we will assume that $N = N' = n = n' = 0$, but our arguments will be valid in the general case.

Let $p_{m'+i}$ be the unique lowest degree monic polynomial in $I \cap \mathcal{K}(x)$ $[x_{m'+i}]$ for each $i$ and $q_{M+j}$ be the lowest degree monic polynomial in $I \cap \mathcal{K}(x', y')$ $[y_{M+j}]$ for each $j$. Define

$$\Delta := \det \left( \frac{\partial \left( p_{m'+i}, q_{M'+j} \right)}{\partial \left( x_{m'+i}, y_{M'+j} \right)} \right)_{i,j} = \frac{\partial p_{m'+i}}{\partial x_{m'+i}} \ldots \frac{\partial p_m}{\partial x_m} \frac{\partial q_{M'+1}}{\partial y_{M'+1}} \ldots \frac{\partial q_M}{\partial y_M},$$

as the Jacobian matrix above is upper triangular. Define also

$$Z := \{ \bar{p} \in V(I)_K : \Delta(\bar{p}) \neq 0 \}.$$

Notice that because each $p_{m'+i}$ is the lowest degree monic polynomial and $I$ is a prime ideal, we have $\Delta \notin I$. Therefore by Theorem 6.2 of [3], the geometric dimension of $Y := V(I)_K \setminus Z = V(I \cup \{\Delta\})_K$ is less than $m' + M' + n' + N'$.

On the other hand, the facts that for all $\bar{p} \in Z$ we have $\frac{\partial p_{m'+i}}{\partial x_{m'+i}}(\bar{p}) \neq 0$ and $\frac{\partial q_{M'+j}}{\partial y_{M'+j}}(\bar{p}) \neq 0$ for all $i, j$, imply that each $p_{m'+i}$ is regular of degree one in $x_{m'+i}$ and each $q_{M'+j}$ is regular of degree one in $y_{M'+j}$ in a rational neighborhood $\bar{W}$ of $\bar{p}$. Here the regularity of these polynomials is in the sense of Definition 2.3.7 of [15]. Furthermore $p_{m'+i}, q_{M'+j}$ generate $I \cdot \mathcal{O}(U)$ for some neighborhood $U \subset \bar{W}$ of $\bar{p}$ by Theorem 30.4 of [16]. Now the statement of the lemma follows from the Weierstrass Division Theorem (Theorem 2.3.8 of [15]).

Before we prove the Parameterized Smooth Stratification Theorem, we also need to clarify what we mean by an analytic manifold. Following [9] and [17], the definition is as follows.

**Definition 4.2.** A $D$-semianalytic subset $X$ of $K^n$ is a $d$-dimensional $K$-analytic manifold if it is endowed with a system of charts $(U, \varphi_U : U \to \mathcal{K}^d)$ such that the transition maps $\varphi_U \circ \varphi_V^{-1}$ are locally given by convergent power series over $K$. Note that here the topology of $X$ is the subspace topology inherited from the metric topology of $\mathcal{K}$.

The goal of the Parameterized Smooth Stratification Theorem is to establish the existence of a uniform stratification of specializations of $D$-semianalytic subsets of $(K^\circ)^{n+M} \times (K^\circ)^{n+N}$ at points from $(K^\circ)^n \times (K^\circ)^n$, which we consider as the parameter space. Its precise statement is the following.

**Theorem 4.3** (Parameterized Smooth Stratification). Let $\text{Char } K = 0$ and let $X$ be a $D$-semianalytic subset of $(K^\circ)^{n+M} \times (K^\circ)^{n+N}$, then we can write $X$ as a finite union of $D$-semianalytic manifolds $X_i$ such that for each $p \in (K^\circ)^n \times (K^\circ)^n$, the fiber $X_i(p)$ of $X_i$ at $p$ is either empty or is a $D$-semianalytic manifold.

**Proof.** We may assume that $X = \text{Dom}_{m+M,n+N} \cdot V(I)_K$ for some generalized ring of fractions $B$ over $S_{m+M,n+N}$ and ideal $I$ of $B$. By Lemma 3.4 we may also assume that there are integers $m', n', M', N'$ and a Weierstrass automorphism $\phi$ respecting the parameters (see Remark 3.5) such that $S_{m'+M', n'+N'} \to B/\phi(I)$ is a finite inclusion and $I = \mathcal{I}(\text{Dom}_{m+M,n+N} \cdot B \cap V(I)_K)$. By Lemma 4.5 of [16] each minimal prime divisor $p$ of $I$ satisfies $p = \mathcal{I}(\text{Dom}_{m+M} \cdot B \cap V(p)_K)$, so we may also assume that $I$ is prime.

The proof will be by induction on $d := \text{g-dim } X = m' + M' + n' + N'$. There is nothing to prove if $X = \emptyset$ or if $d = 0$, as in those cases we have the statement of...
the theorem by Remark 3.2 and Theorem 6.2 of [3]. Therefore, in what follows we will assume that 
$d > 0$.

The following maps between the affinoid spaces will play an important role in understanding the
relations between geometric objects we are interested in. Write $B = S_{m+s+s,n+t+N+T}/J$ and notice that
$\phi$ induces a one-to-one analytic transformation

$$\phi' : (K^\circ)^{m+s+M+S} \times (K^{oo})^{n+t+N+T} \to (K^\circ)^{m+s+M+S} \times (K^{oo})^{n+t+N+T}$$

with non-zero Jacobian so that $\phi'(V(I)K) = V(\phi(I))_K$ where $I$ is the ideal of $S_{m+s+s,n+t+N+T}$ that corresponds to $I$. Another map related to our construction is the projection map

$$\pi : (K^\circ)^{m+s+M+S} \times (K^{oo})^{n+t+N+T} \to (K^\circ)^{m+M} \times (K^{oo})^{n+N}$$

which is one-to-one when restricted to $\pi^{-1}(\text{Dom}_{m+s+s,n+t+N+T}) \cap V(J)_K$.

Note that in our setting there is an open set $U \subset (K^\circ)^{m+s+M+S} \times (K^{oo})^{n+t+N+T}$
which is such that $X = \pi(V(\bar{I})_K \cap U)$ where $\bar{I} = \bar{Z}(V(\bar{I})_K)$. On the other hand, by
Theorem 3.2 we have $g\text{-dim } V(\bar{I})_K \cap U = d$. Apply Lemma 4.1 to $\phi'(V(\bar{I})_K) = V(\phi(\bar{I}))_K$
(replacing $(m,s,n,M,N)$ by $(m+s,n+t,M+S,N+T)$) to get the
$V(\phi(\bar{I}))_K = Z \cup Y$ as described in the lemma so that $Z \subset \phi'(V(\bar{I})_K)$ is a $d$-
dimensional D-semianalytic manifold and $Y$ is the zero-set of the ideal $I'$, for which we have
$k\text{-dim } S_{m+s+s,n+t+N+T}/I' < d$.

Next, we observe that Theorem 2 of §I.III.11.2 of [17] implies that the set
$\pi(\phi'^{-1}(Z) \cap U) \subset X$ is also a $D$-semianalytic $d$-manifold. Observe also that we have $g\text{-dim } \phi'^{-1}(Y) < d$, again by Theorem 3.2, as it is the zero set of the ideal $\phi^{-1}(I')$ and $k\text{-dim } S_{m+s+s,n+t+N+T}/\phi^{-1}(I')$ is less than $d$. Hence we have that
$g\text{-dim } \pi(\phi'^{-1}(Y)) < d$ and therefore we only need to show that $\pi(\phi'^{-1}(Z))$ satisfies the condition about fibers of points in the parameter space before we are done by induction.

Let $\bar{p} \in (K^\circ)^{m} \times (K^{oo})^{n}$, be such that the specialization $X(\bar{p}) \neq \emptyset$, then there
is a unique $\bar{q} \in (K^\circ)^{m+s} \times (K^{oo})^{n+t}$ such that $\pi|_{(K^\circ)^{m+s} \times (K^{oo})^{n+t}}(\bar{q}) = \bar{p}$. Because
$\phi$ respects parameters we also have $\phi'(\bar{q})$ in the space $(K^\circ)^{m+s} \times (K^{oo})^{n+t}$ and by
Lemma 4.1 the specialization $(Z \cap U)(\phi'(\bar{q}))$ is either empty, or is a $D$-semianalytic
manifold. Again by the observation that $\phi'$ takes parameters to parameters and fibers to fibers and by Theorem 2 of §I.III.11.2 of [17], we have the statement of the theorem.

Next we turn our attention to the subanalytic sets. We would like to remind the reader that these are the projections of the $D$-semianalytic sets onto the coordinate hyper-planes and if $K$ is algebraically closed, then they are the same as the $D$-
semianalytic sets by the Quantifier Elimination Theorem of [9]. We are going to prove below, by an argument similar to the one in the proof above, that subanalytic sets share many of the nice geometric and dimension theoretic properties of
$D$-semianalytic sets. But first, we are going to justify the need for a new treatment by showing that the class of subanalytic sets is a strictly larger class of sets than the class of $D$-semianalytic sets in general, through the next easy example.

**Example 4.4.** Consider $K = \mathbb{Q}((t))$ with the $t$-adic valuation, then the set

$$P_2 := \{ x \in K^\circ : (\exists y)(y^2 = x) \}$$
is subanalytic but not $D$-semianalytic as it can not be written as a finite boolean combination of discs (see Definition 6.1), contradicting Theorem 6.7.

Although the previous example shows that it is not possible to extend the concept of complexity of a $D$-semianalytic subset of $K^\circ$ (see Definition 6.2), to work for the subanalytic subsets of $K^\circ$, the next theorem shows that the dimension theory of the subanalytic sets is closely connected to that of the $D$-semianalytic sets.

**Theorem 4.5.** Suppose $\text{Char } K = 0$ and $X$ is a subanalytic subset of $(K^\circ)^m \times (K^\circ)^n$. Suppose $X$ is not contained in any $(d-1)$-dimensional $D$-semianalytic set, then $X$ contains a $d$-dimensional $D$-semianalytic manifold whose charts are given by coordinate projections.

**Proof.** Let $X$ be as in the statement of the lemma, and let $X'$ be a $D$-semianalytic subset of $(K^\circ)^{m+M} \times (K^\circ)^{n+N}$ whose projection onto $(K^\circ)^m \times (K^\circ)^n$ is $X$. Once again we will treat $(K^\circ)^m \times (K^\circ)^n$ as the parameter space and will be interested in the semianalytic set $X' \subset (K^\circ)^{m+s+M+S} \times (K^\circ)^{n+t+N+T}$ whose image is $X'$ under the coordinate projection

$$\pi : (K^\circ)^{m+s+M+S} \times (K^\circ)^{n+t+N+T} \rightarrow (K^\circ)^{m+M} \times (K^\circ)^{n+N}.$$ 

Two other coordinate projection maps

$$\pi_1 : (K^\circ)^{m+s+M+S} \times (K^\circ)^{n+t+N+T} \rightarrow (K^\circ)^{m+s} \times (K^\circ)^{n+t},$$

and

$$\pi_2 : (K^\circ)^{m+s} \times (K^\circ)^{n+t} \rightarrow (K^\circ)^m \times (K^\circ)^n$$

will also help us understand the underlying geometry in our setting. The main idea of the proof is to show that $\pi_1(X')$ contains a $d$-dimensional analytic manifold whose charts are given by the coordinate projection onto a $d$-dimensional coordinate hyperplane of $(K^\circ)^m \times (K^\circ)^n$.

Note that if we partition $X$ into finitely many pieces, then at least one of those pieces can not be contained in a $(d-1)$-dimensional $D$-semianalytic set. Therefore we may assume that $X' = \text{Dom}_{m+M,n+N} B \cap V(I)_K$ for some generalized ring of fractions $B$ over $S_{m+M,n+N}$ and ideal $I$ satisfying $I = \mathcal{I}(\text{Dom}_{m+M,n+N} B \cap V(I)_K)$. By the same fact and Lemma 3.4, we may also assume that

$$S_{m'+M',n'+N'} \rightarrow B/\phi(I)$$

is a finite inclusion where $\phi$ is a Weierstrass change of variables respecting parameters. By Lemma 4.9 of [3] we may assume that $I$ is prime and by Remark 3.5(ii) we may also assume that $m' + n' = d$.

Write $B = S_{m+s+M+s,n+t+N+T}/J$ for

$$J := ((g_i x_{m+i} - f_i)_{i=1}^s \cup (g_j y_{n+j} - f_j')_{j=1}^t \cup (G_k y_{M+k} - F_k)_{k=1}^S \cup (G'_{l(N+l')} - F'_{l(1)})_{l=1}^T,$$

and let $I \supset J$ be the ideal corresponding to $I$ in $S_{m+s+M+s,n+t+N+T}$ so that $X = \pi_2 \circ \pi_1 (V(J)_K \cap U)$, where $U$ is the open set

$$U := \{p \in (K^\circ)^{m+s+M+S} \times (K^\circ)^{n+t+N+T} : \prod_{i=1}^s g_i(p) \cdot \prod_{j=1}^t g_j'(p) \cdot \prod_{k=1}^S G_k(p) \cdot \prod_{l=1}^T G'_l(p) \neq 0 \}.$$
Observe that \( \pi_1(V(\mathcal{I})_K \cap U) \) is contained in the \((m + n)\)-dimensional manifold
\[
R := \{ \dot{p} \in (K^\infty)^{m+s} \times (K^\infty)^{n+t} : \dot{p} \in V(\{ g_i x_{m+i} - f_i \}_{i=1}^s \cup \{ g'_j \rho_{n+j} - f'_j \}_{j=1}^t) \}
\]
and \( \prod_{i=1}^s g_i(\dot{p}) \cdot \prod_{j=1}^t g'_j(\dot{p}) \neq 0 \),
whose charts are given by the restrictions of the projection \( \pi_2 \) to open subsets of \( R \). That is, for each \( \dot{p} \in R \), there is a rational open neighborhood \( W \) of \( \dot{p} \) such that \( W \cap R \) is given by relations of the form
\[
\begin{align*}
\alpha_{m+i}(x', \rho') & = x_{m+i} \\
\beta_{n+j}(x', \rho') & = \rho_{n+j}
\end{align*}
\]
where \( x' = (x_1, \ldots, x_m) \), \( \rho' = (\rho_1, \ldots, \rho_n) \) and \( \alpha_{m+i}, \beta_{n+j} \in \mathcal{O}(\pi_2(W)_K) \).

Let us write \( \bar{x}, \bar{\rho}, \bar{y}, \lambda \) for the images of the variables \( x, \rho, y, \lambda \) under the map \( \phi \) and observe that \( k\text{-dim} \: S_{m+s+M+n+t+\mathcal{I}} / \phi(\mathcal{I}) = m' + n' + M' + N' \). Applying Lemma 4.1 to \( V(\phi(\mathcal{I})) \), we get two sets \( Z \) and \( Y \) such that \( V(\phi(\mathcal{I})) = Z \cup Y \), where \( Z \) is a \((m' + n' + M' + N')\)-dimensional manifold which is locally given by relations of the form
\[
\begin{align*}
\bar{x}_{m'+i} & = \bar{\alpha}_{m'+i}(x', \rho') \\
\bar{\rho}_{n'+j} & = \bar{\beta}_{n'+j}(x', \rho') \\
\bar{y} & = \bar{\gamma}'(x', \rho', \bar{y}', \bar{\lambda}') \\
\bar{\lambda}_{N'+t} & = \bar{\delta}'(x', \rho', \bar{y}', \bar{\lambda})
\end{align*}
\]
where \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_{m'}) \), \( \bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_{n'}) \), \( \bar{y} = (\bar{y}_1, \ldots, \bar{y}_{M'}) \), \( \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{N'}) \), and \( g\text{-dim} \: Y < m' + n' + M' + N' \). Let
\[
\phi' : (K^\infty)^{m+s+M+S} \times (K^\infty)^{n+t+N+T} \to (K^\infty)^{m+s+M+S} \times (K^\infty)^{n+t+N+T}
\]
be the one-to-one analytic map that is induced by \( \phi \) so that \( \phi'(V(\mathcal{I})_K) = V(\phi(\mathcal{I}))(K) \), and let \( \Delta \not\in \phi(\mathcal{I}) \) be the determinant of the Jacobian as in the proof of Lemma 4.1, so that \( Z = \{ \dot{p} \in V(\phi(\mathcal{I})) : \Delta(\dot{p}) \neq 0 \} \). Then we have \( Z \cap \phi'(U) \neq \emptyset \), as otherwise
\[
\phi^{-1}(\Delta) \cdot \prod_{i=1}^s g_i \cdot \prod_{j=1}^t g'_j \cdot \prod_{k=1}^S G_k \cdot \prod_{l=1}^T G'_l \in \mathcal{I}
\]
contradicting the fact that \( \mathcal{I} \) is prime and \( \mathcal{I} = \mathcal{I}(V(\mathcal{I})_K) \).

Observe that, by Lemma 4.1, we have \( \pi_1(Z) \) (and also \( \pi_1(Z \cap \phi'(U)) \)) containing a \( d\)-dimensional manifold of \((K^\infty)^{m+s} \times (K^\infty)^{n+t+1}\), locally given by the relations
\[
\begin{align*}
\bar{x}_{m'+i} & = \bar{\alpha}_{m'+i}(x', \rho') \\
\bar{\rho}_{n'+j} & = \bar{\beta}_{n'+j}(x', \rho')
\end{align*}
\]
Let us write \( \psi \) for the Weierstrass change of variables \( \phi|_{S_{m+s+n+t}} \) and let
\[
\psi' : (K^\infty)^{m+s} \times (K^\infty)^{n+t} \to (K^\infty)^{m+s} \times (K^\infty)^{n+t}
\]
be the non-singular analytic map that it induces which is also the same as the map \( \phi'(K^\infty)^{m+s} \times (K^\infty)^{n+t} \). Let \( W_1 \) be a rational subdomain of \((K^\infty)^{m+s} \times (K^\infty)^{n+t}\) such that \( W_1 \cap \pi_1(Z \cap \phi'(U)) \) contains a manifold whose charts are given by relations of the form (3). Let
\[
I_1 := \mathcal{I}(W_1 \cap \pi_1(Z \cap \phi'(U))) \subset \mathcal{O}(W_1)_K
\]
so that $k$-dim $\mathcal{O}(W_1)/I_1 = m + n$. Similarly let $W_2 \subset \psi^{-1}(W_1)$ be another rational subdomain such that $W_2 \cap R$ is given by relations in Equation (2) and let

$$I_2 := \mathcal{I}(R \cap W_2) \subset \mathcal{O}(W_2)_K$$

so that $k$-dim $\mathcal{O}(W_2)/I_2 = m + n$ and $\{x_{m+i} - \alpha_{m+i}\} \cup \{\rho_{n+j} - \beta_{n+j}\} \subset I_2$. Notice that each $x_{m+i} - \alpha_{m+i}(x', \rho')$ and $\rho_{n+j} - \beta_{n+j}(x', \rho')$ is regular of degree one in $x_{m+i}$ and $\rho_{n+j}$ respectively for each $i$ and $j$. By replacing $W_2$ with a smaller rational open subdomain we may assume that

$$\psi^{-1}(W_1) = W_2.$$

Now notice that the inclusion

$$T_{m+n} \to \mathcal{O}(W_2)/I_2$$

is an isomorphism and that $\psi^{-1}(I_1) \supset I_2$ as $\psi^{-1}$ takes $\pi_1(Z \cap \phi'(U))$ into $R$. Notice also that in this case we have

$$\psi^{-1}(I_1) \cap T_{m+n} = \mathcal{I}(V(\psi^{-1}(I_1) \cap T_{m+n})_K)$$

and therefore by Theorem 3.2, $\pi_2(V(\psi^{-1}(I_1) \cap T_{m+n})_K)$, which is $D$-semianalytic and contained in $X$, is $d$-dimensional. Now the statement of the theorem follows from the Smooth Stratification Theorem (Theorem 4.6) of [3].

We have the following immediate corollaries about the dimension theory of subanalytic sets.

**Corollary 4.6.** For a subanalytic subset $X$ of $(K^o)^m \times (K^o)^n$, we have

$$m\text{-dim } X = g\text{-dim } X = w\text{-dim } X.$$

**Corollary 4.7.** There is no infinite subanalytic set of dimension 0.

**Corollary 4.8.** Let $X$ be a subanalytic subset of $(K^o)^m \times (K^o)^n$ and let $Q \subset (K^o)^{m+M} \times (K^o)^{n+N}$ be a relation such that the graph $\Gamma(Q)$ of $Q$ over $X$ is subanalytic and for all $\bar{p} \in X$ there are finitely many $\bar{q} \in (K^o)^M \times (K^o)^N$ such that $(\bar{p}, \bar{q}) \in Q$. Then we have

$$g\text{-dim } X = g\text{-dim } \Gamma(Q).$$

**Proof.** Put $d := g\text{-dim } \Gamma(Q)$. Note that by Theorem 4.5, $\Gamma(Q) \subset (K^o)^{m+M} \times (K^o)^{n+N}$ contains a $d$-dimensional manifold whose charts are given by coordinate projections. Hence it is easy to see that if $d > g\text{-dim } X$ then the assumption of the finiteness property of $Q$ can not be satisfied.

Next is a property which was proved to hold for $D$-semianalytic sets in Theorem 6.6 of [3]. We now generalize it to subanalytic sets. First we need some notation. Let $X$ be a subset of $(K^o)^{m+M} \times (K^o)^{n+N}$ and let us fix an $(m+n)$-dimensional coordinate hyperplane $(K^o)^m \times (K^o)^n$, then we will write $X^{(d)}$ for the set

$$\{\bar{p} \in (K^o)^m \times (K^o)^n : \text{the fiber of } X \text{ over } \bar{p} \text{ is } d\text{-dimensional}\}.$$

**Corollary 4.9.** Let $X$ be a subanalytic subset of $(K^o)^{m+M} \times (K^o)^{n+N}$ and assume $g\text{-dim } X^{(d)} = m + n$, then $g\text{-dim } X \geq m + n + d$. 
Proof. By Theorem 4.5 there is a $D$-semianalytic $Y \supset X$ such that $Y$ and $X$ are of the same dimension. Observe that for such a $Y$, if $X^{(d)}$ is of dimension $m + n$, then $Y^{(d)}$ is also of dimension $m + n$. Then, by Theorem 6.6 of [3], $g$-$\dim Y \geq m + n + d$, and the statement follows. □

Let $X$ be a $d$-dimensional subanalytic subset of $(K^o)^m \times (K^o)^n$ and let $K' \supset K$ be a complete valued extension of $K$. As the open subsets of $(K^o)^m \times (K^o)^n$ are not necessarily open when treated as subsets of $(K^o)^m \times (K^o)^n$, one expects the dimension of $X$ as a subset of $(K^o)^m \times (K^o)^n$ to be less than $d$. This phenomenon is exemplified in Remark 4.11. However, Corollary 4.10 below explains that, nevertheless $X$ can not be contained in a subanalytic subset of $(K^o)^m \times (K^o)^n$ of dimension less than $d$.

**Corollary 4.10.** Let $X$, $K$ and $K'$ be as above and suppose $Y \subset (K^o)^m \times (K^o)^n$ is a subanalytic set that contains $X$, then we have

$$g$-$\dim Y \geq g$-$\dim X.$$

**Proof.** Let $\pi : (K^o)^m \times (K^o)^n \to (K^o)^{m'} \times (K^o)^{n'}$ be a coordinate projection map and assume that $\pi(X)$, which is also subanalytic, contains an open set $U$. Then for the analogous projection map $\pi' : (K^o)^m \times (K^o)^n \to (K^o)^{m'} \times (K^o)^{n'}$, $\pi(Y)$ also contains $U$. However, it is easy to see that $U$ can not be contained in any $(m' + n' - 1)$-dimensional $D$-semianalytic subset of $(K^o)^m \times (K^o)^n$ and the statement follows from Theorem 4.5. □

**Remark 4.11.** Note that if $Y$ in the above corollary is not subanalytic, then the statement that $g$-$\dim Y \geq g$-$\dim X$ need not be true. For example, $\mathbb{Q}(t)$, which is a 1-dimensional subanalytic subset of itself, is not a subanalytic subset of $\mathbb{R}(t)$ and it is easy to see that its geometric dimension as a subset of $\mathbb{R}(t)$ is 0.

In turn, above results concerning the geometry of subanalytic sets help us understand the geometry of $D$-semianalytic sets better. Once again let $K' \supset K$ be a complete extension of $K$, $B = S_{m + M, n + N}/J$ be a generalized ring of fractions over $S_{m,n}$, and $I$ be an ideal of $B$. Set

$$B' := S_{0,0}(E, K') \otimes_{S_{0,0}(E, K)} B = S_{m + M + N, E, K'} / J \cdot S_{m + M + N, E, K'}.$$

In this case, similar to the situation in Corollary 4.10, one can ask how the dimension of $K'$-$\text{Dom}_{m,n} B' \cap V(I \cdot B')_{K'}$ compares to that of $\text{Dom}_{m,n} B \cap V(I)_{K}$. Theorem 4.12 below, in a way, guarantees that the dimensions of these two $D$-semianalytic sets are equal, provided that we avoid the obvious complications that may arise from the fact that $K$ is not necessarily algebraically closed.

**Theorem 4.12.** Let $\text{Char } K = 0$, $B$, $I$, $J$, $K'$ and $B'$ be as above, and assume that $I = I(\text{Dom}_{m,n}B \cap V(I)_{K})$, then

$$g$-$\dim \text{Dom}_{m,n} B \cap V(I)_{K} = g$-$\dim K'$-$\text{Dom}_{m,n} B' \cap V(I \cdot B')_{K'}.$$

**Proof.** Notice that by Theorem 3.2, we have

$$g$-$\dim \text{Dom}_{m,n} B \cap V(I)_{K} \geq g$-$\dim K'$-$\text{Dom}_{m,n} B' \cap V(I \cdot B')_{K'} \geq g$-$\dim B/I = k$-$\dim B'/I'. $$

The reverse inequality follows from Corollary 4.10. □

As a corollary we see that for each $D$-semianalytic set $X$, there is a "nice" quantifier free formula $\varphi$, in the sense below, which defines it.
Corollary 4.13. Given a $d$-dimensional $D$-semianalytic set $X \subset (K^\circ)^m \times (K^{\circ\circ})^n$, there is a quantifier free formula $\varphi$ of $\mathcal{L}_\text{an}^D(E,K)$ such that

$$X = \{ \bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n : \varphi(\bar{p}) \}$$

and for all complete extensions $K'$ of $K$,

$$X' = \{ \bar{p} \in (K'^\circ)^m \times (K'^{\circ\circ})^n : \varphi(\bar{p}) \}$$

is also $d$-dimensional.

We end this section by proving another desirable property of the dimension functions we have been considering for $D$-semianalytic sets by the help of Corollaries 4.10 and 4.13.

Theorem 4.14. Let $\text{Char } K = 0$ and $X \subset (K^\circ)^m \times (K^{\circ\circ})^n$ be $D$-semianalytic. Let $\overline{X}$ denote the closure of $X$ in the metric topology, then

$$\text{g-dim } \overline{X} \setminus X < \text{g-dim } X.$$

Proof. First we are going to prove the theorem in the case when $K$ is algebraically closed. Note that this case was already proved in Theorem 4.3 of [13] by a different argument.

By Theorem 3.2 it is enough to consider the case $X = \text{Dom}_{m,n} B \cap V(I)_K$, where $B$ is a generalized ring of fractions over $S_{m,n}$ and $I \subset B$ is a prime ideal satisfying $I = \mathcal{I}(\text{Dom}_{m,n} B \cap V(I)_K)$. Write

$$B = S_{m+M,n+N}/\big( \{ g_i x_i - f_i \}_{i=m+1}^{m+M} \cup \{ g'_j \rho_j - f'_j \}_{j=n+1}^{n+N} \big).$$

Let $\pi : (K^\circ)^{m+M} \times (K^{\circ\circ})^{n+N} \to (K^\circ)^m \times (K^{\circ\circ})^n$ be the coordinate projection and $I$ be the ideal of $S_{m+M,n+N}$ that corresponds to $I$. Note that in this case

$$\text{g-dim } X = \text{k-dim } S_{m+M,n+N}/I.$$

Now let $\bar{p} \in \overline{X} \setminus X$. By Lemma 6.3 of [10], the projection of a closed subanalytic set is also a closed subanalytic set. Therefore, as $V(I)_K$ is a closed subset of $(K^\circ)^{m+M} \times (K^{\circ\circ})^{n+N}$, we have $\overline{X} \subset \pi(V(I)_K)$. Hence there is a $\bar{q} \in V(I)_K$ such that $\pi(\bar{q}) = \bar{p}$. Note that for each $\bar{q}$ which projects onto $\bar{p}$, we have $\bar{q} \in V(I \cup \prod_{i=m+1}^{m+M} g_i \cdot \prod_{j=n+1}^{n+N} g'_j)_K$, as otherwise $\bar{p} \in X$. Therefore $\overline{X} \setminus X$ is contained in $\pi(V(I \cup \prod_{i=m+1}^{m+M} g_i \cdot \prod_{j=n+1}^{n+N} g'_j)_K)$. The case in which $K$ is algebraically closed is now proved by the fact that $\text{g-dim } V(I \cup \prod_{i=m+1}^{m+M} g_i \cdot \prod_{j=n+1}^{n+N} g'_j)_K < \text{g-dim } X$.

For the general case, when $K$ is not necessarily algebraically closed, let $K'$ be an algebraically closed complete extension of $K$ and let $\varphi$ be the quantifier-free formula as in Corollary 4.13, so that the dimension of

$$X' := \{ \bar{p} \in (K'^\circ)^m \times (K'^{\circ\circ})^n : \varphi(\bar{p}) \}$$

is the same as that of $X$. Notice that $\overline{X} \setminus X'$ is a definable subset of $(K'^\circ)^m \times (K'^{\circ\circ})^n$ and $\overline{X} \setminus X$ is contained in it. Therefore, by [14], Corollary 4.3, $\overline{X} \setminus X'$ is quantifier free definable in the language $\mathcal{L}_\text{an}^D(E,K)$, say by the formula $\psi$. Now notice that

$$\overline{X} \setminus X' \supset Y := \{ \bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n : \psi(\bar{p}) \} \supset (\overline{X} \setminus X)$$

and therefore by combining the algebraically closed case above with Corollary 4.10 we have

$$\text{g-dim } (\overline{X} \setminus X) \leq \text{g-dim } Y \leq \text{g-dim } (\overline{X} \setminus X') < \text{g-dim } X' = \text{g-dim } X,$$

finishing the proof.
5. Piece Number

In [1] Bartenwerfer introduced the notions of the dimensional filterings and the piece number of a dimensional filtering of analytic varieties over a complete non-Archimedean field \( K \), to obtain a more refined measure of complexity than the number of Zariski irreducible components for these varieties. He showed that for such a variety \( X \subset (K^\circ)^m \times (K^{\circ\circ})^n \), for \( K \) satisfying Char \( K = 0 \) or Char \( K = p > 0 \) and \([K : K^p] < \infty\), there is a bound \( \Gamma \) such that for all \( \bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n \), the fiber \( X(\bar{p}) \) has a dimension filtering with piece number less than \( \Gamma \).

In this section we show that the Parameterized Normalization Lemma (Lemma 3.3) can be used to extend Bartenwerfer’s results from analytic varieties to \( D \)-semianalytic sets in the case Char \( K = 0 \).

The concepts of dimensional filterings and piece number can easily be extended to \( D \)-semianalytic sets, but before we give the corresponding definitions, we repeat an example due to Bartenwerfer to justify the interest in the piece number as a finer measure than the Zariski irreducible components.

**Example 5.1.** Let \( K \) be a discretely valued non-Archimedean complete field of any characteristic with prime element \( \varpi \) and let \( f = y^2 - (\varpi + x)x^2 \), then the reduction of \( f \) to \((K^\circ/K^{\circ\circ})[x,y] \) is irreducible and therefore \( f \) is itself irreducible.

Notice that there is an infinite sequence of points \( \bar{p}_i \in V(f)_K \) which converge to the point \((1,1) \) and therefore \( g\text{-dim} V(f)_K = 1 \) and \( (f) = \mathcal{I}(V(f)_K) \). Nevertheless the point \((0,0) \) is an isolated point of \( V(I)_K \) as \( \varpi \) has no square root in \( K \).

This example shows that the number of irreducible components of an analytic variety may be inadequate as a measure of number of “pieces” in that variety. Now we extend the definitions in [1] to \( D \)-semianalytic sets.

**Definition 5.2.** Let \( X \) be a \( D \)-semianalytic subset of \((K^\circ)^m \times (K^{\circ\circ})^n \), such that \( g\text{-dim} X = d \geq 0 \), then a \( d \)-tuple of sets \( S = (S_d, ..., S_0) \) is called a dimensional filtering of \( X \) if each \( S_i \) is a (not necessarily disjoint) finite union of \( i \)-dimensional \( D \)-semianalytic \( K \)-analytic manifolds and \( X \) is equal to the (again not necessarily disjoint) union of the \( S_i \).

In order to be able to define the piece number we also need to define the number of irreducible components of a \( D \)-semianalytic set.

**Definition 5.3.** Let \( X \) be a \( D \)-semianalytic subset of \((K^\circ)^m \times (K^{\circ\circ})^n \), let

\[
P : X = \bigcup_{i=1}^{r}(\text{Dom}_{m,n}B_i \cap V(I_i)_K)
\]

be a presentation of \( X \), let \( J_i = \mathcal{I}(\text{Dom}_{m,n}B_i \cap V(I_i)_K) \), and let \( p_{i,1}, ..., p_{i,s_i} \) be the minimal prime divisors of \( J_i \) in \( B_i \) for \( 1 \leq i \leq r \), then the number of irreducible components of this presentation \( P \) of \( X \) is defined to be

\[
\#ic P := s_1 + ... + s_r,
\]

and the number of irreducible components of \( X \) is defined to be

\[
\#ic X := \min\{\#ic P : P \text{ is a presentation of } X\}.
\]

Now we are ready to define
Definition 5.4. Given a $D$-semianalytic set $X$ with dimensional filtering $S = (S_d, ..., S_0)$, we define the piece number of $S$ as

$$\text{pn } S := \sum_{i=0}^{d} (\# \text{ic } S_i).$$

We define the piece number of $X$ to be

$$\text{pn } X := \min \{\text{pn } S : S \text{ is a dimensional filtering of } X\}.$$ 

Note that this definition is slightly different from Bartenwerfer’s definition as instead of working with pure dimensional subsets of varieties we work with analytic manifolds. However, as in his definition, pn $X$ is easily seen to dominate the number of irreducible components and isolated points in $X$. It is also easy to see that the analytic variety in Example 5.1 has piece number at least two, confirming what one would intuitively expect from the piece number.

The following theorem is a generalization of the main theorem of [1] which was proved for analytic varieties in the case $\text{Char } K = 0$ or $\text{Char } K = p > 0$ and $[K : K^p] < \infty$. As we usually do throughout this paper, we assume that $\text{Char } K = 0$.

Theorem 5.5. Let $\text{Char } K = 0$ and let $X \subset (K^\circ)^{m+M} \times (K^\circ)^{n+N}$ be a $D$-semianalytic set, then there is a bound $\Gamma \in \mathbb{N}$ such that for all $\bar{p} \in (K^\circ)^m \times (K^\circ)^n$ the piece number of the fiber $X(\bar{p})$ is less than $\Gamma$.

Proof. By definition it is enough to find a $\Gamma$ such that for each $\bar{p} \in (K^\circ)^m \times (K^\circ)^n$, $X(\bar{p})$ has a dimensional filtering with piece number less than $\Gamma$. Our plan is to make use of the Parameterized Normalization Lemma (Lemma 3.3) to first normalize the $D$-semianalytic set we are working on, and then use Parameterized Smooth Stratification Theorem (Theorem 4.3) to find a dimensional filtering for each fiber. The piece number of each such dimensional filtering will be uniformly bounded by the product of the degrees of the minimal polynomials of the integral variables of the normalization.

Let $X_1, ..., X_k$ be as in Theorem 4.3 so that $X = X_1 \cup ... \cup X_k$ and for all $\bar{p} \in (K^\circ)^m \times (K^\circ)^n$, $X_i(\bar{p})$ is either empty or is a $D$-semianalytic manifold. We will make use of the construction in the proof of Theorem 4.3 to further observe that in this case, for each $i$, we have

$$X_i = \text{Dom}_{m+M,n+N} B_i \cap V(p_i) \cap \{q \in (K^\circ)^{m+M} \times (K^\circ)^{n+N} : \Delta_i(q) \neq 0\}$$

for some generalized ring of fractions $B_i$ over $S_{m+M,n+N}$, prime ideal $p_i \subset B_i$ satisfying $p_i = I(\text{Dom}_{m+M,n+N} B_i \cap V(p_i) K)$ and the determinant $\Delta_i \in B_i$ of the Jacobian. Moreover, in this case there are integers $m_i, M_i, n_i, N_i$ and Weierstrass changes of variables $\phi_i$ such that there is a finite injection

$$S_{m_i+M_i,n_i+N_i} \to B_i/\phi_i(p_i).$$

Note that if $z$ is a variable not appearing in the presentation of any $B_i$, then $X_i$ is a $D$-semianalytic set with the presentation

$$X_i = \text{Dom}_{m+M,n+N} (B_i (z) / \Delta_i z - 0) \cap V(p_i \cdot (B_i (z) / \Delta_i z - 0)) K.$$ 

Now for a given $\bar{p} \in (K^\circ)^m \times (K^\circ)^n$ let $g\text{-dim } X(\bar{p})$ be $d$ and set

$$S_j(\bar{p}) := \bigcup \{X_i : X_i(\bar{p}) \text{ is } j\text{-dimensional}\}$$
for all $0 \leq j \leq d$, so that $S := (S_d(\bar{p}), ..., S_0(\bar{p}))$ is a dimensional filtering for $X(\bar{p})$. Now it is enough to show that for each $X_i$, there is a bound $\Gamma$ such that for $\bar{p} \in (K^\circ)^m \times (K^\circ)^n$ we have $\# \text{ic } X_i(\bar{p}) \leq \Gamma$.

Let $B_i = S_{m+s_i+M+S_i,n+t_i+N+T_i}/I_i$ and let $\bar{p}_i$ be the ideal corresponding to $p$ in the quasi-affinoid algebra $S_{m+n+M+S_i,n+t_i+N+T_i}/I_i$. By $\psi_i$, let us denote the restriction of the Weierstrass change of variables $\phi_i$ to $S_{m+s_i,n+t_i}$, so that there is a finite homomorphism

$$S_{m,s_i} \to S_{m+s_i,n+t_i}/\psi_i(\bar{p}_i \cap S_{m+s_i,n+t_i}).$$

For $\bar{p} \in (K^\circ)^m \times (K^\circ)^n$ let $B_i(\bar{p})$ and $p_i(\bar{p})$, $\Delta_i(\bar{p})$ denote the objects we obtain by making the obvious substitutions the notation indicates so that $B_i(\bar{p})$ is a generalized ring of fractions over $S_{M,N}$, $\Delta_i(\bar{p}) \in B_i(\bar{p})$ and $p_i(\bar{p})$ is an ideal of $B_i(\bar{p})$.

Note that the restriction $\phi_i^* \circ \phi_i$ to $y$ and $\lambda$ variables induces a finite homomorphism

$$S_{M_i,N_i} \to B_i(\bar{p})/\phi_i^*(p_i(\bar{p})), $$

and if $X_i(\bar{p})$ is not empty, then $X_i(\bar{p})$ is an $(M_i+N_i)$-dimensional manifold. Now it is easy to see also that $g$-dim $X_i(\bar{p}) = M_i + N_i$ and therefore $k$-dim $B_i(\bar{p})/\phi_i^*(p_i(\bar{p})) \geq M_i + N_i$. Thus we see that in fact the above homomorphism is an injection.

Next, put

$$J_\bar{p} := I(D_{\bar{p}+N}) (B_i(\bar{p}) (z)/\Delta_i(\bar{p})^2 - 0) \cap V(p_i(\bar{p} \cdot (B_i(\bar{p}) (z)/\Delta_i(\bar{p})^2 - 0))_K)$$

and let us write $J_\bar{p}$ for the ideal of $S_{M+S_i+1,N+T_i}$ which corresponds to $J_\bar{p}$. Note that we have $z \in J_\bar{p}$. Therefore $(B_i(\bar{p}) (z)/\Delta_i(\bar{p})^2 - 0)/\phi_i^*(J_\bar{p})$ is finite over $S_{M_i,N_i}$. Notice also that $X_i(\bar{p}) = \text{Dom}_{\bar{p}+N}, N_i \Delta_i(z) \in 0 \cap V(J_\bar{p})_K$ and therefore $g$-dim $X_i(\bar{p}) = M_i + N_i$. In fact, we have a finite inclusion

$$S_{M_i,N_i} \to (B_i(\bar{p}) (z)/\Delta_i(\bar{p})^2 - 0)/\phi_i^*(J_\bar{p}).$$

Now let $r_1, ..., r_{M+S_i-M_i}$, and $r_1', ..., r_{N+T_i-N_i}$ be the degrees of the lowest degree monic polynomials in $\phi_i(\bar{p}_i) \cap S_{m+S_i,n+S_i}[y_{M+S_i}]$ and $\phi_i(\bar{p}_i) \cap S_{m+n+1,n+1}[\lambda_{N,T_i}]$ respectively, for $1 \leq j \leq M + S_i - M_i$ and $1 \leq j' \leq N + T_i - N_i$. Then for all $\bar{p}$, $r_j$ is greater than or equal to the degree of the lowest degree monic polynomial in $\phi_i^*(J_\bar{p}) \cap S_{M_i,N_i}[y_{M+S_i}]$ for all $j$, and the analogous statement also holds for $r_j'$, for all $j'$. Therefore

$$\Gamma := \prod_{j=1}^{M+S_i-M_i} r_j \cdot \prod_{j'=1}^{N+T_i-N_i} r_j'$$

is a bound on the number of irreducible components of $\phi_i^*(\bar{J}_\bar{p})$ for all $\bar{p}$. From this we obtain the bound in the statement of the theorem.

6. **Complexity**

Next we turn our attention to the fibers of a $D$-semianalytic set which are subsets of $K^\circ$. Our aim is to give a simpler proof of the main theorem of [12] using the results of the previous section on the piece number. That is, we are going to prove that given a $D$-semianalytic set $X$, there is a uniform bound $\Gamma$ such that each one-dimensional fiber of $X$ is a boolean combination of at most $\Gamma$ discs (see Definition 6.1 below) and points, by showing that the piece number of each such fiber is closely related to the number of discs and points required for such a boolean combination. This result is analogous to the main theorem of [6] in the sense that both concern a
measure of complexity of one-dimensional fibers of definable sets. Moreover, at the end of this section we will prove a more readily recognizable analog of that theorem in our setting.

Before we state and prove these theorems we need some groundwork.

**Definition 6.1.** A disc in $K^o$ is a set of the form
\[
D^-(a, r) := \{ x \in K^o : |x - a| < r \}, \text{ or } \\
D^+(a, r) := \{ x \in K^o : |x - a| \leq r \}
\]
where $a \in K^o$ and $r \in \sqrt{|K^o|}$.

We follow the terminology of [12] and call a set of the form a disc minus a finite union of discs a $K$-rational special set, or in case $K$ is algebraically closed, a special set.

In [11], Lipshitz and Robinson showed that for an algebraically closed complete non-Archimedean field $\bar{K}$, an $R$-subdomain $X$ of $\bar{K}$ is a boolean combination of positive radius discs. Conversely, with a little effort, one sees that any such combination is an $R$-subdomain of $\bar{K}^o$. Let $K' \subset \bar{K}$ be another non-Archimedean complete field which is not necessarily algebraically closed. Then by the ultrametric inequality, it is easy to see that given a disc $C \subset \bar{K}$, either it is the case that $C \cap K'$ is a $K'$-rational disc or we have $C \cap K' = \emptyset$. Therefore, it is also true that the set of $K$-rational points of an $R$-subdomain of $\bar{K}$ is a boolean combination of $K$-rational discs, some of which are possibly of zero radius. Notice that there may be more than one way of writing such a set as a union of disjoint special sets, as in the case the residue field $K^o/K^\infty$ is finite, the closed unit disc $D^+(0, 1)$ is a disjoint union of the open unit disc $D^-(0, 1)$ and finitely many smaller closed discs centered at the "boundary" of $D^+(0, 1)$.

**Definition 6.2.** Let $X \subset K$ be the set of $K$-rational points in an $R$-domain and
\[
D : X = \bigcup_{i=1}^{r} S_i = \bigcup_{i=1}^{r} (C_i \setminus \bigcup_{j=1}^{s_i} C_{ij})
\]
be a decomposition of $X$ into $K$-rational special sets $S_i = C_i \setminus \bigcup_{j=1}^{s_i} C_{ij}$ then the complexity of $D$ is defined to be
\[
\text{comp } D := r + s_1 + \ldots + s_r,
\]
and the complexity of $X$ is defined to be
\[
\text{comp } X := \min \{ \text{comp } D : D \text{ is a decomposition of } X \text{ into } K\text{-rational special sets} \}.
\]

Our aim is to establish a relation between the piece number and the complexity. For this, we first observe that if $B$ is a generalized ring of fractions over $S_0,1(E, K)$ (or $S_1,0(E, K)$) for which $X = \bar{K}$-Dom$_{m,n}B$ is an $R$-domain and if we put $X = \text{Dom}_{m,n}B$, then by Definition 6.2 and the discussion preceding it, it is clear that the complexity of $X$ dominates the complexity of $X$. Next we are going to show that for a $K$-rational $R$-domain $X \subset K^o$, 
\[
\text{pn } X^+ + \text{pn } (K^o \setminus X) \geq \text{comp } X.
\]
Showing this requires a deeper understanding of some special type of quasi-affinoid algebras. The following two key lemmas which we state without proof are consequences of results in [4] by Cluckers, Lipshitz and Robinson. Both of these lemmas
are well known in the affinoid category and we refer the reader to §7.3.3 of [2] and §2.2 of [7] for further details.

**Lemma 6.3.** Suppose $K$ is an algebraically closed complete field and let $B$ be the ring of analytic functions of an $R$-domain $X \subset K^{\circ}$. Let

$$X = \bigcup_{i=1}^{r} X_i$$

be a decomposition of $X$ into disjoint special sets with minimal complexity, then we have

$$B = \mathcal{O}(X) = \bigoplus_{i=1}^{r} \mathcal{O}(X_i).$$

We would like to point out that one can find an analogue of the above lemma in the affinoid category in Proposition 7 of §7.3.2 of [2].

**Lemma 6.4.** Suppose $K$ is an algebraically closed complete field and suppose $B$ is the ring of analytic functions of a special set, then for any $h \in B$ we have

$$h = u \cdot p(x)$$

where $u \in B$ is a unit and $p(x) \in K[x]$.

As an immediate corollary to the above lemmas we have:

**Lemma 6.5.** Suppose $K$ is an algebraically closed complete field and suppose $B$ is the ring of analytic functions of an $R$-subdomain of $K^{\circ}$, then $B$ is a principal ideal domain.

**Proof.** By separating $\text{Dom}_{1,0}B$ into disjoint special sets and making use of Lemma 6.3, we may assume that $\text{Dom}_{1,0}B$ is a special set. Now it is enough to show that for any $f_1, f_2 \in B$ there is an $f \in B$ such that $(f_1, f_2) = (f)$. Using Lemma 6.4 write $f_i = u_i \cdot p_i(x)$ where $u_i \in B$ is a unit and $p_i(x) \in K[x]$ for $i = 1, 2$. Now put $f$ to be the greatest common divisor of $p_1(x)$ and $p_2(x)$ and the result follows.

The next lemma shows that, when $K$ is algebraically closed, an arbitrary generalized ring of fractions over $S_{1,0}(E, K)$ (or $S_{0,1}(E, K)$) is not too far removed from one that is the ring of analytic functions of an $R$-subdomain of $K^{\circ}$. But before we state and prove this lemma, we need a notation which enables us to keep track of the steps in the inductive construction of a generalized ring of fractions.

Let $B$ be a generalized ring of fractions over $S_{1,0}$ which is constructed in $m + n$ steps. Then there is a sequence $\{B_i\}_{i=0}^{m+n}$ of generalized rings of fractions such that $B_0 = S_{1,0}$, $B_{m+n} = B$ and if $B_i$ is given by the presentation

$$B_i = S_{1+m_i, n_i}/\langle \{g_j x_j - f_j\}_{j=1}^{m_i} \cup \{g''_n p_n - f''_n\}_{n=1}^{n_i} \rangle$$

where $m_i + n_i = i$. Thus, $B_{i+1}$ is given either by $B_i (x_{m_i+1}) / (g x_{m_i+1} - f)$ or by $B_i [\rho_{n_i+1}] / (g \rho_{n_i+1} - f)$ for some $f, g \in S_{1+m_i, n_i}$, for $i = 0, ..., m + n - 1$.

For such an inductive construction of $B$, we will introduce the following notation inductively: at the $(i+1)^{\text{st}}$ step define

$$[x/\rho]_{i+1}^B := \begin{cases} x_{m_i+1} & \text{if } B_{i+1} = B_i \langle x_{m_i+1} \rangle / (f x_{m_i+1} - g) \\ \rho_{n_i+1} & \text{if } B_{i+1} = B_i [\rho_{n_i+1}] / (f \rho_{n_i+1} - g) \end{cases}$$

$$[f]_{i+1}^B := f$$

$$[g]_{i+1}^B := g$$
So that
\[ B = S_{1+m,n}/(\{ [g_i]^B[x/\rho]^B - [f_i]^B \}_{i=1}^m + n). \]
When the generalized ring of fractions $B$ is clear from the context, we will just write $[x/\rho]_i, f_i, g_i$ instead of $[x/\rho]^B_i, [f_i]^B, [g_i]^B$. Now we are ready to state and prove:

**Lemma 6.6.** Suppose $K$ is an algebraically closed complete field and let $B$ be a generalized ring of fractions over $S_{1,0}(E, K)$. Assume $\text{Dom}_{1,0}B \neq \emptyset$ and write
\[ B := S_{1+m,n}/(\{ g_i[x/\rho]_i - f_i \}_{i=1}^m + n) \]
for $g_i, f_i \in S_{1+m,n}$ as above. Then for each $i = 1, ..., m + n$, there exist $G_i, F_i \in S_{1+m,n}$ such that
\begin{enumerate}
\item[(i)] $(G_i[x/\rho]_1 - F_i, ..., G_i[x/\rho]_{i-1} - F_{i-1}, G_i, F_i)$ is the unit ideal of $S_{1+m,n}$ and
\[ \sqrt{\{g_i[x/\rho]_i - f_i \}_{i=1}^m + n} \subseteq \sqrt{\{G_i[x/\rho]_i - F_i \}_{i=1}^m + n}, \]
\item[(ii)] for $\bar{B} := S_{1+m,n}/(\{ G_i[x/\rho]_i - F_i \}_{i=1}^m + n)$, $\text{Dom}_{1,0}B \setminus \text{Dom}_{1,0}\bar{B}$ consists of finitely many points,
\item[(iii)] any minimal prime divisor $p \subset S_{1+m,n}$ of $(\{ G_i[x/\rho]_i - F_i \}_{i=1}^m + n)$ is also a minimal prime divisor of $(\{ g_i[x/\rho]_i - f_i \}_{i=1}^m + n)$.
\end{enumerate}

**Proof.** We start by observing that by Lemma 6.4 if $A$ is a generalized ring of fractions over $S_{1,0}$ and $\text{Dom}_{1,0}A$ is a special $R$-domain, then $A$ is an integral domain and it has Krull dimension 1. Therefore if we found $G_i, F_1, ..., G_{m+n}, F_{m+n}$ satisfying the conditions (i) and (ii) above, then by Lemma 6.3 we have $k\text{-dim } \bar{B} = 1$. Hence for any minimal prime divisor $p$ of $(\{ G_i[x/\rho]_i - F_i \}_{i=1}^m + n)$, $k\text{-dim } S_{1+m,n}/p = 1$.

Next we proceed by induction on $m + n$. The reader may also think of this as an induction on the complexity of $L^n_{\text{an}}$-terms that appear in the definition of $\text{Dom}_{m+1,n}B$, where $L^n_{\text{an}}$ stands for the analytic language of [9]. Suppose the lemma holds for all generalized rings of fractions constructed in less than $m + n$ steps. Put
\[ B' := S_{1+m+n-1,n+m+n-1}/(\{ g_i[x/\rho]_i - f_i \}_{i=1}^{m+n-1}), \]
in which case $B'$ is the the ring we obtain at the penultimate step in the construction of $B$. Next, for simplicity in notation, we assume that $[x/\rho]_{m+n}$ is $\rho_n$ so that $B' = S_{1+m,n-1}/(\{ g_i[x/\rho]_i - f_i \}_{i=1}^{m+n-1})$ and $B = B'[[\rho_n]]/([m+n] \rho_n - f_{m+n})$, where $g_{m+n}$ and $f_{m+n}$ are the images of $g_{m+n}$ and $f_{m+n}$ in $B'$.

Assume that we found $G_1, F_1, ..., G_{m+n-1}, F_{m+n-1}$ satisfying the conclusion of the lemma for $B'$, then
\[ B'' := S_{1+m,n-1}/(\{ G_i[x/\rho]_i - F_i \}_{i=1}^{m+n-1}) \]
is a ring of functions of an $R$-domain and if $U_1, ..., U_k$ are the disjoint special $R$-domains that appear in $\text{Dom}_{1,0}B'$ then $B'' = \bigoplus_{j=1}^k \mathcal{O}(U_j)$. Notice that $k\text{-dim } B'' = 1$ and therefore the possible values for $d := k\text{-dim } S_{1+m,n-1}/(\{ G_i[x/\rho]_i - F_i \}_{i=1}^{m+n-1}, g_{m+n}, f_{m+n})$
are $-1, 0$ and 1.

**Case 1:** $d = -1$.

Then we put $G_{m+n} = g_{m+n}, F_{m+n} = f_{m+n}$ and the statements (i) and (ii) are easily seen to be satisfied. Now assume $p \in S_{1+m,n}$ is a minimal prime divisor of $(\{ G_i[x/\rho]_i - F_i \}_{i=1}^{m+n})$, by observing the Krull dimensions we see that
\( p \cap S_{1+m,n-1} \) is minimal over \( \{(G_i[x]/p_i) - F_i\}_{i=1}^{m+n-1} \) and by inductive hypothesis also over \( \{(g_i[x]/p_i) - F_i\}_{i=1}^{m+n-1} \). Now if \( q \in p \) is a minimal prime divisor of \( \{(g_i[x]/p_i) - F_i\}_{i=1}^{m+n} \) then \( q \cap S_{1+m,n-1} \) contains \( \{(G_i[x]/p_i) - F_i\}_{i=1}^{m+n-1} \) and therefore we have \( p = q \).

Case 2: \( d = 0 \).

By Lemma 6.5 there is an \( h \in S_{1+m,n-1} \) such that

\[
\{(G_i[x]/p_i) - F_i\}_{i=1}^{m+n-1}, g_{m+n}, f_m+n) = \{(G_i[x]/p_i) - F_i\}_{i=1}^{m+n-1}, h
\]

and the image \( h \) of \( h \) in \( B'' \) is of the form \( h = p_1(x) \oplus \ldots \oplus p_k(x) \) where \( p_j(x) \in K[x] \) for all \( j \).

Write

\[
\begin{align*}
f_{m+n} &= F_{m+n}h + a \\
g_{m+n} &= G_{m+n}h + b
\end{align*}
\]

where \( a, b \in \{(G_i[x]/p_i) - F_i\}_{i=1}^{m+n-1} \). By the Nullstellensatz and Lemma 6.4 it is easily seen that \( \{(G_i[x]/p_i) - F_i\}_{i=1}^{m+n-1}, G_{m+n}, F_{m+n} \) is the unit ideal and that \( \text{Dom}_{1,0}B \) and \( \text{Dom}_{1,0}B \) differ by finitely many points. Now by routine checks the problem reduces to Case 1.

Case 3: \( d = 1 \).

Let us write \( f_{m+n} = a_1 \oplus \ldots \oplus a_k \) and \( g_{m+n} = b_1 \oplus \ldots \oplus b_k \) for the images of \( f_{m+n} \) and \( g_{m+n} \) in \( B'' \). Then after a rearrangement of the components there is an \( l \leq k \) such that \( a_j = b_j = 0 \) for \( 1 \leq j \leq l \) and \( k \)-dim \( (U_i)/(a_j, b_j) \) is 0 or -1 for \( l < j \leq k \). Let \( h \in S_{1+m,n-1} \) be such that the image of \( h \) in \( B'' \) is \( 1 \oplus \ldots \oplus 1 \oplus \oplus 1 \oplus \ldots \oplus 0_k \) and set \( F_{m+n} := f_{m+n} + h, G_{m+n} := g_{m+n} \) so that the Krull dimension of \( S_{1+m,n}/\{(G_i[x]/p_i) - F_i\}_{i=1}^{m+n-1}, G_{m+n}, F_{m+n} \) is 0 or -1. By the Nullstellensatz it is again easy to see that the problem reduces to Case 1 or Case 2.

Now we are ready to give another proof of the main theorem (Theorem 1.6) of [12]. Note that this result is analogous to the main theorem of [6] in that it puts a bound on how “complicated” one dimensional fibers of \( D \)-semianalytic sets can get.

**Theorem 6.7.** Let \( \text{Char} \ K = 0 \) and let \( X \subset (K^\circ)^{m+1} \times (K^\circ)^n \) be \( D \)-semianalytic, then there exists a bound \( \Gamma \) such that for each \( \bar{p} \in (K^\circ)^m \times (K^\circ)^n \) there is a \( K \)-rational \( R \)-domain \( Y(\bar{p}) \subset K^\circ \) of complexity at most \( \Gamma \) and a set \( Q(\bar{p}) \subset K^\circ \) of at most \( \Gamma \) points such that

\[
Y(\bar{p}) \setminus Q(\bar{p}) = X(\bar{p}) \setminus Q(\bar{p}).
\]

**Proof.** By the discussion following Definition 6.2 we can assume that \( K \) is algebraically closed. Let \( \Gamma_1 \in \mathbb{N} \) be such that \( \text{pn} \ X(\bar{p}) < \Gamma_1 \) for all \( \bar{p} \in (K^\circ)^m \times (K^\circ)^n \) as in Theorem 5.5. Fix a \( \bar{p} \in (K^\circ)^m \times (K^\circ)^n \) and let \( S = (S_1, S_0) \) be a dimensional filtering of \( X(\bar{p}) \) such that \( \text{pn} \ S = \text{pn} \ X(\bar{p}) \). Write \( S_1 = \bigcup_{i=1}^k \text{Dom}_{1,0}B_i \cap V(I_i) \) for some generalized rings of fractions \( B_i \) over \( S_1, S_0 \) and ideals \( I_i \subset B_i \) where each \( \text{Dom}_{1,0}B_i \cap V(I_i) \) is a one-dimensional \( K \)-analytic manifold.

Let \( B_i \) denote the generalized ring of fractions we obtain from \( B_i \) by applying Lemma 6.6 so that each \( \text{Dom}_{1,0}B_i \) is an \( R \)-domain and by Lemma 6.3 so are \( \text{Dom}_{1,0}B_i \cap V(I_i) \) for all \( i \). Note that the number of special sets in any decomposition of each \( \text{Dom}_{1,0}B_i \cap V(I_i) \) is \#ic \( \text{Dom}_{1,0}B_i \cap V(I_i) \). Define

\[
Y(\bar{p}) := \bigcup_{i=1}^k \text{Dom}_{1,0}B_i \cap V(I_i),
\]
and

\[ Q(\bar{p}) := (Y(\bar{p}) \setminus X(\bar{p})) \cup (X(\bar{p}) \setminus Y(\bar{p})), \]

so that \( Y(\bar{p}) \) is an \( R \)-domain and the number of special sets in any decomposition of \( Y(\bar{p}) \) is bounded by

\[ s := \sum_{i=1}^{k} \# \text{ic} (\text{Dom}_{1,0}B_i \cap V(I_i)) \geq \sum_{i=1}^{k} \# \text{ic} (\text{Dom}_{1,0}B_i \cap V(I_i)). \]

Notice that \( s + |X(\bar{p}) \setminus Y(\bar{p})| \leq s + |Q(\bar{p})| \leq \Gamma_1 \), and \( S_0 \cap (Y(\bar{p}) \setminus S_1) = \emptyset \), as if \( \bar{q} \in S_0 \cap (Y(\bar{p}) \setminus S_1) \) then using the construction in Lemma 6.6, it is easy to see that \( \{\bar{q}\} \cup S_1, S_0 \setminus \{\bar{q}\} \) is another dimensional filtering of \( X(\bar{p}) \) with a smaller piece number. Therefore the points in \( Y(\bar{p}) \setminus S_1 \) are the isolated points of \( K^0 \setminus X(\bar{p}) \). Hence the bound \( \Gamma_2 \) for the piece numbers of the fibers of the \( D \)-semianalytic set \( ((K^0)^{m+1} \times (K^0)^n) \setminus X \) dominates the total number of holes in the special sets that make up \( X(\bar{p}) \) plus the number of points in \( Y(\bar{p}) \setminus S_1 \) and \( \Gamma := \Gamma_1 + \Gamma_2 \) is the desired bound in the statement of the lemma.

As suggested by Leonard Lipshitz we can prove a theorem with a statement more similar to Theorem A of [6] in all characteristics using an argument similar to the one that is used to prove the Parameterized Normalization Lemma. Before we state this theorem, we would like to remind the reader that a \emph{semialgebraic subset} of \((K^0)^m \times (K^0)^n\) is a finite union of sets of the form

\[ \{ \bar{p} \in (K^0)^m \times (K^0)^n : \bigwedge_{i=1}^{s} |f_i(\bar{p})| < |g_i(\bar{p})| \wedge \bigwedge_{j=1}^{t} |f_j(\bar{p})| \leq |g_j(\bar{p})| \} \]

where \( f_i, g_i, f_j, g_j \) are polynomials.

**Theorem 6.8.** Let \( X \subset (K^0)^{m+1} \times (K^0)^n \) be a \( D \)-semianalytic set, then there exist an \( M \in \mathbb{N} \) and a semialgebraic set \( Y \subset (K^0)^{m+M+1} \times (K^0)^{n+N} \) such that for every \( \bar{p} \in (K^0)^{m+1} \times (K^0)^n \) there is a \( \bar{q} \in (K^0)^{m+M} \times (K^0)^{n+N} \) such that the fibers \( X(\bar{p}) \) and \( Y(\bar{q}) \) are the same.

**Proof.** For the moment assume that \( X = \text{Dom}_{m+1,n}B \cap V(I) \) for some generalized ring of fractions \( B \) over \( S_{m+1,n} \) and ideal \( I \subset B \). Write

\[ B = S_{m+1,s,n+t}/(\{g_i x_i - f_i\}_{i=m+1}^{m+s} \cup \{g_j^i \rho_j - f_j^i\}_{j=n+1}^{n+t}), \]

where \( S_{m+1,s,n+t} \) denotes the ring of separated power series over the variables \( x_1, \ldots, x_m, y, x_{m+1}, \ldots, x_{m+s}, \rho_1, \ldots, \rho_{n+t} \) and \( y \) variables correspond to the coordinates representing the fiber space. Let \( I \) be the ideal corresponding to \( I \) in \( S_{m+1,s,n+t} \) and let \( h_1, \ldots, h_k \) generate \( I \). For \( i = 1, \ldots, k \) write

\[ h_i = \sum_{j \in \mathbb{N}} a_{i,j}(x, \rho) y^j, \]

and let \( I' \) be the ideal of \( S_{m+s,n+t} \) generated by \( \{a_{i,j}\}_{i,j} \). Let \( I' \) be the ideal which corresponds to \( I' \) in \( B \). By Lemma 3.1.6 of [15], there is a finite set \( Z \subset \{1, \ldots, k\} \times \mathbb{N} \) such that for any \( \beta \in \{1, \ldots, k\} \times \mathbb{N} \), there are \( b_{\beta,\alpha} \in Z \) such that \( a_{\beta} = \sum_{\alpha \in Z} b_{\beta,\alpha} a_{\alpha} \) and \( \|a_{\beta}\| \geq \|b_{\beta,\alpha} a_{\alpha}\| \) for all \( \alpha \in Z \).

Now for each \( \alpha = (i_0, j_0) \in Z \) define

\[ B_\alpha = B \left\langle \{a_{i_0,j}/a_{i_0,j_0}\}_{i_0,j_0} : (i_0,j) \in Z, j < j_0 \right\rangle \left( \{a_{i_0,j}/a_{i_0,j_0}\}_{i_0,j_0} : (i_0,j_0) \in Z, j_0 > j \right) \right]. \]
X_\alpha = \text{Dom}_{m+1,n}B_\alpha \cap V(I \cdot B_\alpha)_K \text{ and } X' = \text{Dom}_{m+1,n}B \cap V(I')_K. \text{ Notice that }

X = X' \cup \bigcup_{\alpha \in \mathbb{Z}} X_\alpha.

Our plan is to show that the statement of the theorem holds if we replace \(X\) by both \(X_\alpha\) and \(X'\). For \(\bar{p} \in (K^\circ)^m \times (K^\circ)^n\) the fiber \(X' (\bar{p})\) is either empty or all of \(K^\circ\) and so the statement holds trivially for \(X'\). On the other hand if we write \(B_\alpha = S_{m+1,s_\alpha,n+t_\alpha}/J_\alpha\) and \(I_\alpha\) for the ideal of \(S_{m+1,s_\alpha,n+t_\alpha}\) corresponding to \(I \cdot B_\alpha\) then for \(\alpha = (i_0, j_0) \in \mathbb{Z}\) we have \(h_{i_0} \in I_{i_0,j_0}\) and we can write

\[
h_{i_0} \equiv a_0(y^{i_0} + \sum_{j \neq j_0} b_{i_0,j} y^j) \mod \{(a_0x_1 - a_{i_0,j})_{i=1}^{n+1} \cup \{a_0p_l - a_{i_0,j}\}_{l=1}^{n+1}\}
\]

for some \(b_{i_0,j} \in S_{m+s_\alpha,n+t_\alpha}\) which make \(y^{i_0} + \sum_{j \neq j_0} b_{i_0,j} y^j\) regular in \(y\) of degree \(j_0\).

Therefore by the Weierstrass Preparation Theorem, we see that there is a monic regular polynomial \(p_\alpha(y)\) of degree \(j_0\) in \(S_{m+s_\alpha,n+t_\alpha}[y]\) which, when treated as a function over \((K^\circ)^{m+1} \times (K^\circ)^n\), vanishes at \(X_\alpha\). Let \(\bar{p}_\alpha(y)\) be the image of \(p_\alpha(y)\) in \(B_\alpha\) and write \(\bar{p}_\alpha(y) = y^{i_0} + c_{\alpha,j_0-1}y^{j_0-1} + \ldots + c_{\alpha,0}\) for some \(D\)-functions \(c_{\alpha,j_0-1}, \ldots, c_{\alpha,0}\) over \((K^\circ)^{m+1} \times (K^\circ)^n\).

Now let \(\bar{g}_i, \bar{f}_j, \bar{f}'_j, \bar{a}_{i_0,j}, \bar{h}_i\) denote the images of the remainders of \(\bar{f}_i, \bar{f}_j, \bar{f}'_j, \bar{a}_{i_0,j}, \bar{h}_i\) after an application of the Weierstrass Division Theorem to divide by \(\bar{p}_\alpha(y)\) in the generalized ring of fractions \(B_\alpha\). Then \(X_\alpha\) consists of \(\bar{p} \in (K^\circ)^{m+1} \times (K^\circ)^n\) such that

\[
|\bar{f}_i(\bar{p})| \leq |\bar{g}_i(\bar{p})| \neq 0 \quad \text{for } i = m+1, \ldots, m+s,
|\bar{f}'_j(\bar{p})| < |\bar{g}_i(\bar{p})| \quad \text{for } j = n+1, \ldots, n+t,
|\bar{a}_{i_0,j}(\bar{p})| \leq |\bar{a}_{i_0,j}(\bar{p})| \neq 0 \quad \text{for } (i_0,j) \in \mathbb{Z}, j < j_0,
|\bar{a}_{i_0,j}(\bar{p})| < |\bar{a}_{i_0,j}(\bar{p})| \quad \text{for } (i_0,j) \in \mathbb{Z}, j > j_0,
\bar{h}_i(\bar{p}) = 0 \quad \text{for } l = 1, \ldots, k,
\bar{p}_\alpha(y) = 0.
\]

The variable \(y\) appears polynomially on each of the lines above, and therefore by introducing new variables for each of the non-polynomial \((D\text{-function})\) terms in the above description we get a semialgebraic set \(Y_\alpha\). Given a \(\bar{p} \in (K^\circ)^{m+1} \times (K^\circ)^n\), if we substitute the values determined by those \(D\)-functions of \(\bar{p}\) for those new variables, the resulting fiber is the same as \(X_\alpha(\bar{p})\) and we have the statement of the theorem for \(X_\alpha\).

On the other hand, by introducing different variables for each \(X_\alpha\) in the above process we can get the fiber \(X(\bar{p})\) as the union of fibers \(Y_\alpha(\bar{p})\). As a \(D\)-semianalytic set is a finite union of sets of the form \(\text{Dom}_{m+1,n}B \cap V(I)_K\), this argument proves the theorem.

\[\square\]

**Remark 6.9.** We would like to note that the above proof can easily be modified to actually give another proof of Theorem A of [6]. There, the authors prove that for a subanalytic subset \(X \subset \mathbb{Z}^m\) there is a semialgebraic set \(X' \subset \mathbb{Z}^m\) such that for each \(p \in \mathbb{Z}_p\) there is a \(\bar{q} \in \mathbb{Z}_p^{m'+1}\) with \(X(\bar{p}) = X'(\bar{q})\). In their context, subanalytic is in the sense of [5] and semialgebraic is in the sense of Macintyre’s Language. By the Quantifier Elimination theorem of [5], one only needs to consider quantifier free definable subset of \(\mathbb{Z}_p^{m'+1}\), and one follows the argument of the proofs of 6.8 above and Basic Lemma (1.2) of [5] to get \(y\) – the variable which corresponds to the fiber space— to appear polynomially in each term of the formula that defines
X, at the expense of introducing new variables that corresponds to terms that do not involve y.

References


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