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Algebraic Properties of Separated Power Series

Abstract We study the commutative algebra of rings of separated power series over a ring E and that of their extensions: rings of separated (and more specifically convergent) power series from a field K with a separated E-analytic structure. Both of these collections of rings already play an important role in the model theory of non-Archimedean valued fields and we establish their algebraic properties. This will make a study of the analytic geometry over such fields through the classical methods of algebraic geometry possible.

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1 Introduction

We investigate the commutative algebraic properties of two specific types of power series rings with a view toward establishing a basis for future geometric and model theoretic investigations of non-Archimedean fields with analytic structures. These rings are the separated power series rings over an integral domain E (which is complete and separated in the topology induced by an ideal I) and extensions of those with parameters from a field K which has a separated E-analytic structure. Both of these classes of rings were introduced in [5] by Cluckers, Lipshitz and Robinson, and were used to prove important geometric and model theoretic results such as the Cell Decomposition Theorems for analytically definable sets. In general, the results in [5] can be seen as a part of a preparation of a foundation for the analytic motivic integration theory where one of the key tasks is to understand properties of sets over

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which the integration occurs. In algebraic motivic integration theory, such an understanding comes from several results bordering model theory and geometry, such as the Cell Decomposition Theorem for algebraically definable sets over non-Archimedean valued fields in [17] by Pas, which was also the initial inspiration for the main result in [5]. In particular, the importance of this theorem and other geometric results in algebraic motivic integration theory can be seen in [8] by Denef and Loeser, and [6] by Cluckers and Loeser and one expects that a study of analytically definable sets over non-Archimedean fields to enable similar results in analytic motivic integration theory.

However, cell decomposition theorems and their corollaries, such as the quantifier elimination theorems, are only a part of the necessary tools for understanding definable sets. One also needs an elaborate dimension theory as well as several topological results, and deeper results in this direction in the analytic setting above require some tools not readily available in [5]. In particular, one needs to know the existence of some of the algebraic properties that these rings share with the polynomial algebras, such as regularity or the existence of a suitable Nullstellensatz. The purpose of this paper is to establish those properties so that the familiar methods of algebraic geometry can be applied to study various types of sets defined over fields with separated E-analytic structures using the separated power series. In that purpose, but not in scale, this paper is comparable to [12] by Lipshitz and Robinson which established many algebraic properties of quasi-affinoid algebras. As for applications, a quantifier elimination theorem for the theory of algebraically closed valued fields with analytic structure, in a language which utilizes the rings of separated power series discussed here as a source of analytic functions, is reported by this author in [4] and another article on the dimension theory of certain definable sets and subassignments over non-Archimedean valued fields with analytic structures is in its final stages. The results in the latter project will generalize those in [3] by the author and in [9] by van den Dries and will be reported in an upcoming paper by this author.

Let us continue with a brief overview of the history of model theory and geometry over non-Archimedean fields. After the ground-breaking quantifier elimination theorems for the algebraic theories of various non-Archimedean valued fields by Ax and Kochen in [1] and Macintyre in [14], and for the analytic theories by Denef and van den Dries in [7] and by Lipshitz in [11], there has been a growing literature in the definable geometry and model theory of such fields. Here, one should note that given a model M of a theory T in a language $\mathcal{L}$, a subset X of M^n is called definable if there is a formula ψ of $\mathcal{L}$ such that

$$X = \{\bar{p} \in M^n : \psi(\bar{p})\},$$

and is called quantifier-free definable if there is such a formula ψ which does not contain quantifiers ($\forall$ and $\exists$). Oftentimes one works with a subclass of the class of definable sets, such as algebraic varieties, semialgebraic sets, or their analytic counterparts. The quantifier elimination theorems above guarantee that the class of quantifier-free definable sets are closed under coordinate projections.

One difficulty in the study of the analytic properties of non-Archimedean valued fields, compared to the study of algebraic ones, is coming up with the most useful notion of analyticity for the problem at hand. Because the metric topology of a non-Archimedean field is totally disconnected and because many questions concerning the analytic functions over non-Archimedean fields involve power series, one often uses a suitable power series ring as the source of analytic functions over a bounded domain as opposed to the classical definition of analyticity. This was also the case in [7] and [11]. However the problem of how one picks the ring of power series to work with is also non-trivial. An often preferred method is to construct a ring where there is a suitable version of the Weierstrass Preparation Theorem so that it is possible to reduce some questions about analytic functions to questions about polynomials. In order to guarantee that the Weierstrass Preparation Theorem holds, most of the times one requires that the coefficients of the power series in the utilized ring to belong to a ring R which is complete. On the other hand, while studying the geometry of definable sets, there are many cases where one wants to be able to substitute values from a suitable domain in for parameter variables in a power series and obtain another power series in a suitable ring where Weierstrass Preparation Theorem still holds. This is problematic if the values which are substituted do not come from the complete ring R. So one way to satisfy both of these needs, if one is only interested in the definable sets over a single complete field K, is to work with the Tate algebra $T_m(K)$ over K, which was the approach in [7]. For example, there the ring $T_m(\mathbb{Q}_p)$ was the source of analytic functions over $\mathbb{Z}_p^m$. However this approach is not helpful if one is interested in the definable geometry over non-complete non-Archimedean valued fields, especially over those whose valuations are not of rank 1. Also the above mentioned approach does not allow us to fully utilize the tools from model theory as completeness is not a property expressible in the first order languages we use.

An alternative strategy, which is an extension of the ideas initiated in [10] by van den Dries and in [13] by Lipshitz and Robinson, and which was further developed in [5], is to have two separate classes of power series rings: one in which power series which we interpret as our universal analytic functions live and another class we obtain by substituting values in for the parameter variables for working on the definable geometry of a fixed field K. Here, we follow this particular strategy and use the power series rings introduced in [5].

Definition 1.1 Let E be a Noetherian integral domain, I be an ideal of E, and let E be complete and separated in the I-adic topology. Let $(x_1, ..., x_m)$ and $(\rho_1, ..., \rho_n)$ be variables. The ring of *strictly convergent power series in x over E* is

$$T_m(E) = E\langle x\rangle := \{\sum_{\alpha\in\mathbb{N}^m} a_\alpha x^\alpha : \lim_{|\alpha|\to\infty} a_\alpha = 0\}$$

where the limit is taken in the I-adic topology. The ring of separated power series in (x, ρ) over E is the ring

$$S_{m,n}(E) := E\langle x\rangle [[\rho]] .$$

We postpone the definition of the second class of power series we mentioned earlier, the ones which will be used for geometric investigations over a fixed field K, until Section 3, but continue with observations about the rings defined above.

Remark 1.2 We will make use of the following facts without further reference.

i. E comes naturally with an order function $\mathrm{ord}_E : E \to \mathbb{N} \cup \{\infty\}$, which takes the value i at any $a \in I^i \setminus I^{i+1}$. Note that the semi-norm on E which is defined by $|a|_E := 2^{-\mathrm{ord}_E(a)}$ satisfies the ultrametric inequality. Note also that ord_E can be extended to act on $S_{m,n}(E)$ as follows: for $f = \sum_{\mu,\nu} a_{\mu,\nu} x^\mu \rho^\nu \in S_{m,n}(E)$, define $\mathrm{ord}_E(f)$ as

$$\mathrm{ord}_E(f) := \inf_{\mu,\nu} \mathrm{ord}_E(a_{\mu,\nu}) = \min_{\mu,\nu} \mathrm{ord}_E(a_{\mu,\nu}).$$

ii. If one gives $E[[\rho]]$ the topology induced by I and (ρ), then $S_{m,n}(E)$ is isomorphic to $E[[\rho]]\langle x \rangle$.
iii. $E\langle x \rangle$ is the completion of of $E[x]$ in the topology induced by $I \cdot E[x]$, $E\langle x \rangle[[\rho]]$ is the completion of $E[x,\rho]$ in the topology induced by the ideal $(I \cup \{\rho\}) \cdot E[x,\rho]$.

We are now going to explain the usefulness of the rings $S_{m,n}(E)$ as a source of analytic functions over $(K^\circ)^m \times (K^{\circ\circ})^n$ where K is a non-Archimedean valued field with a separated E-analytic structure. The first step is to explain the terminology and the notation in the preceding sentence.

Definition 1.3 A field K is called a non-Archimedean, non-trivially valued field if there is an ordered group $(\Gamma, +, <)$ and a function $\mathrm{ord}_K : K \to \Gamma \cup \{\infty\}$ such that, when $+$ and $<$ are extended to $\Gamma \cup \{\infty\}$ in an obvious way, we have

i. $\mathrm{ord}_K(x) = \infty$ if and only if $x = 0$,
ii. $\mathrm{ord}_K(xy) = \mathrm{ord}_K(x) + \mathrm{ord}_K(y)$ for all $x, y \in K$,
iii. $\mathrm{ord}_K(x + y) \geq \min\{\mathrm{ord}_K(x), \mathrm{ord}_K(y)\}$,
iv. there is an element $a \in K$ such that $\mathrm{ord}_K(a) \neq 0$.

When the last condition above is not satisfied then ord_K is said to be the trivial valuation on K. From now on we are going to use the term "non-Archimedean valued field" to mean "non-Archimedean, non-trivially valued field". Such a field K comes equipped with a topology determined by the function ord_K. That is the topology whose basis elements are given by the sets of the form

$$D_K(\bar{p}, r)^- := \{\bar{q} \in K : \mathrm{ord}_K(\bar{p} - \bar{q}) > r\},$$

where $\bar{p} \in K$ and $r \in \Gamma$. Note that in this topology both the sets above and also the sets of the form

$$D_K(\bar{p}, r)^+ := \{\bar{q} \in K : \mathrm{ord}_K(\bar{p} - \bar{q}) \geq r\},$$

are simultaneously open and closed. It is customary to have special notation for two of these sets, namely

$$K^\circ := D_K(0,0)^+, \text{ and } K^{\circ\circ} := D_K(0,0)^-.$$

We are now ready to explain the meaning of the term "separated E-analytic structure". Once again, following [5]:

Definition 1.4 Let E be as in Definition 1.1. A non-Archimedean valued field K is said to have a *separated E-analytic structure* if there is a collection of homomorphisms $\sigma_{m,n}$ from $S_{m,n}(E)$ into the ring of K°-valued functions on $(K^\circ)^m \times (K^{\circ\circ})^n$ for each m, $n \in \mathbb{N}$ such that

i. $(0) \neq I \subset \sigma_0^{-1}(K^{\circ\circ})$,
ii. $\sigma_{m,n}(x_i)$ is the i-th coordinate function on $(K^\circ)^m \times (K^{\circ\circ})^n$ for $i = 1, ..., m$ and $\sigma_{m,n}(\rho_j)$ is the $(m+j)$-th coordinate function on $(K^\circ)^m \times (K^{\circ\circ})^n$ for $j = 1, ..., n$,
iii. $\sigma_{m,n+1}$ extends $\sigma_{m,n}$, where we identify, in the obvious way, functions on $(K^\circ)^m \times (K^{\circ\circ})^n$ with functions on $(K^\circ)^m \times (K^{\circ\circ})^{n+1}$ that do not depend on the last coordinate, and $\sigma_{m+1,n}$ extends $\sigma_{m,n}$ similarly.

For example if K is a complete non-Archimedean field containing E and if I is contained in $K^{\circ\circ}$, then K is a field with separated E-analytic structure. However, we are not going to assume the completeness of fields with separated E-analytic structures. Below are some properties of separated E-analytic structures.

Remark 1.5 i. By Proposition 2.8 of [5], separated E-analytic structures preserve composition. Furthermore by Lemma 2.12 and Theorem 2.13 of [5], if E' is a finitely generated $\sigma_0(E)$-subalgebra of K° generated by $a_1, ..., a_m$, and if $E' \cap K^{\circ\circ}$ is generated by $b_1, ..., b_n$, then

$$E^{\sigma,E'} := \{\sigma_{m,n}(f)(a,b) : f \in S_{m,n}(E)\}$$

is independent of the choices for a and b. Also, $E^{\sigma,E'}$ is Noetherian complete and separated with respect to the J-adic topology where $J = \sigma_0(I) \cdot E^{\sigma,E'}$, and σ induces a unique $E^{\sigma,E'}$-analytic structure τ on K. Moreover, σ and τ are compatible and each $\tau_{m,n}$ is injective. We are going to abuse the notation and use σ instead of τ when we consider such algebras throughout in this paper.

ii. By Theorem 2.18 of [5], a separated E-analytic structure on K extends uniquely to give an analytic structure on the algebraic closure of K.

iii. By Theorem 2.17 of [5], if K is a valued field with a separated E-analytic structure for some $E \neq 0$, then K° is a Henselian valuation ring.

We end this section by briefly discussing the relation between the functions in $\sigma(S_{m,n}(E))$ and the classical definition of analyticity. Put

$$\mathcal{S}(E) := \bigcup_{m,n} \sigma(S_{m,n}(E)),$$

and let $F(x,\rho) \in \mathcal{S}(E)$, say $F = \sigma(f(x,\rho))$ for some $f \in S_{m_0,n_0}(E)$ for some m_0 and n_0. Let $x' = (x'_1, ..., x'_{m_0})$ and $\rho' = (\rho'_1, ..., \rho'_{n_0})$ be new variables. Then we have

$$F(x+x', \rho+\rho') = \sigma(f(x,\rho) + g(x,x',\rho,\rho')),$$

for some $g \in (x',\rho') \cdot S_{2m_0,2n_0}(E)$ and hence we can write

$$F(x+x', \rho+\rho') = F(x,\rho) + G(x,x',\rho,\rho')$$

where $G \in (x',\rho') \cdot \mathcal{S}(E)$. Therefore, given $\varepsilon \in \Gamma$, for all $\bar{p} \in (K^\circ)^{m_0} \times (K^{\circ\circ})^{n_0}$ and $\bar{q} \in D(\bar{p},\varepsilon)^+$ we have $(F(\bar{p}) - F(\bar{q})) \in D(0,\varepsilon)^+$. This shows the uniform continuity of the members of $\mathcal{S}(E,K)$.

Now let $\bar{p} \in (K^\circ)^{m_0} \times (K^{\circ\circ})^{n_0}$ be given and let us write $F_{(\bar{p},k)}$ for the k^{th}-Taylor polynomial of F at $\bar{p}$. Then one can argue as above to show that given $\varepsilon \in \Gamma$, for all $\bar{q} \in D(\bar{p},\varepsilon)^+$, we have $(F(\bar{q}) - F_{(\bar{p},k)}(\bar{q})) \in D(0,(n+1)\varepsilon)$. This version of Taylor's Theorem is a justification for our implicit designation of members of $\mathcal{S}(E,K)$ as "analytic functions".

2 Algebraic Properties of $S_{m,n}(E)$

In this section we discover some of the algebraic properties of the rings $S_{m,n}(E)$ which make them useful for model theoretic and geometric investigations. We start by proving the crucial property of the strong Noetherianness and hence the Noetherianness of the $S_{m,n}(E)$. Establishing this property was a key step in proving the main theorems in [7] and [11]. However as the power series rings used in [7] and [11] were quite different from what we use here, we provide a statement and a proof for the version of this property for $S_{m,n}(E)$. First we define a new valuation, ord_ρ, on the members of $S_{m,n}(E)$ as follows. We put $\text{ord}_\rho 0 = \infty$ and for any $f \in (\rho)^i \setminus (\rho)^{i+1}$ we define $\text{ord}_\rho(f) := i$.

Theorem 2.1 (Strong Noetherian Property) *Let $J \subset S_{m,n}(E)$ be an ideal, then there are $g_1, ..., g_k \in J$ such that for all $g \in J$, there are $a_1, ..., a_k \in S_{m,n}(E)$ such that*

$$g = a_1 g_1 + ... + a_k g_k,$$

and $ord_E(g) = \min\{ord_E(a_i g_i)\}_i$ and $ord_\rho(g) = \min\{ord_\rho(a_i g_i)\}_i$.

Proof We will prove the statement by treating three cases separately.
Case 1: $m = n = 0$.

Because E is separated and Noetherian, it has a natural filtration

$$E_0 := E \supset E_1 := I \supset E_2 := I^2 \supset .... \supset 0.$$

Let

$$J_0 := J \supset J_1 := I \cap J \supset J_2 := I^2 \cap J \supset ... \supset 0$$

be the induced filtration. For each $i \in \mathbb{N}$, let $g_{i1}, ..., g_{i,k_i} \in J_i$ be such that their canonical images generate J_i/J_{i+1}. Observe that by the Artin-Rees

Lemma, there is an $N \in \mathbb{N}$ such that $J_{N+i} = I^i \cdot J_N$ and therefore, for $g \in J_{N+i} \setminus J_{N+i+1}$, there are $a_1, ..., a_{k_N} \in E$ and $h_1, ..., h_{k_N} \in I^i$ such that

$$g - h_1 a_1 g_{N,1} + ... + h_{k_N} a_{k_N} g_{N,k_N} \in J_{N+i+1}.$$

Hence by completeness of E, $\{g_{i,1}, ..., g_{i,k_i}\}_{i=0}^{N}$ is a generating set which satisfies the desired property.
Case 2: $n = 0$.

In this case we induct on m. Notice that $E\langle x_1, ..., x_m\rangle$ has a natural filtration

$$E\langle x_1, ..., x_m\rangle = E_0\langle x_1, ..., x_m\rangle \supset E_1\langle x_1, ..., x_m\rangle \supset .... \supset 0$$

which is inherited from the filtration of E. Now let us write $\widetilde{g}$ for the image of a $g \in E_i\langle x_1, ..., x_m\rangle$ in $E_i\langle x_1, ..., x_m\rangle / E_{i+1}\langle x_1, ..., x_m\rangle$. Define

$$\begin{aligned} L_i := \{f \in E\langle x_1, ..., x_{m-1}\rangle :\ & \text{there exists } g \in J \text{ such that} \\ & \widetilde{g} = \widetilde{f} x_m^i + \widetilde{b}_{i-1} x_m^{i-1} + ... + \widetilde{b}_0, \\ & \qquad \text{for some } b_{i-1}, ..., b_0 \in E_i\langle x_1, ..., x_{m-1}\rangle\} \end{aligned}$$

and let J_i denote the ideal generated by L_i. It is clear that $J_1 \subset J_2 \subset ...$ and by inductive hypothesis there is an N such that $J_N = J_{N+i}$ for all i and each J_i has a finite generating set of elements consisting of members of L_i. Now let $g_{i,1}, ..., g_{i,k_i} \in J$ be such that the leading coefficients of $g_{i,1}, ..., g_{i,k_i}$ generate J_i for $i = 0, ..., N$. Then again by the inductive hypothesis, for any $g \in J$, there are $a_{1,1}, ..., a_{N,k_N} \in E\langle x_1, ..., x_{m-1}\rangle$ such that $\mathrm{ord}_E(g) \le \mathrm{ord}_E(a_{i,j} g_{i,j})$ for all i and j, and

$$g = a_{1,1} x_m^{l_{1,1}} g_{1,1} + ... + a_{i,k_i} x_m^{l_{i,k_i}} g_{i,k_i} + g',$$

for some g' with $\mathrm{ord}_E(g') > \mathrm{ord}_E(g)$. Now the result follows as the completeness of E in the I-adic topology implies the completeness of $E\langle x_1, ..., x_m\rangle$ in the I-adic topology.
Case 3: The general case.

Now we will induct on n. For $i \in \mathbb{N}$, define

$$\begin{aligned} J_i = \{f \in E\langle x\rangle [[\rho_1, ..., \rho_{n-1}]] : f\rho_n^i + h\rho_n^{i+1} \in J \\ \text{for some } h \in E\langle x\rangle [[\rho_1, ..., \rho_{n-1}]]\}. \end{aligned}$$

It is clear that each J_i is an ideal and that $... \supset J_2 \supset J_1 \supset J_0$. By the inductive hypothesis there is an $N \in \mathbb{N}$ such that $J_N = J_{N+i}$ for all i. Also by the inductive hypothesis, for each $i \in \mathbb{N}$, there are $f_{i,1}, ..., f_{i,k_i}$ generating J_i and satisfying the statement of the theorem. Now for each $0 \le i \le N$ and $1 \le j \le k_i$ let $g_{i,j} \in J$ be such that $g_{i,j} = f_{i,j}\rho_n^i + h\rho_n^{i+1}$ for some $h \in E\langle x\rangle [[\rho_1, ..., \rho_{n-1}]]$. Then the set $\{g_{i,1}, ..., g_{i,k_i}\}_{0 \le i \le N}$ satisfies the statement of the theorem. □

From the theorem above follows another important property of $S_{m,n}(E)$.

Proposition 2.2 *Let $J \subset S_{m,n}(E)$ be an ideal, then J is a closed set in the $(I \cup \{\rho\})$-adic topology of $S_{m,n}(E)$.*

Proof Let $a \in S_{m,n}(E)$ be a limit point of J. Then there is a sequence $\{a_i\}_i$ of elements of J which converges to a in the $(I \cup \{\rho\})$-adic topology. Set $b_0 := a_0$, and for $i > 0$ put $b_i = a_i - a_{i-1} \in J$, so that $\{b_i\}_i$ converge to 0 and $\{b_0 + ... + b_i\}_i$ converge to a, again in the $(I \cup \{\rho\})$-adic topology. Now let $\{g_1, ..., g_k\}$ be a generating set of J satisfying the statement of Theorem 2.1. For each i, let $c_{i1}, ..., c_{ik} \in E$ be such that

$$\begin{aligned} b_i &= c_{i1}g_1 + ... + c_{ik}g_k, \\ \mathrm{ord}_E(b_i) &= \min\{\mathrm{ord}_E(c_{ij}g_j)\}_{1\leq j\leq k}, \\ \mathrm{ord}_\rho(b_i) &= \min\{\mathrm{ord}_\rho(c_{ij}g_j)\}_{1\leq j\leq k}. \end{aligned}$$

Now notice that for each $1 \leq j \leq k$, $\{c_{ij}\}_i$ converges to 0 in the $(I \cup \{\rho\})$-adic topology. Hence by Remark 1.2 (iii), $c_j = \sum_i c_{ij} \in S_{m,n}(E)$ for all j. Furthermore, clearly we have $a = c_1g_1 + ... + c_kg_k$ and $c_jg_j \in J$ for all $1 \leq j \leq k$ and therefore we have $a \in J$. □

The proposition above easily yields the following useful fact about extension ideals.

Corollary 2.3 *Let J be an ideal of $S_{m,n}(E)$ and let z be a variable not appearing in in the presentation of $S_{m,n}(E)$, then*

$$J \cdot S_{m,n}(E)\langle z\rangle = \{\sum_{i=0}^{\infty} a_i z^i \in S_{m,n}(E)\langle z\rangle : a_i \in J \text{ for all } i\}, \text{ and}$$

$$J \cdot S_{m,n}(E)[[z]] = \{\sum_{i=0}^{\infty} a_i z^i \in S_{m,n}(E)[[z]] : a_i \in J \text{ for all } i\}.$$

Now suppose that $J \subset E$ is a prime ideal so that E/J is an integral domain. As Proposition 2.2 guarantees that J is closed in the I-adic topology of E, it is easy to see that E/J is separated in the $(I \cdot E/J)$-adic topology. Also, by Theorem 8.1 of [16] we see that E/J is complete in this topology. Hence it is possible to use the method of Definition 1.1 to construct rings of separated power series $S_{m,n}(E/J)$ over E/J. Then it is natural to ask about the relation between the rings $S_{m,n}(E)/J \cdot S_{m,n}(E)$ and $S_{m,n}(E/J)$.

Proposition 2.4 *Let J and $S_{m,n}(E/J)$ be as above, then*

$$S_{m,n}(E)/J \cdot S_{m,n}(E) \simeq S_{m,n}(E/J).$$

Proof Clearly it is enough to show that $E\langle x\rangle / J \cdot E\langle x\rangle \simeq (E/J)\langle x\rangle$. Observe that by Theorem 8.1 of [16] and Proposition 2.2 we have

$$E\langle x\rangle / J \cdot E\langle x\rangle \simeq (E[x]/J \cdot E[x])^\frown,$$

where $^\frown$ denotes the completion with respect to the quotient topology of the $I \cdot E[x]$-adic topology. This is the same as the topology induced by the ideal $I \cdot (E[x]/J \cdot E[x])$. But $E[x]/J \cdot E[x] \simeq (E/J)[x]$ and $((E/J)[x])^\frown \simeq (E/J)\langle x\rangle$ where $^\frown$ denotes the completion with respect to $I \cdot (E/J)[x]$, and hence the statement of the proposition. □

Next we turn our attention to a flatness result. Working with flat extensions is often very advantageous and the result below is a step in establishing one of our main results in Theorem 2.6.

Proposition 2.5 *Let $m' \leq m$, and $n' \leq n$, then the natural inclusion $S_{m',n'}(E) \to S_{m,n}(E)$ is faithfully flat.*

Proof Because $S_{m,n}(E) = S_{m',n'} \langle x_{m'+1}, ..., x_m \rangle [[\rho_{n'+1}, ..., \rho_n]]$, it is enough to show that the inclusion $E \to S_{m,n}(E)$ is faithfully flat. On the other hand, because the inclusion $E\langle x \rangle \to E\langle x \rangle [[\rho]]$ is faithfully flat, by induction we only need to show that

$$E \to E\langle x \rangle$$

is faithfully flat in the case x is a single variable. Also, because for any maximal ideal $\mathfrak{m}$ of E, $\mathfrak{m} \cdot E\langle x \rangle \neq (1)$, by Theorem 7.2 of [16], it is enough to prove the flatness of the above map to have faithful flatness. For proving that, we start by noting that by completeness of $E\langle x \rangle$ in the I-adic topology, for any $a \in I \cdot E\langle x \rangle$, $1 + a$ is a unit of $E\langle x \rangle$ and so $I \cdot E\langle x \rangle$ is contained in the nilradical of $E\langle x \rangle$. Therefore, in the terminology of [16], $I \cdot E\langle x \rangle$ is I-adically ideal separated as an E-module and hence, by Theorem 22.3 of [16], it is enough to check that the canonical map $E/I^r \to E\langle x \rangle / (I \cdot E\langle x \rangle)^r$ is flat for each r. But this is clear as $E\langle x \rangle / (I \cdot E\langle x \rangle)^r$ is a free module over E/I^r. □

In addition to being an important ingredient of algebraic investigations, Krull dimension is an important tool in model theory as it provides an instrument of induction. Hence we consider the Theorem 2.6 below as one of the main results of this paper. Also, note that as a pair, Theorem 2.6 and Proposition 3.9 form a potent tool in the study of the geometry of definable sets.

Theorem 2.6 *Let d denote the Krull dimension of E, then $S_{m,n}(E)$ has Krull dimension $d + m + n$.*

Proof If the Krull dimension of $S_{m,0}(E)$ is e, then by Theorem 15.4 of [16], the Krull dimension of $S_{m,n}(E)$ is $e + n$. Therefore, by induction, we only need to show that the Krull dimension of $E\langle x \rangle$ is $d + 1$, when x is a single variable.

For this, let $\mathfrak{m}$ be a maximal ideal of $E\langle x \rangle$ and note that $I \cdot E\langle x \rangle \subset \mathfrak{m}$, as for any $a \in I \cdot E\langle x \rangle$, $1 + a$ is a unit of $E\langle x \rangle$. Define

$$S := \{a_r x^r + ... + a_0 \in E[x] \cap \mathfrak{m} : a_i \notin \mathfrak{m} \cap E \text{ for all } i\}.$$

We claim that for any $f \in \mathfrak{m}$, there are $f_1 \in S$ and $f_2 \in (\mathfrak{m} \cap E) \cdot E\langle x \rangle$ such that $f = f_1 + f_2$. This is easy to see, because writing $f = \sum_i a_i x^i$, and setting

$$f_1 := \sum_{a_i \notin \mathfrak{m} \cap E} a_i x^i, \text{ and } f_2 := \sum_{a_i \in \mathfrak{m} \cap E} a_i x^i,$$

we immediately have that $f_1 \in S$, as $I \cdot E\langle x \rangle \subset \mathfrak{m}$, and $f_2 \in (\mathfrak{m} \cap E) \cdot E\langle x \rangle$ by Corollary 2.3.

Next we observe that $(\mathfrak{m}\cap E)\cdot E\langle x\rangle \neq \mathfrak{m}$, as $((\mathfrak{m}\cap E)\cup\{x\})\cdot E\langle x\rangle$ is a non-unit ideal strictly containing $(\mathfrak{m}\cap E)\cdot E\langle x\rangle$, and thus $S\neq\emptyset$. Now, let $g\in S$ be of lowest degree; we claim that g generates $\mathfrak{m}\cdot(E\langle x\rangle/(E\cap\mathfrak{m})\cdot E\langle x\rangle)_{\mathfrak{m}}$. To see this, let $h\in\mathfrak{m}$ and write $h=h_1+h_2$ where $h_1\in S$ and $h_2\in(\mathfrak{m}\cap E)\cdot E\langle x\rangle$. Then by the division algorithm it is clear that there are $u,h_3\in E_{\mathfrak{m}\cap E}[x]$, and an $h_4\in(\mathfrak{m}\cap E)\cdot E_{\mathfrak{m}\cap E}[x]$, such that $h_1-ug=h_3+h_4$. But if it is the case that $h_3\neq 0$, then there is an $h_3'\in S$ which is of lower degree than g. This contradicts with g being of minimal degree and the claim follows. Now as a corollary to the claim, we easily see that $(E\langle x\rangle/(E\cap\mathfrak{m})\cdot E\langle x\rangle)_{\mathfrak{m}}$ has Krull dimension 1. Now by Theorem 15.1 of [16], the height of the ideal $\mathfrak{m}$ is one more than the height of the ideal $\mathfrak{m}\cap E$ and therefore the dimension of $E\langle x\rangle$ is at most $d+1$. On the other hand, for any maximal ideal $\mathfrak{n}$ of E, $\mathfrak{n}\cup\{x\}$ is a non-unit ideal of $E\langle x\rangle$ which is of height at least $d+1$ by the Going Down Theorem for flat extensions (Theorem 9.5 of [16]) and the assertion follows. □

As is the case with most of the power series rings one considers in model theory of valued fields, having suitable Weierstrass Division and Preparation Theorems for the rings $S_{m,n}(E)$ is essential. Before we discuss these theorems, we recall a definition from [5].

Definition 2.7 Let $f\in S_{m,n}(E)$, then f is said to be *regular in x_m of degree d* if f is congruent modulo $(I\cup\{\rho\})\cdot S_{m,n}(E)$ to a monic polynomial in x_m of degree d. Similarly f is said to be *regular in ρ_n of degree d* if f is congruent modulo $I\cup\{\rho_1,...,\rho_{n-1}\}$ to $\rho_n^d g(x,\rho)$ for some unit g of $S_{m,n}(E)$.

Next we continue with the Weierstrass Division and Preparation Theorems themselves. Both of these theorems are from [5] and we repeat their full statements here as they are central to our later discussions. We are especially interested in extending these theorems to a more general class of power series with a more general sense of regularity in the next section. The proof of the division theorem below follows from a standard inductive argument making use of the completeness property stated in Remark 1.2.

Theorem 2.8 (Weierstrass Division) *Let $f,g\in S_{m,n}(E)$.*

i. Suppose f is regular in x_m of degree d. Then there exist uniquely determined elements $q\in S_{m,n}(E)$ and $r\in S_{m-1,n}(E)[x_m]$ of degree at most $d-1$ such that $g=qf+r$. If $g\in J\cdot S_{m,n}(E)$ for some ideal J of $S_{m-1,n}$, then $q,r\in J\cdot S_{m,n}(E)$.

ii. Suppose f is regular in ρ_n of degree d. Then there exist uniquely determined elements $q\in S_{m,n}(E)$ and $r\in S_{m,n-1}(E)[\rho_n]$ of degree at most $d-1$ such that $g=qf+r$. If $g\in J\cdot S_{m,n}(E)$ for some ideal J of $S_{m,n-1}$, then $q,r\in J\cdot S_{m,n}(E)$.

We immediately get the following theorem by setting $g=x_m^d$ and applying the above theorem.

Theorem 2.9 (Weierstrass Preparation) *Let $f\in S_{m,n}(E)$.*

i. *If f is regular in x_m of degree d, then there exist a unique unit u of $S_{m,n}(E)$ and a unique monic polynomial $P \in S_{m-1,n}[x_m]$ of degree d such that $f = uP$.*
ii. *If f is regular in ρ_n of degree d, then there exist a unique unit u of $S_{m,n}(E)$ and a unique monic polynomial $P \in S_{m,n-1}[\rho_n]$ of degree d such that $f = uP$, moreover P is regular in ρ_n of degree d.*

Let $f = \sum a_{\alpha\beta}(x,\rho)y^\alpha\lambda^\beta \in S_{m+M,n+N}(E)$, where $x = (x_1, ..., x_m)$, $\rho = (\rho_1, ..., \rho_n)$, $y = (y_1, ..., y_M)$ and $\lambda = (\lambda_1, ..., \lambda_N)$. In geometric and model theoretic investigations, one often needs to consider the behavior of f when the parameter variables x and ρ are specialized at a point $\bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n$. We finish this section by establishing the existence of a finite set of candidates for the "dominant" coefficient of $f(\bar{p}, y, \lambda)$ in terms of ord_K. Similar results were the key steps in [7] and [11] where one applies the Weierstrass Division and Preparation Theorems to power series with parameters. In addition to establishing this theorem for the first time for $S_{m,n}(E)$, the version below improves on the earlier versions as it guarantees the existence of a candidate set to work uniformly over any field K with an E-analytic structure.

Proposition 2.10 *Let f be above, then there is a finite set $\mathcal{Z} \subset \mathbb{N}^M \times \mathbb{N}^N$ such that for each non-Archimedean valued field K with a separated E-analytic structure σ and for each point $\bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n$, either we have $\bar{p} \in V(\{a_{\alpha,\beta}(x,\rho)\}_{\alpha,\beta})_K$ or there is an $(\alpha_0, \beta_0) \in \mathcal{Z}$ such that*

$$\begin{aligned}
&ord_K(\sigma(a_{\alpha_0\beta_0})(\bar{p})) \leq ord_K(\sigma(a_{\alpha\beta})(\bar{p})), \text{ for all } \alpha, \beta;\\
&ord_K(\sigma(a_{\alpha_0\beta_0})(\bar{p})) < ord_K(\sigma(a_{\alpha\beta})(\bar{p})), \text{ for all } \beta < \beta_0 \text{ and all } \alpha;\\
&ord_K(\sigma(a_{\alpha_0\beta_0})(\bar{p})) < ord_K(\sigma(a_{\alpha\beta_0})(\bar{p})), \text{ for all } \alpha > \alpha_0,
\end{aligned}$$

with respect to the lexicographical ordering of $\mathbb{N}^m$ and $\mathbb{N}^n$.

Proof We start by observing that $S_{m,n}(E)$ is a Noetherian ring which is complete and separated with respect to the $I' := (I \cup \{(\rho)\}) \cdot S_{m,n}(E)$-adic topology, and that $S_{m+M,n+N}(E) = S_{M,N}(S_{m,n}(E))$. Let $g_1, ..., g_k$ be as in Theorem 2.1 for the ideal $J := (\{a_{\alpha,\beta}\}_{(\alpha,\beta)\in\mathbb{N}^M\times\mathbb{N}^N}) \cdot S_{m,n}(E)$, and let the indices $(\alpha_1, \beta_1), ..., (\alpha_r, \beta_r) \in \mathbb{N}^M \times \mathbb{N}^N$ be such that for all $i = 1, ..., k$, there exist $c_{i,1}, ..., c_{i,r} \in S_{m,n}(E)$ such that

$$g_i = c_{i,1}a_{\alpha_1,\beta_1} + ... + c_{i,r}a_{\alpha_r,\beta_r}.$$

Then $\{a_{\alpha_i,\beta_i}\}_{i=1}^r$ generate J, and by the ultrametric inequality, for any non-Archimedean valued field K with separated E-analytic structure σ and any point $\bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n$, there is an $i_0 \in \{1, ..., r\}$ such that

$$\mathrm{ord}_K(\sigma(a_{\alpha_{i_0},\beta_{i_0}})(\bar{p})) \leq \mathrm{ord}_K(\sigma(a_{\alpha,\beta})(\bar{p})), \tag{2.1}$$

for all α and β. Put

$$\mathcal{Z}_1 := \{\beta \in \mathbb{N}^N : |\beta| \leq \max\{|\beta_i|\}_{i\in\{1,...,r\}}\},$$

and for each $\beta \in \mathcal{Z}_1$, let $\gamma_\beta \in \mathbb{N}$ be such that $|\alpha| > \gamma_\beta$ implies

$$\mathrm{ord}_{S_{m,n}(E)}(g_i) < \mathrm{ord}_{S_{m,n}(E)}(a_{\alpha,\beta}),$$

for all $i \in \{1, ..., k\}$. We claim that the index set

$$\mathcal{Z} := \{(\alpha, \beta) : \beta \in \mathcal{Z}_1, |\alpha| < \gamma_\beta\}$$

satisfies the statement of the lemma.

To see this, we first fix a field K as in the statement. It is clear that for any $\beta \in \mathcal{Z}_1$ and $|\alpha| > \gamma_\beta$ one has

$$a_{\alpha,\beta_j} = d_1 a_{\alpha_1,\beta_1} + ... + d_r a_{\alpha_r,\beta_r},$$

where $\mathrm{ord}_{S_{m,n}(E)}(d_i) > 0$. Then, by the ultrametric inequality, for $\bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n$ we see that if $\bar{p} \notin V(\{a_{\alpha,\beta}(x,\rho)\}_{\alpha,\beta})_K$, then

$$\mathrm{ord}_K(\sigma(a_{\alpha_i,\beta_i})(\bar{p})) < \mathrm{ord}_K(\sigma(a_{\alpha,\beta})(\bar{p})), \tag{2.2}$$

for all i. Next we choose β_0 to be the lexicographically the first element of $\mathcal{Z}_1$ such that there is an α_0, such that

$$\mathrm{ord}_K(\sigma(a_{\alpha_0,\beta_0})(\bar{p})) \leq \mathrm{ord}_K(\sigma(a_{\alpha,\beta})(\bar{p})), \tag{2.3}$$

for all α and β. Note that such a β_0 exists by Inequality (2.1), and by Inequality (2.2) such an α_0 necessarily satisfies $|\alpha_0| < \gamma_0$. Then we choose α_0 to be the lexicographically the last α which makes Inequality (2.3) true for all α and β, and see that the conditions in the statement of the proposition are all satisfied. □

3 Power Series with Parameters from *K*

Although the rings $S_{m,n}(E)$ exhibit very nice algebraic properties, for purposes of geometric investigations it is essential to use a larger class of power series. This is especially so if one needs results like the Nullstellensatz (Theorem 3.8) and Noether Normalization (Corollary 3.7). For this, we fix a non-Archimedean valued field K with a separated E-analytic structure and study rings of separated and strictly convergent power series with parameters from K. The definition of our main objects of study once again come from [5].

Definition 3.1 Let K be as above and let us write $\mathcal{F}(\sigma, K)$ to denote the collection of all finitely generated E-subalgebras of K°. Using the notation of Remark 1.5, define

$$S^\circ_{m,n}(\sigma, K) := \varinjlim_{E^{\sigma,E'} \in \mathcal{F}(\sigma,K)} E^{\sigma,E'} \langle x \rangle [[\rho]] .$$

Then the *rings of separated power series with parameters from* K are defined to be

$$S_{m,n}(\sigma, K) := K \otimes S^\circ_{m,n}(\sigma, K).$$

We will call a ring of the form $S_{m,0}(\sigma, K)$ a *ring of strictly convergent power series with parameters from* K, and use the more customary notation $T_m(\sigma, K)$ to denote it. Similarly $T^\circ_m(\sigma, K)$ will denote $S^\circ_{m,0}(\sigma, K)$. Moreover, for $\varepsilon \in K$, we will write $T_{m,\varepsilon}(\sigma, K)$ and $T^\circ_{m,\varepsilon}(\sigma, K)$ for the respective images of $T_m(\sigma, K)$ and $T^\circ_m(\sigma, K)$ under the map

$$\varphi_\varepsilon : K\,[[x]] \to K\,[[x]]\,; \sum_\alpha a_\alpha x^\alpha \to \sum_\alpha \frac{a_\alpha}{\varepsilon^{|\alpha|}} x^\alpha.$$

Although we are not going to be working with the rings $T_{m,\varepsilon}(\sigma, K)$ in this paper, they are included in the above definition as they are indispensable for investigations of local properties of definable sets. A few easy observations are in order.

Remark 3.2 i. Note that $\ker(\varphi_\varepsilon) = 0$, and for any $\varepsilon \in K^{\circ\circ}$, $T_{m+n,\varepsilon}(\sigma, K)$ contains $S_{m,n}(\sigma, K)$.
ii. Given an $f \in T_{m,\varepsilon}(\sigma, K)$, we can naturally treat f as an analytic function over $D^+_K(0, \mathrm{ord}_K\varepsilon)^m$ by the following conventional interpretation: given $\bar{p} \in (D^+_K(0, \mathrm{ord}_K\varepsilon))^m$, define $f(\bar{p}) = \sigma(\varphi_\varepsilon^{-1}(f))(\frac{1}{\varepsilon}\bar{p})$.

Our first task is to establish the Weierstrass Division and Preparation Theorems for $S_{m,n}(\sigma, K)$. For this, we start by defining a sense of regularity which is more useful when working with the members of $S_{m,n}(\sigma, K)$.

Definition 3.3 Let $f(x) \in S_{m,n}(\sigma, K)$. We say that f is *regular of degree d in x_m (or ρ_n)* if there are $a_1, ..., a_M \in K^\circ$, $b_1, ..., b_N \in K^{\circ\circ}$, a $c \in K$, an $E' \in \mathcal{F}(\sigma, K)$ and an $F(x, y, \rho, \lambda) \in S_{m+M,n+N}(E^{\sigma,E'})$ such that F is regular of degree d in x_m (or ρ_n) and

$$cf = F(x, \rho, a_1, .., a_M, b_1, ..., b_N).$$

Using the above definition we arrive at the following generalization of Theorems 2.8 and 2.9.

Theorem 3.4 (Weierstrass Division and Preparation for $S_{m,n}(\sigma, K)$) *Theorems 2.8 and 2.9 still hold if one replaces $S_{m,n}(E)$ by $S_{m,n}(\sigma, K)$ and uses the notion of regularity in Definition 3.3.*

Proof We are just going to show the Weierstrass Division Theorem assuming f is regular of degree d in x_m; the case when f is regular in ρ_n is similar. This is going to imply that any $f \in S_{m,n}(\sigma, K)$ which is regular in any variable of degree 0 is a unit and then the preparation theorem follows from the division theorem through a standard argument.

Let $E' \in \mathcal{F}(\sigma, K)$, $c \in K$, $F \in S_{m+M,n+N}(E^{\sigma,E'})$, $a = (a_1, ..., a_M)$ and $b = (b_1, ..., b_N)$ be such that $cf = F(x, \rho, a, b)$, and let $e \in K$ be such that $eg \in S^\circ_{m+M,n+N}(\sigma, K)$. We may, for some $G(x, \rho, y, \lambda) \in S_{m+M,n+N}(E^{\sigma,E'})$, assume that $eg(x, \rho) = G(x, \rho, a, b)$. Let $E'' \in \mathcal{F}(\sigma, K)$ contain a_i and b_j for all i and j. Then by Theorem 2.8, there exists a $q \in S_{m+M,n+N}(E^{\sigma,E''})$ and a polynomial $r \in S_{m-1+M,n+N}(E^{\sigma,E''})[x_m]$ of degree at most $d-1$ such that

$$G(x, \rho, y, \lambda) = h(x, \rho, y, \lambda)F(x, \rho, y, \lambda) + r(x, \rho, y, \lambda).$$

From this, by substituting a and b for y and λ, and dividing by e, we get

$$g(x, \rho) = \frac{c}{e}h(x, \rho, a, b)f(x, \rho) + \frac{1}{e}r(x, \rho, a, b).$$

Now the theorem follows as $\frac{c}{e}h(x,\rho,a,b)$ and $\frac{1}{e}r(x,\rho,a,b)$ are clearly members of $S_{m+M,n+N}(\sigma,K)$. □

Weierstrass Division and Preparation Theorems yield many algebraic properties of $T_m(\sigma,K)$, but first we need a lemma.

Lemma 3.5 *For any non-zero $f\in T_m(\sigma,K)$, there exist a constant $c\in K$ and an automorphism ϕ of $T_m(\sigma,K)$ such that $\phi(cf)$ is regular in x_m.*

Proof Let $c\in K$ be such that $cf\in T_m(E^{\sigma,E'})$ for some $E'\in\mathcal{F}(\sigma,K)$, and write $cf=\sum_\alpha a_\alpha x^\alpha$. Let $g_1,...,g_k$ be the generators for the ideal $(\{a_\alpha\}_{\alpha\in\mathbb{N}^m})\subset E^{\sigma,E'}$ satisfying the statement of Theorem 2.1. Also, let $a_{\alpha_1},...,a_{\alpha_r}\in E^{\sigma,E'}$ be such that for each $i\in\{1,...,k\}$ there are $c_{i,1},...,c_{i,r}\in E^{\sigma,E'}$, such that

$$g_i=c_{i,1}a_{\alpha_1}+...+c_{i,r}a_{\alpha_r}.$$

Observe that by Proposition 2.10 there is an index $\alpha_0\in\mathbb{N}^m$ such that

$$\begin{aligned}&\mathrm{ord}_K(\sigma(a_{\alpha_0}))\le\mathrm{ord}_K(\sigma(a_\alpha))\text{ for all }\alpha,\text{ and}\\&\mathrm{ord}_K(\sigma(a_{\alpha_0}))<\mathrm{ord}_K(\sigma(a_\alpha))\text{ for all }\alpha>\alpha_0.\end{aligned}$$

Next, define

$$\begin{aligned}\mathcal{Z}_1&:=\{\alpha_i:\mathrm{ord}_K(\sigma(a_{\alpha_i}))=\mathrm{ord}_K\sigma(a_{\alpha_0}),1\le i\le r\},\\\mathcal{Z}_2&:=\{\alpha_i:\mathrm{ord}_K(\sigma(a_{\alpha_i}))>\mathrm{ord}_K\sigma(a_{\alpha_0}),1\le i\le r\},\end{aligned}$$

and note that $\alpha_i<\alpha_0$ for all $\alpha_i\in\mathcal{Z}_1$ lexicographically. Note also that, after a Weierstrass automorphism of $T_m(E^{\sigma,E'})$ given by

$$\phi:x_m\to x_m,x_i\to x_i+x_m^s\text{ for }i\ne m,$$

for a suitable s, we may assume that $\alpha_0=(0,...,0,d)$ for some d.

Now by our assumption on the set $\{a_{\alpha_1},...,a_{\alpha_r}\}$, for all $\alpha\in\mathbb{N}^m$, there are $c_{\alpha,1},...,c_{\alpha,r}$ such that

$$a_\alpha=\sum_{\alpha_i\in\mathcal{Z}_1}c_{\alpha,i}a_{\alpha_i}+\sum_{\alpha_i\in\mathcal{Z}_2}c_{\alpha,i}a_{\alpha_i},$$

and $\lim_{|\alpha_i|\to\infty}c_{\alpha,i}=0$ for all i. Put

$$F(x,y,\rho):=\sum_{\alpha\ne(0,...,0,d)}\left(\sum_{\alpha_i\in\mathcal{Z}_1}c_{\alpha,i}y_i+\sum_{\alpha_i\in\mathcal{Z}_2}c_{\alpha,i}\rho_i\right)x^\alpha+x_m^d,$$

and observe that

$$\frac{c}{a_{\alpha_0}}f(x)=F\left(x,\left\{\frac{\sigma(a_{\alpha_i})}{\sigma(a_{\alpha_0})}\right\}_{\alpha_i\in\mathcal{Z}_1},\left\{\frac{\sigma(a_{\alpha_i})}{\sigma(a_{\alpha_0})}\right\}_{\alpha_i\in\mathcal{Z}_2}\right),$$

and also that $F(x,y,\rho)$ is regular in x_m of degree d as desired.

From the above lemma and Theorem 3.4 the following easily follows.

Proposition 3.6 $T_m(\sigma, K)$ *is Noetherian.*

Proof The proof is by induction on m. Assume that $T_{m-1}(\sigma, K)$ is Noetherian and let J be an ideal of $T_m(\sigma, K)$. By Lemma 3.5, we may assume that there is an $f \in J$ which is regular of degree d in x_m.

For $i \geq 0$, define

$$J_i := \{a \in T_{m-1}(\sigma, K) : a \text{ is the leading coefficient of some } g \in J \cap T_{m-1}(\sigma, K)[x_m] \text{ of degree } i\}.$$

Note that each J_i is an ideal and, by Theorem 3.4, $J_i = (1)$ for $i \geq d$. By the inductive hypothesis there exist $g_{i,1}, ..., g_{i,k_i} \in J \cap T_{m-1}(\sigma, K)[x_m]$ such that leading coefficients of $g_{i,1}, ..., g_{i,k_i}$ generate J_i for $i < d$. Then, by Theorem 3.4, $\{f\} \cup \{g_{i,1}, ..., g_{i,k_i}\}_{i=0,...,d-1}$ is a generating set for J. □

An immediate and important corollary to Lemma 3.5 is the following version of the Noether Normalization Lemma for $T_m(\sigma, K)$.

Corollary 3.7 (Noether Normalization) *Let J be an ideal of $T_m(\sigma, K)$, then there is an integer $m' \leq m$ and a Weierstrass change of variables ϕ of $T_m(\sigma, K)$ such that $T_{m'}(\sigma, K) \to T_m(\sigma, K)/\phi(J)$ is finite.*

Of course a suitable version of the Nullstellensatz is also essential for any geometric investigation. That is one of the most important reasons for working with $T_m(\sigma, K)$ as opposed to $S_{m,n}(E)$.

Theorem 3.8 (Nullstellensatz) *Let K_{alg} denote the algebraic closure of K with the valuation extending that of K, then the following hold.*

i. For any maximal ideal $\mathfrak{m}$ of $T_m(\sigma, K)$, $T_m(\sigma, K)/\mathfrak{m}$ is an algebraic extension of K.
ii. For any maximal ideal $\mathfrak{m}$ of $T_m(\sigma, K)$ there is a point $\bar{p} \in (K_{alg}^\circ)^m$ such that $\mathfrak{m} = \mathfrak{m}_{\bar{p}} \cdot T_m(\sigma, K)$ where $\mathfrak{m}_{\bar{p}}$ is the maximal ideal of $K[x]$ corresponding to $\bar{p}$.
iii. For any proper ideal $J \subset T_m(\sigma, K)$,

$$\sqrt{J} = \bigcap \{\mathfrak{m} : \mathfrak{m} \textit{ is a maximal ideal of } T_m(\sigma, K)\}.$$

Proof Let $\mathfrak{m}$ be a maximal ideal of $T_m(\sigma, K)$. By Lemma 3.5, we may assume that there is an $f \in \mathfrak{m}$ such that f is regular in x_m, in which case by Theorem 3.4 one has a finite inclusion

$$T_{m-1}(\sigma, K)/(\mathfrak{m} \cap T_{m-1}(\sigma, K)) \to T_m(\sigma, K)/\mathfrak{m}.$$

Notice that by the Going-up Theorem for integral extensions (Theorem 9.4 of [16]), $\mathfrak{m} \cap T_{m-1}(\sigma, K)$ is maximal, and by induction the first assertion follows. The proof of the second assertion is exactly as in the proof of Theorem 4.1.1 (iii) of [12].

To prove the third assertion, we will assume that the statement holds for $T_{m-1}(\sigma, K)$ and argue inductively. First, observe that by the second assertion of the theorem, the intersection of all maximal ideals of $T_m(\sigma, K)$ is the zero ideal. Define

$$W := \{\text{monic regular polynomials in } T_{m-1}(\sigma, K)[x_m]\},$$

and observe also that by Theorem 3.4 and Lemma 3.5, W satisfies the conditions in Definition 5.2.5.1 of [2]. Therefore $T_m(\sigma, K)$ is Rückert over $T_{m-1}(\sigma, K)$. But then, By Proposition 5.2.5.3 of [2], the third assertion is satisfied for all non-zero ideals of $T_m(\sigma, K)$, and by the above observation and by induction the theorem follows. □

Proposition 3.9 *For any maximal ideal* $\mathfrak{m}$ *of* $T_m(\sigma, K)$*,* $(T_m(\sigma, K))_{\mathfrak{m}}$ *is a regular local ring of dimension* m*.*

Proof Let $\mathfrak{m}$ be as above. Then by Theorem 3.8, we see that $\mathfrak{n} := \mathfrak{m} \cap K[x]$ is also maximal and that $\mathfrak{n} \cdot T_m(\sigma, K) = \mathfrak{m}$. Therefore for each $r \in \mathbb{N}^+$ we have

$$K[x]/\mathfrak{n}^r \simeq T_m(\sigma, K)/\mathfrak{m}^r.$$

But, because each $K[x]/\mathfrak{n}^r$ is local, we have $(K[x]_{\mathfrak{n}})^\frown \simeq (T_m(\sigma, K)_{\mathfrak{m}})^\frown$, where $^\frown$ denotes the completion with respect to the maximal ideal of a local ring. Note that $(K[x]_{\mathfrak{n}})^\frown$ is a local ring which is faithfully flat over $K[x]_{\mathfrak{n}}$. Notice also that the maximal ideal $\mathfrak{n} \cdot (K[x]_{\mathfrak{n}})^\frown$ of $(K[x]_{\mathfrak{n}})^\frown$ is generated by at most m elements. Hence, by the Going Down Theorem for flat extensions (Theorem 9.5 of [16]) and the Principal Ideal Theorem (Theorem 13.5 of [16]), we see that the Krull dimension of $(K[x]_{\mathfrak{n}})^\frown$ is m. Now the assertion follows by Theorem 19.3 of [16]. □

Next we state a series of properties of the rings $T_m(\sigma, K)$ which easily follow from the regularity result above. The first one below is an immediate corollary of Theorem 17.8 of [16].

Corollary 3.10 $T_m(\sigma, K)$ *is a Cohen-Macaulay ring.*

In addition, from Theorem 17.9 of [16] follows:

Corollary 3.11 $T_m(\sigma, K)$ *and all its quotients are universally catenary.*

With a little work we also get:

Proposition 3.12 $T_m(\sigma, K)$ *is a UFD.*

Proof By Theorem 20.1 of [16], we only need to show that every height one prime ideal of $T_m(\sigma, K)$ is principal. Let $\mathfrak{p}$ be such a prime, then $T_m(\sigma, K)_{\mathfrak{p}}$ has Krull dimension 1. Now we are done by Proposition 3.9 which guarantees that $\mathfrak{p}_{\mathfrak{p}}$ is principal. □

Finally, by combining Theorem 3.8, Proposition 3.9 and Theorem 2.7 of [15] by Matsumura we obtain:

Proposition 3.13 *If* Char K=0*, then* $T_m(\sigma, K)$ *is an excellent ring.*

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