Comparing Classes of Structures

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Motivation

There is a body of work in mathematical logic dealing with comparing the complexity of the classification problem for various classes of structures.

Example (model theory)

By looking at the cardinality of the set of isomorphism types, we know that the classification problem for the class of countable linear orderings ($2^\aleph_0$ many isomorphism types) must be more complicated than the classification problem for the class of $\mathbb{Q}$-vector spaces ($\aleph_0$ many isomorphism types).

Example (descriptive set theory)

Using Borel embeddings and the $\leq_B$ partial ordering induced by the embeddings, we can make distinctions among classes with $2^\aleph_0$ many isomorphism types. For instance, we know that the class of Abelian $p$-groups of length $\omega$ lies strictly below the class of countable linear orderings in the $\leq_B$ partial ordering.
Our structures all have computable languages - that is, the set of symbols in the language is computable, and we can computably determine the type (function, relation) and arity of a given symbol.

Structures are countable - that is, they have a countable universe \( A \subseteq \omega \).

Classes contain structures for only one language, and the classes of structures are closed under isomorphism (modulo the restriction on the universes).

We work in computable \( \mathcal{L}_{\omega_1\omega} \) - a logic that allows infinite c.e. disjunctions (count as an existential quantifier) and infinite c.e. conjunctions (count as a universal quantifier).
The embeddings

**Definition**

A **computable embedding** of $K$ into $K'$ is a c.e. set $\Phi$ of pairs $(\alpha, \varphi)$, where $\alpha$ is a finite set of basic $(L \cup \omega)$-sentences, and $\varphi$ is a basic $(L' \cup \omega)$-sentence, such that

1. for each $A \in K$, there is a corresponding $\Phi(A) = B \in K'$ with $D(B) = \{ \varphi : (\exists \alpha \subseteq D(A)) (\alpha, \varphi) \in \Phi \}$, and
2. for $A, A' \in K$, $A \cong A'$ iff $\Phi(A) \cong \Phi(A')$.

**Definition**

A **Turing computable embedding** of $K$ into $K'$ is an operator $\Phi = \varphi_e$ such that

1. for each $A \in K$, there exists $\Phi(A) = B \in K'$ such that $\varphi_{e^{D(A)}} = \chi_{D(B)}$, and
2. for $A, A' \in K$, $A \cong A'$ iff $\Phi(A) \cong \Phi(A')$. 
Landmark classes

From each of these effective embeddings, we obtain a partial ordering, \( \leq_c \) and \( \leq_{tc} \), respectively.

**Fact (Greenberg, Kalimullin)**
\[ \leq_c \Rightarrow \leq_{tc} \text{ but } \leq_{tc} \not\Rightarrow \leq_c \]

Let \( PF \) be the class of finite prime fields, \( FLO \) be the class of finite linear orderings, \( LO \) the class of linear orderings, \( FVS \) the class of finite dimensional \( \mathbb{Q} \)-vector spaces, and \( VS \) the class of \( \mathbb{Q} \)-vector spaces.
Let \( e \in \{ c, tc \} \). We write \( K <_e K' \) if \( K \leq_e K' \) but \( K' \not>_e K \).

**Theorem (Calvert-Cummins-Knight-M., Knight-M.-Vanden Boom)**
\[
PF <_e FLO <_e FVS <_e VS <_e LO
\]

Notice that the classes \( PF, FLO, FVS, \) and \( VS \) have only countably many isomorphism types.
These partial orderings that arise from the effective embeddings do allow us to order the complexity of the classification problem. Suppose $K \leq_e K'$. Then the classification problem for $K$ must be no more complicated than the classification problem for $K'$, since structures in $K$ can be described by finding their images in $K'$ and describing them there.

The following “Pull-back Theorem” allows us to describe members of $K$ in their own language.

**Theorem (Knight-M.-Vanden Boom)**

If $K \leq_e K'$ via $\Phi = \varphi_e$, then for any computable infinitary sentence $\varphi$ in the language of $K'$, we can find a computable infinitary sentence $\varphi^*$ in the language of $K$ such that for all $A \in K$, $\Phi(A) \models \varphi$ iff $A \models \varphi^*$.

Moreover, if $\varphi$ is computable $\Sigma_\alpha$, or computable $\Pi_\alpha$, for $\alpha \geq 1$, then so is $\varphi^*$.
Let $T$ be a countable complete theory with infinite models. We say that $T$ is strongly minimal if for all models $A$ of $T$ and all formulas $\varphi(\bar{a}, x)$ with parameters $\bar{a}$, the set of elements satisfying $\varphi(\bar{a}, x)$ in $A$ is either finite or co-finite.

The algebraic closure of a set $C \subseteq A$, denoted by $acl_A(C)$, is the set of elements satisfying some formula (with parameters from $C$) that is satisfied by only finitely many elements.

A set $C \subseteq A$ is algebraically independent if for all $c \in C$, $c \notin acl_A(C \setminus \{c\})$.

Let $A$ be a model of a strongly minimal theory. The dimension of $A$ is the cardinality of a maximal algebraically independent set, called a basis.

**Theorem (Baldwin-Lachlan)**

*For a strongly minimal theory $T$, each model is determined up to isomorphism by its dimension.*
Triviality and a difference

The classes $ACF$, $VS$, and $ZS = Th(\mathbb{Z}, S)$ (where $S$ is the successor relation) are all examples of classes of models of strongly minimal theories. Using the Pull-back theorem, we can show that all such classes are $tc$-equivalent.

**Theorem (Knight-M.-Vanden Boom)**

$$ACF \equiv_{tc} VS \equiv_{tc} ZS$$

**Definition**

A strongly minimal theory is **trivial** if for any model $\mathcal{A}$ and any set $C \subseteq \mathcal{A}$, $acl_\mathcal{A}(C) = \bigcup_{c \in C} acl_\mathcal{A}(c)$.

The structures in $ZS$ have a trivial theory, while the structures in $VS$ and $ACF$ have non-trivial theories.

**Theorem (Chisholm-Knight-M.)**

$$VS \not\preceq_c ZS \text{ and } ACF \not\preceq_c ZS$$
Open questions

Still unknown is the $\leq_c$-relation between $ACF$ and $VS$.

**Question**
Do we have $ACF \leq_c VS$? Do we have $VS \leq_c ACF$?

There are nice properties of Turing computable embeddings that make it easier to characterize when we will have an embedding into a particular class or not. These embedding characterizations are strongly related to the sentences that distinguish between non-isomorphic members of the class being considered. On the other hand, computable embeddings make finer distinctions than the Turing computable embeddings do.

**Question**
Which is the better notion, $\leq_c$ or $\leq_{tc}$?
