
Quantifier elimination for the theory of algebraically closed valued fields with analytic structure

Yalın Firat Çelikler*¹

¹ Department of Mathematics
East Carolina University
Greenville, NC 27858, USA

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The theory of algebraically closed non-Archimedean valued fields is proved to eliminate quantifiers in an analytic language similar to the one used by Cluckers, Lipshitz and Robinson. The proof makes use of a uniform parameterized normalization theorem which is also proved in this paper and which has far reaching consequences in the geometry of definable sets. This method of proving quantifier elimination in an analytic language does not require the algebraic quantifier elimination theorem of Weispfenning unlike the customary method of proof used in similar earlier analytic quantifier elimination theorems.

1 Introduction

In this paper we consider the theory of algebraically closed non-Archimedean valued fields in the language $\mathcal{L}_{\text{sep}}^D(E)$, which is the language introduced by Cluckers, Lipshitz and Robinson in [4] minus the angular component functions and show that it admits elimination of quantifiers. This language is an analytic language in the sense that its terms are built up from power series, just as the language in [7] by Lipshitz or the one in [5] by Denef and van den Dries. The power series used in $\mathcal{L}_{\text{sep}}^D(E)$ come from the ring of separated power series $S_{m,n}(E)$ which was introduced in [4] and have coefficients in a Noetherian integral domain E which is complete and separated with respect to the I -adic topology on it. There are important differences in our setting and the one in [7], as there rings of separated power series based on quasi-Noetherian algebras were used and only the complete valued fields were considered. Here, as it was done in [4], instead of complete fields we consider valued fields with an analytic E -structure. These are valued fields K for which there exists a certain collection of homomorphisms from the rings $S_{m,n}(E)$ into the ring of K° -valued functions over $(K^\circ)^m \times (K^{\circ\circ})^n$. As one no longer assumes completeness of the valued fields, which is not a first-order definable property, it is possible to use basic model theoretic tools in further investigations in our setting. Another advantage in this approach is the potential for applications in analytic motivic integration of the power series and the language used here as explained in [4]. Note that similar approaches were first used by van den Dries in [6] and later by Lipshitz and Robinson in [9], and was also recently utilized in [11] by Scanlon.

We use a somewhat unusual method of proving quantifier elimination in this paper. There are a number of well known analytic quantifier elimination theorems for non-Archimedean valued fields in the literature and most of them were proved by utilizing a parameterized version of Weierstrass Preparation Theorem and an induction on some sort of a "rank" of a formula. This way, one reduces the problem to the problem of quantifier elimination in algebraic languages and using results like the quantifier elimination theorem by Weispfenning in [12] for algebraically closed valued fields completes the proof. Here we avoid using an algebraic quantifier elimination theorem and instead prove and use Uniform Parameterized Normalization Theorem (Theorem 2.4) for $\mathcal{L}_{\text{sep}}^D(E)$ -quantifier-free definable sets. This theorem is a complete analogue of Lemma 5.3 of [3] by the author, and it is a quite powerful tool for investigating geometric properties of definable sets even if the considered valued field is not algebraically closed. This is especially true for geometric properties which are related to projections of quantifier-free definable sets and in fact, as the images of those projections are exactly the existentially definable

* e-mail: celiklery@ecu.edu

sets, parameterized normalization yields a very geometry oriented proof for quantifier elimination. That is in accordance with our general point of view that quantifier elimination is a part of the investigation of the geometric properties of various sorts of definable sets and parameterized normalization is a key step in further efforts in this direction. Applications of parameterized normalization of [3] in [2] nicely exemplifies this view and generalizing the results there to the setting in this paper is an ongoing project of the author.

We now give the basic definitions of our study. At the center is the non-Archimedean (non-trivially) valued fields with an E -analytic structure. First we recall:

Definition 1.1 A field K is called a non-Archimedean, (non-trivially) valued field if there is an ordered group $(\Gamma, +, <)$ and a function $\text{ord} : K \rightarrow \Gamma \cup \{\infty\}$ such that when $+$ and $<$ are extended to $\Gamma \cup \{\infty\}$ in an obvious way we have

- i. $\text{ord}(x) = \infty$ if and only if $x = 0$,
- ii. $\text{ord}(xy) = \text{ord}(x) + \text{ord}(y)$ for all $x, y \in K$,
- iii. $\text{ord}(x + y) \geq \min\{\text{ord}(x), \text{ord}(y)\}$,
- iv. there is an element $a \in K$ such that $\text{ord}(a) \neq 0$.

For such a field K ,

$$K^\circ := \{\bar{p} \in K : \text{ord}(\bar{p}) \geq 0\},$$

is called the valuation ring of K and $K^{\circ\circ}$ denotes the maximal ideal of K° , i.e.

$$K^{\circ\circ} = \{\bar{p} \in K : \text{ord}(\bar{p}) > 0\}.$$

Among the non-Archimedean valued fields, we are interested in those which have an analytic E -structure. In order to be able to give a definition for those, we first need to define a class of power series rings which will be the initial source of analytic functions for us.

Definition 1.2 Let E be Noetherian domain which is complete and separated in I -adic topology for some ideal I . Let (x_1, \dots, x_m) and (ρ_1, \dots, ρ_n) be variables, then the ring of *strictly convergent power series in x over E* is

$$T_m(E) = E \langle x \rangle := \left\{ \sum_{\alpha \in \mathbb{N}^m} a_\alpha x^\alpha : \lim_{|\alpha| \rightarrow \infty} a_\alpha = 0 \right\}$$

where the limit is taken in the I -adic topology. The ring of separated power series in (x, ρ) over E is then the ring

$$S_{m,n}(E) := E \langle x \rangle [[\rho]].$$

The rings $S_{m,n}(E)$ were first introduced in [4] and some of their algebraic properties were investigated in [1] by the author. We will use results from both of these sources in our investigation. Now using these rings, we are ready to repeat the definition of fields with analytic structure from [4].

Definition 1.3 Let E be as in Definition 1.2. A non-Archimedean valued field K is said to have a (*separated*) E -analytic structure if there is a collection of homomorphisms $\sigma_{m,n}$ from $S_{m,n}(E)$ into the ring of K° -valued functions on $(K^\circ)^m \times (K^{\circ\circ})^n$ for each $m, n \in \mathbb{N}$ such that

- i. $(0) \neq I \subset \sigma_0^{-1}(K^{\circ\circ})$,
- ii. $\sigma_{m,n}(x_i)$ is the i -th coordinate function on $(K^\circ)^m \times (K^{\circ\circ})^n$ for $i = 1, \dots, m$ and $\sigma_{m,n}(\rho_j)$ is the $(m+j)$ -th coordinate function on $(K^\circ)^m \times (K^{\circ\circ})^n$ for $j = 1, \dots, n$,
- iii. $\sigma_{m,n+1}$ extends $\sigma_{m,n}$, where we identify in the obvious way functions on $(K^\circ)^m \times (K^{\circ\circ})^n$ with functions on $(K^\circ)^m \times (K^{\circ\circ})^{n+1}$ that do not depend on the last coordinate, and $\sigma_{m+1,n}$ extends $\sigma_{m,n}$ similarly.

In other words, a non-Archimedean valued field K has an analytic E -structure if members of $S_{m,n}(E)$ can be interpreted as functions over $(K^\circ)^m \times (K^{\circ\circ})^n$ in a natural way. However one can make a more model theoretic description using the language $\mathcal{L}_{\text{sep}}^D(E)$.

Definition 1.4 Define $\mathcal{L}_{\text{sep}}^D(E)$ to be the multi-sorted language which consists of

- i. Three sorts \mathcal{K}° , $\mathcal{K}^{\circ\circ}$, and \mathcal{V} ;
- ii. A function $\text{ord} : \mathcal{K}^\circ \rightarrow \mathcal{V}$;
- iii. The symbols $\{0, +, -, <, \infty\}$ belonging to the sort \mathcal{V} ;
- iv. A function symbol for each member of $S_{m,n}(E)$ for all $m, n \geq 0$ to act on $(\mathcal{K}^\circ)^m \times (\mathcal{K}^{\circ\circ})^n$;
- v. Two function symbols D_0 and D_1 .

By convention we will use x_1, x_2, \dots and y_1, y_2, \dots for variables which range over the sort \mathcal{K}° and ρ_1, ρ_2, \dots and $\lambda_1, \lambda_2, \dots$ for variables that range over $\mathcal{K}^{\circ\circ}$.

The following are the theories in $\mathcal{L}_{\text{sep}}^D(E)$ which we consider.

Definition 1.5 We write $\mathcal{T}_{\text{an}}(E)$ for the theory in the language $\mathcal{L}_{\text{sep}}^D(E)$ which describes the fact that the sort \mathcal{V} is an ordered monoid; ord is a non-Archimedean valuation which is non-trivial; \mathcal{K}° is a valuation ring; $\mathcal{K}^{\circ\circ}$ is the set of elements of valuation greater than 0; $D_0 : \mathcal{K}^\circ \times \mathcal{K}^\circ \rightarrow \mathcal{K}^\circ$ and $D_1 : \mathcal{K}^\circ \times \mathcal{K}^\circ \rightarrow \mathcal{K}^{\circ\circ}$ are functions that satisfy

$$D_0(a, b) = \begin{cases} a/b & \text{if } \infty \neq \text{ord}(b) \leq \text{ord}(a), \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad D_1(a, b) = \begin{cases} a/b & \text{if } \text{ord}(b) < \text{ord}(a), \\ 0 & \text{else;} \end{cases}$$

and all of the identities of the form

$$f = F(g_1, \dots, g_m, h_1, \dots, h_n)$$

where $F \in S_{m,n}(E)$ and $g_1, \dots, g_m \in S_{M,N}(E)$, $h_1, \dots, h_n \in (\lambda)S_{M,N}(E)$ for some separated power series ring $S_{M,N}$ over the variables y and λ . To these, if we add the axioms that describe the fact that the quotient field of \mathcal{K}° is algebraically closed, we arrive at the theory $\mathcal{T}_{\text{an}}^{\text{ac}}(E)$.

So alternatively, we say that a valued field K has an analytic E -structure if $(K^\circ, K^{\circ\circ}, \text{ord}(K^\circ))$ is a model of $\mathcal{T}_{\text{an}}(E)$. Note that by Lemma 2.12 and Theorem 2.13 of [4], if E' is a finitely generated $\sigma(E)$ -subalgebra of K° , generated by a_1, \dots, a_m , and if $E' \cap K^{\circ\circ}$ is generated by b_1, \dots, b_n , then

$$E^{\sigma, E'} := \{\sigma_{m,n}(f)(a, b) : f \in S_{m,n}(E)\}$$

is independent of the choices for a and b . Furthermore $E^{\sigma, E'}$ is Noetherian, complete and separated with respect to the J -adic topology where $J = \sigma(I) \cdot E^{\sigma, E'}$, and σ induces a unique $E^{\sigma, E'}$ -analytic structure τ on K . Moreover σ and τ are compatible and each $\tau_{m,n}$ is injective. With the help of the above observation we can give the definition of another class of power series which was first introduced in [4]. Note that we are abusing notation and using σ instead of τ in what follows.

Definition 1.6 Let K be a valued field with an analytic E -structure σ and let us write $\mathcal{F}(\sigma, K)$ to denote the collection of all finitely generated E -subalgebras of K° . Using the notation above, define

$$S_{m,n}^\circ(\sigma, K) := \varinjlim_{E^{\sigma, E'} \in \mathcal{F}(\sigma, K)} E^{\sigma, E'} \langle x \rangle [[\rho]].$$

Then the *rings of separated power series with parameters from K* are defined to be

$$S_{m,n}(\sigma, K) := K \otimes S_{m,n}^\circ(\sigma, K).$$

We will call a ring of the form $S_{m,0}(\sigma, K)$ a *ring of strictly convergent power series with parameters from K* , and use the more customary notation $T_m(\sigma, K)$ to denote it.

These rings come in handy when one investigates the geometry of definable sets over a fixed valued field K with an analytic E -structure. Of course in order to understand properties of sets defined over such fields, one also needs to understand the properties of the terms of $\mathcal{L}_{\text{sep}}^D(E)$. The following class of algebras are our tool in this pursuit and their definition is a generalization of Definition 5.3.1 of [8] by Lipshitz and Robinson.

Definition 1.7 A *generalized ring of fractions* (shortened to GRF henceforth) over $S_{m,n}(E)$ is inductively defined as follows:

- i. $S_{m,n}(E)$ is a generalized ring of fractions over $S_{m,n}(E)$.
- ii. If A is a generalized ring of fractions over $S_{m,n}(E)$ and y and λ are variables not appearing in the presentation of A , and $f, g \in A$, then

$$A \langle f/g \rangle := A \langle y \rangle [[\lambda]] / (gy - f) \text{ and } A [[f/g]] := A [[\lambda]] / (g\lambda - f)$$

are both generalized rings of fractions over $S_{m,n}(E)$.

Note that oftentimes we will find it more useful to consider a generalized ring of fractions as an algebra together with a particular presentation or inductive construction.

Now notice that every quantifier-free formula $\psi(x, \rho)$ of $\mathcal{L}_{\text{sep}}^D(E)$ can be written as a finite disjunction of formulas of the form

$$\varphi(x, \rho) \Leftrightarrow \bigwedge_i (h_i(x, \rho) = 0) \wedge \bigwedge_i (\text{ord}(f_i(x, \rho)) \leq \text{ord}(g_i(x, \rho)) \neq \infty) \wedge \bigwedge_i (\text{ord}(f'_i(x, \rho)) < \text{ord}(g'_i(x, \rho))), \quad (1.1)$$

where f_i, g_i, f'_i, g'_i and h_i are $\mathcal{L}_{\text{sep}}^D(E)$ -terms. To a formula in the form of (1.1) we associate a GRF as follows.

- i. Start with $A_0 = S_{m,n}(E)$ and adjoin a fraction for each occurrence of a D_0 or D_1 as follows: at the k -th step take an innermost occurrence of either of the D -functions in φ which is not yet processed and set $A_k := A \langle f/g \rangle$ if it is $D_0(f, g)$, or $A_k := A [[f/g]]$, if it is $D_1(f, g)$.
- ii. Consider the inequalities in φ in any order. For each inequality $\text{ord}(f_i(x, \rho)) \leq \text{ord}(g_i(x, \rho)) \neq \infty$ that appears in φ , at the k -th step, set $A_k = A \langle f_i/g_i \rangle$ and continue similarly for each inequality $\text{ord}(f'_i(x, \rho)) < \text{ord}(g'_i(x, \rho))$ and $A_k = A [[f'_i/g'_i]]$.

Then one can use A to uniformly rewrite the sets which φ defines over models of $\mathcal{T}_{\text{an}}(E)$ in a way that enables an understanding of the geometry of such sets through the study of algebraic properties of A .

Definition 1.8 Given a field K with an analytic E -structure and a generalized ring of fractions A over $S_{m,n}(E)$ with a presentation

$$A = S_{m,n} \langle y_1, \dots, y_M \rangle [[\lambda_1, \dots, \lambda_N]] / (\{g_i y_i - f_i\}_{i=1}^M \cup \{g'_i \lambda_i - f'_i\}_{i=1}^N) \quad (1.2)$$

we define the K -domain of A , which is denoted by $K\text{-Dom}_{m,n}A$, to be the projection of the set

$$X := \{\bar{p} \in (K^\circ)^{m+M} \times (K^{\circ\circ})^{n+N} : \bar{p} \in V(\sigma(\{g_i y_i - f_i\}_{i=1}^M \cup \{g'_i \lambda_i - f'_i\}_{i=1}^N))_K \setminus V(\sigma(\prod_i g_i \cdot \prod_i g'_i)_K)\}$$

onto the coordinate hyperplane $(K^\circ)^m \times (K^{\circ\circ})^n$.

Observe that the projection above is one-to-one when restricted to X and so each member of A can be thought of as a function on $K\text{-Dom}_{m,n}A$ via the analytic structure σ . For an ideal $J \subset A$, we use the notation

$$K\text{-Dom}_{m,n}A \cap V(J)_K := \{\bar{p} \in K\text{-Dom}_{m,n}A : \sigma(f)(\bar{p}) = 0 \text{ for all } f \in J\}.$$

Note that if A is the GRF associated with φ of (1.1) and J is the ideal generated by h_i , then

$$K\text{-Dom}_{m,n}A \cap V(J)_K = \{\bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n : \varphi(\bar{p})\}.$$

So we reach at the following observation.

Observation 1.9 For any quantifier free formula $\psi(x, \rho)$ of $\mathcal{L}_{\text{sep}}^D(E)$, there are GRFs A_i over $S_{m,n}(E)$ and ideals $J_i \subset A_i$ such that

$$\{\bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n : \psi(x, \rho)\} = \bigcup_i K\text{-Dom}_{m,n} A_i \cap V(J_i)_K$$

for any field K with an analytic E -structure.

2 Quantifier-free Definable Sets

In this section we prove two theorems concerning quantifier-free definable sets. First is the parameterized normalization theorem which, given a quantifier-free formula $\psi(x, \rho)$, constructs finitely many formulas ψ_i which define "parametrically normalized sets" which cover the set defined by ψ in any model of \mathcal{T}_{an} . This result, after some some ramifications, is the key to the proof of our main theorem of quantifier elimination for $\mathcal{T}_{\text{an}}^{\text{ac}}$. We start our investigation with a discussion of sets of the form $K\text{-Dom}_{m,n} A \cap V(J)_K$. We are especially interested in the relation between such a set and the algebra A/J , but to get the most of that relation, we first need to choose the ideal J in a special way.

Definition 2.1 Let A be a GRF over $S_{m,n}(E)$ given by Equation (1.2) and let $\mathfrak{p} \subset A$ be a prime ideal. We say that \mathfrak{p} is a *restricted minimal prime* of B if $g_i, G_j \notin \mathfrak{p}'$ for all i and j , where \mathfrak{p}' denotes the ideal corresponding to \mathfrak{p} in $S_{m+M, n+N}(E)$. Given an ideal $J \subset B$, if there is a minimal prime divisor of J which is a restricted prime, then we set

$$\mathfrak{RR}(J) := \bigcap \{\mathfrak{p} : \mathfrak{p} \text{ is a restricted minimal prime divisor of } J\},$$

else we define $\mathfrak{RR}(J) := (1)$, and call the ideal $\mathfrak{RR}(J)$ the *restricted radical* of J .

The next lemma shows that $\mathfrak{RR}(J)$ and J vanish on the same set inside $K\text{-Dom}_{m,n} A$.

Lemma 2.2 Let A be a GRF over $S_{m,n}(E)$ and let $J \subset A$ be an ideal. Then for any field K with an analytic E -structure,

$$K\text{-Dom}_{m,n} A \cap V(J)_K = K\text{-Dom}_{m,n} A \cap V(\mathfrak{RR}(J))_K.$$

Proof. The inclusion of $K\text{-Dom}_{m,n} A \cap V(\mathfrak{RR}(J))_K$ inside $K\text{-Dom}_{m,n} A \cap V(J)_K$ is clear. In order to see the other inclusion, we will proceed by making use of the Nullstellensatz for $T_m(\sigma, K)$ which is Theorem 3.8 of [1]. In order to apply this version of Nullstellensatz, we start by writing A as in Equation (1.2) and with J' we denote the ideal of $S_{m+M, n+N}(E)$ which corresponds to J . Let $\bar{p} \in K\text{-Dom}_{m,n} A \cap V(J)_K$ and let $\bar{q} \in V(J')_K$ be the corresponding point in $(K^\circ)^{m+M} \times (K^{\circ\circ})^{n+N}$ which projects onto \bar{p} , so that $g_i(\bar{q}) \neq 0, G_j(\bar{q}) \neq 0$ for all i and j . Through a translation we may assume that \bar{q} is the origin. Next we choose an $\varepsilon \in K^{\circ\circ}$ and look at the isomorphic image $T_{m+M+n+N, \varepsilon}(\sigma, K)$, of $T_{m+M, n+N}(\sigma, K)$ under the map

$$\phi_\varepsilon : \sum_\alpha a_\alpha(x, y, \rho, \lambda)^\alpha \rightarrow \sum_\alpha \frac{\sigma(a_\alpha)}{\varepsilon^{|\alpha|}}(x, y, \rho, \lambda)^\alpha,$$

and observe that $T_{m+M+n+N, \varepsilon}(\sigma, K)$ contains $\sigma(S_{m,n}(E))$. Let us write \mathfrak{m} for the maximal ideal corresponding to the origin in $T_{m+M+n+N, \varepsilon}(\sigma, K)$, then by Nullstellensatz, we clearly have $\mathfrak{m} \supset J' \cdot T_{m+M+n+N, \varepsilon}(\sigma, K)$ and $g_i \notin \mathfrak{m}, G_j \notin \mathfrak{m}$ for all i and j . Now for any \mathfrak{p} which is a minimal prime divisor of J which is contained in the canonical image of \mathfrak{m} in A we easily see that \mathfrak{p} has to be a restricted minimal prime divisor of J and $\bar{p} \in K\text{-Dom}_{m,n} A \cap V(\mathfrak{p})_K$, proving the lemma. \square

Both of our main results, Uniform Parameterized Normalization and Quantifier Elimination theorems will require an induction on Krull dimension of algebras. This is made possible by the Theorem 2.6 of [1] which states that the Krull dimension of $S_{m,n}(E)$ is $d + m + n$ where d is the Krull dimension of E . Yet before we can use the Krull dimension we need to discover the effect of adjoining fractions to a GRF on the Krull dimension of algebras we are considering. The next lemma handles that.

Lemma 2.3 *Let E be as above, then for a GRF B over $S_{m,n}(E)$, ideal $J \subset B$, and a GRF A which extends B , the Krull dimension of $A/\mathfrak{R}\mathfrak{R}(J \cdot A)$ is dominated by both the Krull dimension of B/J and $d + m + n$.*

Proof. To see that the Krull dimension of $A/\mathfrak{R}\mathfrak{R}(J \cdot A)$ is less than equal to that of B/J observe that by induction it is enough to prove the following: If z is a variable not appearing in $S_{m,n}(E)$, f and g are two elements of $S_{m,n}(E)$, and \mathfrak{m} is a maximal ideal of $S_{m,n}(E)$ containing J with $gz - f \in \mathfrak{m}$, $g \notin \mathfrak{m}$ and $\mathfrak{n} := \mathfrak{m} \cap S_{m,n}(E)$, then the Krull dimension of $S_{m,n}(E)_{\mathfrak{n}}/J_{\mathfrak{n}}$ is the same as the Krull dimension of $(S_{m,n}(E) \langle z \rangle)_{\mathfrak{m}}/(J \cup \{gz - f\})_{\mathfrak{m}}$ and also the same as that of $(S_{m,n}(E) [[z]])_{\mathfrak{m}}/(J \cup \{gz - f\})_{\mathfrak{m}}$. This statement follows exactly as in the proof of Lemma 3.6 of [3]. To see that the Krull dimension of $A/\mathfrak{R}\mathfrak{R}(J \cdot A)$ is less than or equal to $d + m + n$, write A as in Equation 1.2 and let J' be the ideal that corresponds to $\mathfrak{R}\mathfrak{R}(J \cdot A)$ in $S_{m+M,n+N}(E)$ and set $J'' := J' \cap S_{m,n}(E)$. Then clearly, because the Krull dimension of $S_{m,n}(E)/J''$ is at most $d + m + n$, by the first part of the statement we have that the Krull dimension of $A/\mathfrak{R}\mathfrak{R}(J'' \cdot A)$ is also at most $d + m + n$. Now the statement follows as $J'' \cdot A \subset J \cdot A$. \square

Now we have the tools we need to prove the Uniform Parameterized Normalization Theorem. However we still need to explain our terminology related to parameters before we state and prove it. Let B be a generalized ring of fractions over $S_{m+M,n+N}(E) = E \langle x, y \rangle [[\rho, \lambda]]$, where x and ρ variables are considered as parameters. Then we follow the following convention for presenting B . Suppose

$$B_{s+S+t+T} := S_{m+s+M+n+t+N+T}(E)/J$$

is the generalized ring of fractions that we obtained at the $(s + S + t + T)$ -th step of the inductive construction of B , where $S_{m+s+M+n+t+N+T}(E)$ is the ring of separated power series over the variables x_1, \dots, x_{m+s} , $\rho_1, \dots, \rho_{n+t}$, y_1, \dots, y_{M+S} and $\lambda_1, \dots, \lambda_{N+T}$. Then in the next step of the construction we have either

- i. $B_{s+S+t+T+1} = B_{s+S+t+T} \langle f/g \rangle$, or
- ii. $B_{s+S+t+T+1} = B_{s+S+t+T} [[f/g]]$.

Now we adopt the convention that if both f and g are members of $S_{m+s,n+t}(E)$, then we set

$$B_{s+S+t+T+1} = B_{s+S+t+T} \langle x_{m+s+1} \rangle / (gx_{m+s+1} - f),$$

if it is the case (i) above, or

$$B_{s+S+t+T+1} = B_{s+S+t+T} [[\rho_{n+t+1}]] / (g\rho_{n+t+1} - f),$$

if it is the case (ii). Otherwise, in which case we have either $f \notin S_{m+s,n+t}(E)$ or $g \notin S_{m+s,n+t}(E)$, we substitute y_{M+S+1} for x_{m+s+1} and λ_{N+T+1} for ρ_{n+t+1} in the above process.

Given a such constructed presentation of B , we can write

$$B = (S_{m,n}(E) \langle x_{m+1}, \dots, x_{m+s} \rangle [[\rho_{n+1}, \dots, \rho_{n+t}]] / J_1) \langle y_1, \dots, y_{M+S} \rangle [[\lambda_1, \dots, \lambda_{N+T}]] / J_2.$$

In this case we call $S_{m+s,n+t}(E)/J_1$ the *parameter ring of the presentation of B* . With the terminology above we are ready for the Uniform Parameterized Normalization Theorem.

Theorem 2.4 (Uniform Parameterized Normalization) *Let B be a GRF over $S_{m+M,n+N}(E)$ with a parameter ring A which is a GRF over $S_{m,n}(E)$, and let J be a ideal of B . Then there are GRFs B_1, \dots, B_k over $S_{m+M,n+N}(E)$ with parameter rings A_1, \dots, A_k which are GRFs over $S_{m,n}(E)$, ideals $J_1 \subset B_1, \dots, J_k \subset B_k$, integers M_1, \dots, M_k and N_1, \dots, N_k , and Weierstrass changes of variables ϕ_1, \dots, ϕ_k among the y and λ variables separately such that*

- i. $K\text{-Dom}_{m+M,n+N} B \cap V(J)_K = \bigcup_{i=1}^k (K\text{-Dom}_{m+M,n+N} B_i \cap V(J_i)_K)$ for any valued field K with an analytic E -structure,
- ii. For all i , $(A_i/J_i \cap A_i) \langle y_1, \dots, y_{M_i} \rangle [[\lambda_1, \dots, \lambda_{N_i}]] \rightarrow B_i/\phi_i(J_i)$ is a finite inclusion,
- iii. Each J_i is a restricted prime ideal,

iv. $M_i + N_i \leq M + N$ for all i .

Proof. Here we give only the highlights of the proof of the existence of a covering of $K\text{-Dom}_{m+M,n+N}B \cap V(J)_K$ satisfying the conditions (i) and (ii) as the details are exactly as in the proof of Lemma 5.3 of [3]. This proof is based on a consequence of the strong Noetherian property of the rings $S_{m,n}(E)$ and similar properties were the key in quantifier elimination theorems in [5], [6], [7], [9] and [11]. The version useful for us is Proposition 2.10 of [1], which guarantees that given $f_1, \dots, f_k \in S_{m+M,n+N}(E)$, if for each $i \in \{1, \dots, k\}$ we write

$$f_i = \sum_{\alpha, \beta} a_{i,\alpha,\beta}(x, \rho) y^\alpha \lambda^\beta,$$

then we have a finite set $\mathcal{Z} \subset \mathbb{N}^M \times \mathbb{N}^N$ such that for any field K with an analytic E -structure and for any $\bar{p} \in (K^\circ)^m \times (K^{\circ\circ})^n$, either $\bar{p} \in V(\{a_{i,\alpha,\beta}\})_K$ or there is an $(i_0, \alpha_0, \beta_0) \in \{1, \dots, k\} \times \mathcal{Z}$ such that $a_{i_0,\alpha_0,\beta_0}(\bar{p})$ "dominates" every $a_{i,\alpha,\beta}(\bar{p})$, in a specific meaning of the word.

We start by assuming that $B = A \langle y_1, \dots, y_M \rangle [[\lambda_1, \dots, \lambda_N]]$, as the general case easily follows from this case. By Proposition 2.10 of [1], given the generators f_1, \dots, f_k of an ideal J , by fixing one $(i_0, \alpha_0, \beta_0) \in \{1, \dots, k\} \times \mathcal{Z}$ and adjoining fractions of the type $D_0(a_{i,\alpha,\beta}, a_{i_0,\alpha_0,\beta_0})$ or $D_1(a_{i,\alpha,\beta}, a_{i_0,\alpha_0,\beta_0})$ to A for all $(i, \alpha, \beta) \in \{1, \dots, k\} \times \mathcal{Z}$ we obtain finitely many GRFs A_{i_0,α_0,β_0} , such that the union of $K\text{-Dom}_{m,n}A_{i_0,\alpha_0,\beta_0}$ cover $K\text{-Dom}_{m,n}A \setminus V(\{a_{i,\alpha,\beta}\})_K$. Note that if one substitutes $J \cup \{a_{i,\alpha,\beta}\}$ for J_i , (ii) of the theorem's statement is satisfied trivially.

Next we consider B_{i_0,α_0,β_0} which we obtain from B by adjoining the extra D -terms of A_{i_0,α_0,β_0} . Note that we can replace each f_{i_0} with $(1/a_{i_0,\alpha_0,\beta_0})f_{i_0}$ in B_{i_0,α_0,β_0} to assume that $a_{i_0,\alpha_0,\beta_0} = 1$. Set

$$g_{i_0,\alpha_0,\beta_0} := \sum_{\alpha} a_{i_0,\alpha,\beta_0} y^\alpha,$$

and define

$$\begin{aligned} B'_{i_0,\alpha_0,\beta_0} &:= B_{i_0,\alpha_0,\beta_0} \langle y_{M+1} \rangle / (g_{i_0,\alpha_0,\beta_0} y_{M+1} - 1) \\ B''_{i_0,\alpha_0,\beta_0} &:= B_{i_0,\alpha_0,\beta_0} [[\lambda_{N+1}]] / (\lambda_{N+1} - g_{i_0,\alpha_0,\beta_0}). \end{aligned}$$

Observe that for any field K with an analytic E -structure, $K\text{-Dom}_{m+M,n+N}B'$ and $K\text{-Dom}_{m+M,n+N}B''$ cover $K\text{-Dom}_{m+M,n+N}B$. Observe also that

$$G_{i_0,\alpha_0,\beta_0} := \lambda^{\beta_0} + \sum_{\beta \neq \beta_0, \alpha} a_{i_0,\alpha,\beta} y_{M+1} y^\alpha \lambda^\beta \equiv y_{M+1} f_{i_0} \pmod{(g_{i_0,\alpha_0,\beta_0} y_{M+1} - 1) B_{i_0,\alpha_0,\beta_0}}$$

and $g_{i_0,\alpha_0,\beta_0} y_{M+1} - 1$ are preregular in y and λ respectively. That is, after a Weierstrass change of variables ϕ among y and λ separately, they become regular (cf. Definition 2.3 of [4]) in λ_N and y_{M+1} . Therefore, by Weierstrass Division Theorem (Proposition 2.4 of [4]) we have a finite inclusion

$$A_{i_0,\alpha_0,\beta_0} \langle y_1, \dots, y_M \rangle [[\lambda_1, \dots, \lambda_{N-1}]] / (\phi(J) \cap A_{i_0,\alpha_0,\beta_0} \langle y_1, \dots, y_M \rangle [[\lambda_1, \dots, \lambda_{N-1}]]]) \rightarrow B'_{i_0,\alpha_0,\beta_0} / \phi(J).$$

A similar argument works for $\lambda_{N+1} - g_{i_0,\alpha_0,\beta_0}$ in $B''_{i_0,\alpha_0,\beta_0}$ and the existence of a break-up satisfying (i) and (ii) now follows by induction on pairs (M, N) and Krull dimension.

Now observe that (iv) is an immediate consequence of the process described above once the condition (iii) is satisfied. So we only need to see that we can satisfy (iii) simultaneously with (i) and (ii). Fix an i and consider $\mathfrak{R}\mathfrak{R}(J_i)$. If $\mathfrak{R}\mathfrak{R}(J_i) = (1)$ then $K\text{-Dom}_{m+M,n+N}B_i \cap V(J_i)_K = \emptyset$ for any valued field K with an analytic E -structure by Lemma 2.2 and hence by discarding such J_i and B_i we can assume that $\mathfrak{R}\mathfrak{R}(J_i) \neq 1$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_l$ be restricted minimal prime divisor of J_i , then clearly

$$K\text{-Dom}_{m+M,n+N}A_i \cap V(J_i)_K = \bigcup_{j=1}^l K\text{-Dom}_{m+M,n+N}A_i \cap V(\mathfrak{p}_j)_K.$$

Therefore if $\mathfrak{p}_j \cap A_i \langle y_1, \dots, y_{M_i} \rangle [[\lambda_1, \dots, \lambda_{N_i}]]$ is generated by $\mathfrak{p}_j \cap A_i$, then we can replace J_i by \mathfrak{p}_j and have statements (ii), (iv) and (iii) satisfied simultaneously. Else, clearly the Krull dimension of B_i/\mathfrak{p}_j is less than that of B_i/J_i and the claim follows by induction on Krull dimension and Lemma 2.3. \square

Now we only need one more lemma before we can state and prove quantifier elimination.

Lemma 2.5 *Let A be an integral domain and let $A \rightarrow A[z_1, \dots, z_k]/J$ be a finite inclusion, where J is a prime ideal. Then there is an $h \in A$ and monic polynomials $f_i \in A_h[z_i]$ for $1 \leq i \leq k$ such that $J \cdot A_h[z_1, \dots, z_k]$ is generated by f_1, \dots, f_k .*

Proof. Let us write F for the quotient field of A and observe that $J \cdot F[z_1, \dots, z_k]$ is not the unit ideal as otherwise $J \cap A \neq (0)$. Hence by Going Down Theorem for flat extensions (Theorem 9.5 of [10]), there is a prime ideal $\mathfrak{p} \subset F[z_1, \dots, z_k]$ such that $\mathfrak{p} \cap A[z_1, \dots, z_k] = J$. For each $i \in \{1, \dots, k\}$, let f_i denote the lowest degree monic polynomial in $\mathfrak{p} \cap F[z_i]$, then clearly $\{f_1, \dots, f_k\}$ is a generating set for \mathfrak{p} . Now let $c \in A$ be such that $cf_i \in A[z_1, \dots, z_k]$ for all $1 \leq i \leq k$ and observe that if g_1, \dots, g_l are the generators of J , then there are $c_1, \dots, c_l \in A$ such that $cc_i g_i \in (cf_1, \dots, cf_k) \cdot A[z_1, \dots, z_k]$ for all i . Now setting $h := cc_1 \dots c_l$, we see that $J \subset (cf_1, \dots, cf_k) \cdot A_h[z_1, \dots, z_k]$. \square

Theorem 2.6 *The theory $\mathcal{T}_{an}^{ac}(E)$ admits elimination of quantifiers in the language $\mathcal{L}_{sep}^D(E)$.*

Proof. How to eliminate the quantifiers which apply to the variables of the valued field sort are explained in Weispfenning's proof of the quantifier elimination theorem for the algebraic theory of algebraically closed non-Archimedean valued fields (Theorem 3.1) in [12]. Hence it is enough to show that for a quantifier-free formula $\psi(x, \rho, y, \lambda)$ of \mathcal{L}_{an}^D , the quantified formula $(\exists y, \lambda)\psi$ is equivalent to a quantifier-free formula of \mathcal{L}_{an}^D .

For showing that, we start by rewriting ψ in a special form. Let $x = (x_1, \dots, x_m)$, $\rho = (\rho_1, \dots, \rho_n)$, $y = (y_1, \dots, y_M)$ and $\lambda = (\lambda_1, \dots, \lambda_N)$ be the variables appearing in ψ . Clearly we can assume that ψ satisfies

$$\psi(\bar{q}) \text{ if and only if } \bar{q} \in K\text{-Dom}_{m+M, n+N} B \cap V(J)_K$$

for each valued field K with an analytic E -structure, where B is a GRF over $S_{m+M, n+N}(E)$ and $J \subset B$ is an ideal. If it is the case that the Krull dimension of B/J is zero, then by Theorem 2.4 and Lemma 2.3, we may assume that

$$A/J \cap A \rightarrow B/J$$

is a finite inclusion, where A is a parameter ring for a presentation of B and the Krull dimension of $A/J \cap A$ is 0. From this, it follows that $K\text{-Dom}_{m, n} A \cap V(J \cap A)_K$, which contains $(\exists y, \lambda)\psi(x, \rho, y, \lambda)$, is also the zero set of some $J' \subset K[x_1, \dots, x_m, \rho_1, \dots, \rho_n]$ with $K[x_1, \dots, x_m, \rho_1, \dots, \rho_n]/J'$ being a zero-dimensional ring and the statement of the theorem follows.

To treat the general case we again utilize Theorem 2.4 which guarantees that we may assume that there are integers M' and N' , satisfying $M' + N' \leq M + N$, such that

$$(A/J \cap A) \langle y_1, \dots, y_{M'} \rangle [[\lambda_1, \dots, \lambda_{N'}]] \rightarrow B/J$$

is a finite inclusion, where A is the parameter ring of B , J is a restricted prime ideal and the Krull dimension of $A/J \cap A$ is at most $m + n$, and induct on the dimension of $R := (A/J \cap A) \langle y_1, \dots, y_{M'} \rangle [[\lambda_1, \dots, \lambda_{N'}]]$. Write

$$B = S_{m+s+M+S, n+t+N+T}(E) / (\{g_i x_i - f_i\}_{i=m+1}^{m+s} \cup \{g'_i y_i - f'_i\}_{i=M+1}^{M+S} \cup \{G_i \rho_i - F_i\}_{i=n+1}^{n+t} \cup \{G'_i \lambda_i - F'_i\}_{i=N+1}^{N+T}),$$

and let us rename some of the variables for notational convenience as

$$z := (z_1, \dots, z_k) := (y_{M'+1}, \dots, y_{M+S}, \lambda_{N'+1}, \dots, \lambda_{N+T}),$$

with k standing for $M + S - M' + N + T - N'$. Next, when we apply Lemma 2.5 with R and J , we obtain an $h \in R$ and monic polynomials $f_i \in R_h[z_i]$ which generate $(J \cap R[z]) \cdot R_h[z]$. Choose an $H \in A \langle y_1, \dots, y_{M'} \rangle [[\lambda_1, \dots, \lambda_{N'}]]$ such that the image of H in R is h . Similarly, define

$$g := \left(\prod_{i=m+1}^{m+s} g_i \right) \left(\prod_{i=M+1}^{M+S} g'_i \right) \left(\prod_{i=n+1}^{n+t} G_i \right) \left(\prod_{i=N+1}^{N+T} G'_i \right),$$

and let G be the image of g in B .

Now using the elements we obtained above we look at the set

$$X_K := (K\text{-Dom}_{m,n}A \cap V(J \cap A)_K) \times (K^\circ)^{M'} \times (K^{\circ\circ})^{N'} \setminus \\ (V((J \cup \{G\}) \cap A \langle y_1, \dots, y_{M'} \rangle [[\lambda_1, \dots, \lambda_{N'}]])_K \cup V(H)_K).$$

We claim that, if K is algebraically closed then for each $\bar{p} \in X_K$ there is a $\bar{q} \in K\text{-Dom}_{m+M,n+N}B \cap V(J)_K$ which projects onto \bar{p} . Assume the contrary, then, as $\bar{p} \notin V(H)_K$, we can make the substitution

$$\bar{f}_i(z) := f_i(\bar{p}, z_i) \in K[z_i], \text{ for } i \in \{1, \dots, k\},$$

to get non-constant monic polynomials $\bar{f}_1(z_1), \dots, \bar{f}_k(z_k)$. Therefore if we set J' to be the ideal corresponding to J in $S_{m+s+M+S,n+t+N+T}(E)$, we see that the fiber of $V(J')_K$ over \bar{p} is given by $V(\bar{f}_1(z_1), \dots, \bar{f}_k(z_k))_K$, which is clearly non-empty. Thus it must be the case that for all $\bar{q} \in V(J')_K$ which projects onto \bar{p} , $G(\bar{q}) = 0$ holds. Then for each $g \in (J \cup \{G\}) \cap A \langle y_1, \dots, y_{M'} \rangle [[\lambda_1, \dots, \lambda_{N'}]]$, because $g(\bar{p}) \in (\bar{f}_1(z_1), \dots, \bar{f}_k(z_k))$, $g(\bar{p}) = 0$ must hold. However this contradicts our assumption, proving the claim.

Now let $\varphi_1(x, \rho)$ be the quantifier-free formula which satisfies

$$\varphi_1(\bar{p}) \text{ if and only if } \bar{p} \in K\text{-Dom}_{m,n}A \cap V(J \cap A)_K \setminus V((J \cup \{G \cdot H\}) \cap A)_K$$

for each valued field K with an E -analytic structure. As a corollary to the above claim we see that if K is algebraically closed then for all \bar{p} such that $\varphi_1(\bar{p})$ holds, there is a $\bar{q} \in K\text{-Dom}_{m+M,n+N}B \cap V(J)_K$ which projects onto \bar{p} . We put $\varphi_2(x, \rho, y, \lambda)$ to be the quantifier-free formula which satisfies

$$\varphi_2(\bar{q}) \text{ if and only if } \bar{q} \in K\text{-Dom}_{m+M,n+N}B \cap V(J \cup \{H\})_K$$

for each non-Archimedean valued field with an analytic E -structure and observe that

$$\mathcal{T}_{\text{sep}}^{\text{ac}}(E) \vdash (\exists y, \lambda) \psi(x, \rho, y, \lambda) \Leftrightarrow (\varphi_1(x, \rho) \wedge (\exists y, \lambda) \varphi_2(x, \rho, y, \lambda)).$$

Therefore it is enough to show that $(\exists y, \lambda) \varphi_2(x, \rho, y, \lambda)$ is equivalent to a quantifier-free formula. On the other hand, as we have assumed J to be a prime ideal, we have

$$(J \cup \{H\}) \cap A / (J \cap A) \langle y_1, \dots, y_{M'} \rangle [[\lambda_1, \dots, \lambda_{N'}]] \neq (0).$$

Hence there are finitely many GRFs B_i and ideals J_i such that φ_2 can be written as a finite union of formulas which define $K\text{-Dom}_{m+M,n+N}B_i \cap V(J_i)_K$ for each valued field K with an analytic E -structure and by Lemma 2.3 and Theorem 2.4, each J_i can be chosen such that the Krull dimension of B_i/J_i is strictly less than that of B/J . Now the assertion follows by induction. \square

Next we consider the language $\mathcal{L}_{\text{sep}}^D(E, K)$ that we obtain by adding a function symbol for each member of $\bigcup_{m,n} S_{m,n}^\circ(\sigma, K)$ to $\mathcal{L}_{\text{sep}}^D(E)$, and the theory $\mathcal{T}_{\text{an}}^{\text{ac}}(E, K)$ which we obtain instead of $\mathcal{T}_{\text{an}}^{\text{ac}}(E)$ if we substitute $S_{m,n}^\circ(\sigma, K)$ for $S_{m,n}(E)$ in Definition 1.5. As any formula in $\mathcal{L}_{\text{sep}}^D(E, K)$ is a formula of $\mathcal{L}_{\text{sep}}^D(E^{\sigma, E'})$ for some $E' \in \mathcal{F}(\sigma, K)$, we at once have the following corollary.

Corollary 2.7 *The theory $\mathcal{T}_{\text{an}}^{\text{ac}}(E, K)$ admits elimination of quantifiers in the language $\mathcal{L}_{\text{sep}}^D(E, K)$.*

We finish with an important geometric consequence of quantifier elimination. This result, Corollary 2.8, is a complete analogue of Lemma 6.3 of [9] and is crucial in understanding the topological properties of definable sets over any valued field with analytic E -structure. In fact, the proof of Lemma 6.3 of [9] only depends on quantifier elimination stated in Theorem 5.2 of [9] which is a complete analogue of Corollary 2.7 above and can be applied directly to prove our Corollary 2.8. Note that just as it was done in Theorem 6.4 of [9], the result below can be used to prove Łojasiewicz inequalities for definable sets over algebraically closed valued fields with analytic E -structure.

Corollary 2.8 *Let K be a valued field with an analytic E -structure and let $X \subset (K^\circ)^m$ be a closed definable set. Suppose that for every $\varepsilon \in K^\circ \setminus \{0\}$ there is an $(x_1, \dots, x_m) \in X$ such that $\text{ord}(x_m) > \text{ord}(\varepsilon)$. Then there are $x_1, \dots, x_{m-1} \in K^\circ$ such that $(x_1, \dots, x_{m-1}, 0) \in X$.*

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