The Tree Property

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König’s Lemma. Every finitely branching infinite tree has an infinite path.
What about taller trees?
$\aleph_1$ is the first uncountable cardinal.

An $\aleph_1$-tree is a tree of height $\aleph_1$ with countable levels.

**Thm.** [Aronszajn] There is an $\aleph_1$-tree with no path of length $\aleph_1$.

Such a tree is called an $\aleph_1$-Aronszajn tree. An $\aleph_1$-Aronszajn tree can be constructed from the rationals, $\mathbb{Q}$. 
What about trees of height greater than $\aleph_1$?
$\aleph_2$ is the second uncountable cardinal.

**Def.** A tree $T$ is an $\aleph_2$-tree if $T$ has height $\aleph_2$ and every level of $T$ has cardinality less than $\aleph_2$.

**Def.** An $\aleph_2$-tree is an $\aleph_2$-Aronszajn tree if it has no paths of length $\aleph_2$. 
$\aleph_2$ is the second uncountable cardinal.

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**Def.** An $\aleph_2$-tree is an $\aleph_2$-Aronszajn tree if it has no paths of length $\aleph_2$.

**Thm.** [Silver 71] If there is no $\aleph_2$-Aronszajn tree, then $\aleph_2$ is weakly compact in $L$.

So “large cardinals” become necessary even at this low level.
Some Elements of Set Theory

**ZFC:** Standard axioms of Set Theory

**Large Cardinals:** Extra Axioms to be used in addition to ZFC. Large cardinals are used to decide some statements which are neither provable nor disprovable from ZFC

**Model of ZFC:** A Universe (collection of sets) in which the axioms of ZFC are all true

**Gödel’s Constructible Hierarchy** $L$: All sets constructible from ordinals by set operations. $L$ is a model of ZFC $+$ CH.
Hierarchy of relevant Large Cardinals

supercompact

\[ \Downarrow \]

weakly compact hypermeasurable

\[ \Downarrow \]

measurable
\[ (\exists \text{ a } \kappa\text{-complete ultrafilter on } \kappa) \]

\[ \Downarrow \]

weakly compact

\[ \Downarrow \]

strongly inaccessible
\[ (|\mathcal{P}(\alpha)| < \kappa, \text{ for all } \alpha < \kappa) \]
Forcing

A method for constructing a new universe of ZFC from a given universe of ZFC and an oracle.

Ingredients: 1. A universe $V$ of ZFC,
2. a partial ordering $P$,
3. a ‘generic’ set $G \subseteq P$ which meets every dense subset of $P$.

- $G$ acts as an oracle to give information to build new sets.
- $V[G]$ is the class of sets constructible from $G$ along with sets in $V$, using set operations.
Equiconsistency

Statements A and B are said to be *equiconsistent* if there is a universe (of ZFC) in which A holds iff there is a universe (of ZFC) in which B holds.

Example. ‘There is no $\aleph_2$-Aronszajn tree’ is equiconsistent with ‘There is a weakly compact cardinal’.
The Tree Property

Let $\kappa$ be an infinite cardinal.

Def. A $\kappa$-tree is a tree $T$ of height $\kappa$ such that every level of $T$ has size less than $\kappa$.

Def. A tree $T$ is a $\kappa$-Aronszajn tree if $T$ is a $\kappa$-tree which has no cofinal branches.

Def. We say that the tree property holds at $\kappa$, or $\text{TP}(\kappa)$ holds, if every $\kappa$-tree has a cofinal branch through it.

• $\text{TP}(\kappa)$ holds iff there is no $\kappa$-Aronszajn tree.
Results on the Tree Property

[König 27] TP(\aleph_0) holds: Every finitely branching tree of height \omega has a cofinal branch; i.e. there is no \aleph_0-Aronszajn tree.

[Aronszajn 34] TP(\aleph_1) fails: There is an \aleph_1-Aronszajn tree.

[Specker 51] If \kappa^{<\kappa} = \kappa, then there is a \kappa^+-Aronszajn tree.

• Hence, if TP(\kappa^{++}) holds, then 2^\kappa must be at least \kappa^{++}. 
[Silver 71] TP(\(\kappa\)) for \(\kappa > \aleph_0\) implies \(\kappa\) is weakly compact in \(L\).

**Thm.** The following are equivalent:

1. \(\kappa\) is weakly compact;

2. Any collection of \(L_{\kappa \kappa}\) sentences using at most \(\kappa\) non-logical symbols, if \(\kappa\)-satisfiable, is satisfiable;

3. \(\kappa\) is strongly inaccessible and \(\text{TP}(\kappa)\);

4. \(\kappa\) is strongly inaccessible and for every transitive model \(M\) of \(ZF^-\) such that \(\kappa \in M\), \(M\) is \(< \kappa\)-closed and \(|M| = \kappa\), there is an elementary embedding \(j : M \rightarrow N\), \(N\) transitive, with \(\text{crit}(j) = \kappa\);

5. \(\kappa \rightarrow (\kappa)^2\).
[Mitchell 72]: Given a weakly compact cardinal $\lambda$ above a regular cardinal $\kappa \geq \aleph_1$, one can collapse $\lambda$ to obtain $\text{TP}(\kappa^+)$. 

[Baumgartner/Laver 79]: Countable support iterated Sacks forcing of weakly compact length forces $\text{TP}(\aleph_2)$.

[Kanamori 80]: Assume $\lambda > \kappa$, where $\lambda$ is weakly compact, $\kappa$ is regular, and $\diamondsuit_\kappa$ holds. Then $\lambda$-length iterated Sacks($\kappa$) with $\kappa$ support forces $\text{TP}(\kappa^{++})$.

[Abraham 83]: Assuming the existence of a supercompact cardinal and a weakly compact cardinal above it, there is a generic extension where there are no Aronszajn trees of height $\aleph_2$ or $\aleph_3$. 
[Magidor/Shelah 96]: If a singular cardinal is the limit of strongly compact cardinals, then its successor has the TP. If it is consistent that sufficiently large cardinal numbers exist, then it is consistent that ZFC holds and $\text{TP}(\aleph_{\omega+1})$.

[Cummings/Foreman 98]: If ZFC + ‘there exist infinitely many supercompact cardinals’ is consistent, then ZFC + ‘$\text{TP}(\aleph_n)$ for all $2 \leq n < \omega$’ is also consistent. If ZFC + ‘there exists a supercompact cardinal with a weakly compact cardinal above it’ is consistent then also ZFC + ‘there exists a strong limit cardinal $\kappa$ of cofinality $\omega$ such that $\text{TP}((\kappa)^{++})$’ is consistent.

[Foreman/Magidor/Schindler 2001]: A lower bound on the consistency strength of the TP holding at all $\aleph_n$, $n \geq 2$. 

Results of Dobrinen and Friedman

**Main Theorem.** The following are equiconsistent:
1. $\kappa$ is measurable and $\text{TP}(\kappa^{++})$ holds.
2. $\kappa$ is weakly compact hypermeasurable.

**Cor.** The following are almost equiconsistent:
1. $\kappa$ is measurable and there is no special Aronszajn tree on $\kappa^{++}$.
2. $\kappa$ is Mahlo hypermeasurable.
Thm. If $V$ has a weakly compact hypermeasurable cardinal $\kappa$ and a measurable cardinal $\mu$ sufficiently large above $\kappa$, then there is an inner model of $V$ in which there is a proper class of hypermeasurable cardinals, and in which the tree property holds at the double successor of each strongly inaccessible cardinal.

Thm. If $0^\#$ exists, then there is an inner model in which the tree property holds at the double successor of each strongly inaccessible cardinal.
We give a sketch of the proof of the Main Theorem.
Let $\kappa$ be strongly inaccessible.

$\text{Sacks}(\kappa) =$ set of all perfect subtrees of $2^\kappa$ with splitting levels on a club subset of $\kappa$.

**Fusion**

For $\alpha < \kappa$, $q \leq_\alpha p$ iff $q$ and $p$ agree on all splitting levels below the $\alpha$-th splitting level.

Given a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ such that for each $\alpha < \kappa$, $p_{\alpha+1} \leq_\alpha p_\alpha$, then $\bigcap\{p_\alpha : \alpha < \kappa\} \in \text{Sacks}(\kappa)$. 
The Forcing Sacks($\kappa, \lambda$)

$\kappa < \lambda$, $\kappa$ strongly inaccessible, $\lambda$ weakly compact. Sacks($\kappa, \lambda$) is the $\lambda$ length iteration of Sacks($\kappa$) with supports of size $\kappa$.

Fusion for Sacks($\kappa, \lambda$)

For $\alpha < \kappa$, $X_\alpha \in [\lambda]^{<\kappa}$, say $q \leq_{\alpha, X_\alpha} p$ iff for each $i \in X_\alpha$, $q \upharpoonright i \Vdash q(i) \leq_{\alpha} p(i)$.

Given a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ in Sacks($\kappa, \lambda$) such that for each $\alpha < \kappa$, $p_{\alpha+1} \leq_{\alpha, X_\alpha} p_\alpha$, where $\bigcup\{X_\alpha : \alpha < \kappa\} = \bigcup\{\text{supp}(p_\alpha) : \alpha < \kappa\}$; Then $\bigcap\{p_\alpha : \alpha < \kappa\} \in \text{Sacks}(\kappa, \lambda)$. 
Properties of Sacks($\kappa, \lambda$).

**Fact.** ($2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$) Let $\kappa_0 \leq \kappa < \lambda$ where $\kappa$ is regular and $\lambda$ is weakly compact (or just $\geq \kappa^{++}$). Then Sacks($\kappa, \lambda$)

1. is $< \kappa$-closed;
2. satisfies generalized $\kappa$-fusion;
3. is $\lambda$-c.c.; preserves all cardinals $\leq \kappa^+$;
4. collapses $\lambda$ to $\kappa^{++}$;
5. blows up $2^\kappa$ to $\kappa^{++}$. 
Lemma. [Kanamori 80, Baumgartner/Laver 79] Assume GCH in $V$. Let $\kappa$ be a strongly inaccessible cardinal and $\lambda$ be a weakly compact cardinal above $\kappa$. Let $G$ be $\text{Sacks}(\kappa, \lambda)$ generic over $V$. Then in $V[G]$, $\text{TP}(\kappa^{++})$. 
Upper Bound on the Consistency Strength of $\text{TP}(\kappa^{++})$ for $\kappa$ measurable.

• In order to have the tree property at the double successor of a measurable cardinal $\kappa$, GCH must fail at $\kappa$. By results of [Gitik 93], this requires at least a weakly compact hypermeasurable cardinal.

• A supercompact with a weakly compact above it suffices to obtain $\text{TP}(\kappa^{++})$ where $\kappa$ is measurable.
**Thm.** [D./F. 07] Suppose GCH holds and $\kappa$ is a weakly compact hypermeasurable cardinal in $V$. Then there is a forcing extension $V[G]$ of $V$ in which $\kappa$ is still measurable and TP($\kappa^{++}$) (i.e. there is no $\kappa^{++}$-Aronszajn tree).

**Def.** Let $\kappa$ be a strongly inaccessible cardinal. We say that $\kappa$ is **weakly compact hypermeasurable** if there is a weakly compact cardinal $\lambda > \kappa$ and an elementary embedding $j : V \rightarrow M$ such that $\kappa = \text{crit}(j)$ and $(H(\lambda))^V = (H(\lambda))^M$. 
Ideas of the Proof

Let $\kappa$ be a weakly compact hypermeasurable cardinal and $\lambda$ be the least weakly compact above $\kappa$.

- The Forcing: $\mathbb{P}$ is the $\kappa + 1$ length reverse Easton iteration of $\text{Sacks}(\rho_i, \lambda_i)$’s where all $\rho_i$ are inaccessible, all $\lambda_i$ are weakly compact, and $\rho_0 < \lambda_0 < \rho_1 < \lambda_1 < \cdots$

Notation: $\mathbb{P} = \mathbb{P}_\kappa \ast \dot{\mathbb{Q}}_\kappa$, where $\mathbb{P}_\kappa$ denotes the iteration of length $\kappa$, and $\dot{\mathbb{Q}}_\kappa$ denotes a $\mathbb{P}_\kappa$-name for $\text{Sacks}(\kappa, \lambda)$.

Let $G$ be generic for $\mathbb{P}_\kappa$ over $V$, and let $g$ be generic for $\dot{\mathbb{Q}}_\kappa^G$ over $V[G]$. i.e. $G \ast g$ is $\mathbb{P}$-generic over $V$.

- By the Lemma, $\text{TP}(\kappa^{++})$ holds in $V[G][g]$. 
The Hard Part: Show that $\kappa$ remains measurable in $V[G][g]$.

Let $j : V \to M$ be an elementary embedding witnessing the weakly compact hypermeasurability of $\kappa$.

Without loss of generality, we assume $M = \{ j(f)(a) \mid f \in V, f : H(\kappa) \to V, \text{ and } a \in H(\lambda) \}$, and $V$ and $M$ have the same $H(\lambda)$.

• Find a suitable generic $I$ for $j(\mathbb{P})$ over $M$ so that $j : V \to M$ lifts to $j^* : V[G][g] \to M[I]$. 
The Generic for $j(\mathbb{P})$ over $M$.

$j(\mathbb{P})$ is the $j(\kappa) + 1$ length iteration of $\text{Sacks}(\rho, \lambda_\rho)$ in $M$ with $\leq j(\kappa)$ size supports.

$$j(\mathbb{P}) = P_\kappa \ast Q_\kappa \ast R \ast Q_j^*(\kappa).$$

The generic $I$ will have the form $G \ast g \ast H \ast h$.

An argument similar to [Friedman/Thompson 07] gives a generic $H$ for $\mathbb{R} = j(\mathbb{P}) \upharpoonright [\kappa + 1, j(\kappa))$ over $M[G][g]$ such that $j[G] \subseteq G \ast g \ast H$.

Thus $j$ lifts to $j^* : V[G] \to M[G][g][H]$.

Let $M^*$ denote $M[G][g][H]$.

**Last Step:** Find a generic $h$ for $Q_{j(\kappa)}^* = \text{Sacks}(j(\kappa), j(\lambda))$ over $M^*$. 
The real work begins.

- Tuning Forks: A curious property of product Sacks forcings realized by Friedman and Thompson.

We want to use the Tuning Fork Lemma of [Friedman/Thompson 07] to obtain a generic $h$ for $\mathbb{Q}^*_{\kappa}(j)$ over $M^*$ such that $h$ lives in $V[G][g]$. However, we are working with an iteration, not a product. The analogous Tuning Fork Lemma for the iteration only yields names for branches through $2^j(\kappa)$. Our challenge is to pin them down to being actual ground model (i.e. $M^*$) objects up to specified heights. This will allow us to prove genericity.
Definition of $h$

Let $\bar{i} < \lambda$. Let $\dot{x}(\bar{i})$ denote the $\mathbb{Q}_\kappa \upharpoonright i$ name $\bigcap \{p(i) : p \in g\}$ in $V[G]$. Let $i = j(\bar{i})$. Let $\bar{x}(i)$ denote the element of $2^\kappa$ which $g \upharpoonright i$ forces $\dot{x}(i)$ to be. Let $r_i$ be the condition in $\mathbb{Q}^*_j(\kappa)$ which is trivial everywhere except on index $i$. $r_i(i)$ is the tree with stem $\bar{x}(\bar{i}) \upharpoonright 0$, where $j(\bar{i}) = i$.

Let $h$ be the filter generated by

$$j^*[g] \cup \{\bigcap \{r_i : i \in J\} : J \subseteq j(\lambda) \cap \text{ran}(j), \ |J| = j(\kappa)\}.$$ 

• Goal: Show that $h$ is $\mathbb{Q}$-generic over $M^*$. 
Two steps to show that $h$ is generic. Let $D$ be a dense subset of Sacks($j(\kappa), j(\lambda)$) in $M^*$.

1. Find a $q \in h$ such that there is an $S \subseteq j(\lambda)$ of size less than $j(\kappa)$ and an $\alpha < j(\kappa)$ such that any $(S, i)$-thinning of $q$ is in $D$.

2. Find an $r \in h$ such that $r \leq q$ and $r$ has enough of itself determined in the ground model that it can give concrete branches through which to thin $q$. 
Lower Bound on the Consistency Strength of $\text{TP}(\kappa^{++})$ for $\kappa$ measurable

**Thm.** [Silver 71] If a regular cardinal $\kappa$ has the tree property in $V$ then it also has the tree property in $L$.

**Thm.** [Dobrinen/Friedman 07] Suppose that 0-pistol does not exist and let $K$ denote the core model. If an ordinal $\kappa$ of uncountable cofinality is inaccessible in $K$ and $T \in K$ is a $\kappa$-tree, then there is another $\kappa$-tree $T^*$ in $K$ such that $T^*$ has a $\kappa$-branch in $V$ iff $T$ has a $\kappa$-branch in $K$.

**Proof Idea.** A node of $T^*$ consists of a triple $(M, \alpha, b)$ in $K$ such that $M$ is a mouse which agrees with $K$ below $\alpha < \kappa$, $b \in M$ is a branch through $T$ of length at least $\alpha$ and $M$ is the $\Sigma_1$ Skolem hull of $\alpha \cup \{b\}$. A node $(M_1, \alpha_1, b_1)$ extends another node $(M_0, \alpha_0, b_0)$ of $T^*$ iff there is a (unique) $\Sigma_1$ elementary embedding of $M_0$ into $M_1$ which is the identity on $\alpha_0$ and sends $b_0$ to $b_1$. 
Cor. [Dobrinen/Friedman 07] Suppose that 0-pistol does not exist and let $K$ be the core model. Then for any regular cardinal $\kappa$, $\text{TP}(\kappa)$ implies $\kappa$ is weakly compact in $K$. 
**Thm.** [Dobrinen/Friedman 07] Let $\kappa$ be measurable. If $\text{TP}(\kappa^{++})$ holds, then there is an inner model in which $o^K(\kappa) \geq \lambda + 1$ for some weakly compact $\lambda > \kappa$ in that inner model.

**Proof.** If 0-pistol exists, then by iterating it through the ordinals, we obtain an inner model with a strong cardinal together with a proper class of weakly compact cardinals. In particular, this cardinal is weakly compact hypermeasurable.

If 0-pistol does not exist, then the core model $K$ exists. By Gitik, $o^K(\kappa) \geq \kappa^{++}$. By the Corollary, $\kappa^{++}$ is weakly compact in $K$. Therefore, $o^K(\kappa) \geq \lambda$ for $\lambda$ a weakly compact cardinal in $K$ above $\kappa$.

Thus, in either case, we obtain the consistency of a measurable cardinal $\kappa$ with $o^K(\kappa)$ at least the next weakly compact cardinal above $\kappa$. $o^K(\kappa)$ cannot be a weakly compact cardinal (thanks Joel Hamkins). Hence, actually $o^K(\kappa) \geq \lambda + 1$, where $\lambda$ is the least weakly compact cardinal above $\kappa$. 

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Thm. If \( V \) has a weakly compact hypermeasurable cardinal \( \kappa \) and a measurable cardinal \( \mu \) sufficiently large above \( \kappa \), then there is an inner model of \( V \) in which there is a proper class of hypermeasurable cardinals, and in which the tree property holds at the double successor of each strongly inaccessible cardinal.

Thm. If 0\# exists, then there is an inner model in which the tree property holds at the double successor of each strongly inaccessible cardinal.