Hyperbolic Graphs of Surface Groups

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Notations

Let $S$ be a closed hyperbolic surface. Let $T(S)$ be the Teichmuller space of $S$. Let $PML$ be the projective measured lamination space of $S$. Let $MCG(S)$ be the mapping class group of $S$.

Let $\phi : S \to S$ be a pseudo-Anosov homeomorphism, and let $\Phi \in MCG(S)$ be the mapping class of $\phi$. Let $\Lambda^s, \Lambda^u \in PML$ be the stable and unstable measured geodesic laminations of $\Phi$.

Let $H$ be a subgroup of a group $G$. The subgroup $VC(H) = \{ g \in G : \exists$ finite index subgroup $H' < H$, s.t., $ghg^{-1} = h, \forall h \in H' \}$ is called the virtual centralizer of $H$.

Let $\Lambda^s, \Lambda^u \subset PML$ be the fixed points of a pseudo-Anosov mapping class $\Phi$, and let $Fix\{ \Lambda^s, \Lambda^u \}$ denote the subgroup in $MCG(S)$ whose elements fix $\Lambda^s, \Lambda^u$ point wise.

It is well-known that $Fix\{ \Lambda^s, \Lambda^u \} = VC(\Phi)$. 

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Combination Theorem

The fundamental group of the mapping torus of a pseudo-Anosov homeomorphism of an oriented closed hyperbolic surface is hyperbolic. This was first proved by Thurston. A direct proof was given by Bestvina and Feighn. The Combination Theorem in their paper is an important tool to prove hyperbolicity of a graph of spaces.

Theorem 1 (Bestvina and Feighn, Combination Theorem). Let $S\Gamma$ be a finite graph of spaces with hyperbolic fundamental groups. Suppose that $S\Gamma$ satisfies the quasi-isometrically embedded condition and the hallways flare condition, then the fundamental group of $S\Gamma$ is hyperbolic.
A Hyperbolic-by-Hyperbolic hyperbolic group

Theorem 2 (Mosher). Let $S$ be an oriented closed hyperbolic surface, let $\Phi_1, \ldots, \Phi_m \in MCG(S)$ be an independent set of pseudo-Anosov mapping classes of $S$, and let $\phi_1, \ldots, \phi_m \in Homeo(S)$ be the pseudo-Anosov representatives of $\Phi_1, \ldots, \Phi_m$ respectively. If $i_1, \ldots, i_m$ are large enough positive integers, then the fundamental group of the graph of spaces $\mathcal{K}$, as shown in Figure 1, is a hyperbolic group.

![Figure 1](image-url)
By saying a set $B$ of pseudo-Anosov mapping classes is \textit{independent}, we mean the sets $Fix(\Phi)$ are pairwise disjoint for $\Phi \in B$, where $Fix(\Phi)$ consists of the attractor and the repeller of $\Phi$ on the space of projective measured laminations $\mathbf{PML}(S)$.

Mosher applied the Combination Theorem as follows. Let $B$ be a set of $m$ independent pseudo-Anosov mapping classes, then for sufficiently large integers $i_1, \cdots, i_m$, the property ‘$2m - 1$ out of $2m$ stretch’ holds for $\mathcal{K}$. This implies that the hallways flare condition holds.

Let $\phi_i$ be the pseudo-Anosov homeomorphism representatives of $\Phi_i$. We say the property ‘$2m - 1$ out of $2m$ stretch’ holds if for any $\lambda > 1$, there exist sufficiently large integers $i_1, \cdots, i_m$, such that for any closed geodesic path $\gamma$ on $S$, $\gamma$ must be stretched by a factor $\lambda$, by at least $2m - 1$ out of $2m$ elements $\{\phi_1^{i_1}, \phi_1^{-i_1}, \cdots, \phi_m^{i_m}, \phi_m^{-i_m}\}$. 


Hyperbolic graphs of surface groups

A particular case studied here is: let $G_{\phi^n}$ be a graph of spaces as in Figure 2, where $S$ and $F$ are oriented closed hyperbolic surfaces, $p, q : S \to F$ are degree $r$ covering maps, and $\phi$ is a pseudo-Anosov homeomorphism of $S$. We shall find sufficient conditions under which the fundament group of $G_{\phi^n}$ is hyperbolic.
Let $S_1$ and $S_2$ be two copies of $S$ equipped with the pullback metrics by the covering maps $p$ and $q$ respectively.

There exist the derivative maps $D_p : PS_1 \rightarrow PF$ and $D_q : PS_2 \rightarrow PF$ of covering maps $p$ and $q$ respectively, where $PS_1$, $PS_2$ and $PF$ are the projective tangent bundles of $S_1$, $S_2$ and $F$ respectively.

Let $\Lambda^s$ and $\Lambda^u$ denote the stable and unstable geodesic laminations of $\phi$. Let $T\Lambda^s$ and $T\Lambda^u$ be the unit tangent vector spaces of $\Lambda^s$ and $\Lambda^u$ respectively.
**Theorem 3.** If $D_p|T\Lambda^s$ and $D_q|T\Lambda^u$ are injections, and their images are disjoint compact subsets of $PF$, then $\pi_1(\mathcal{G}_{\phi^n})$ is a hyperbolic group, when $n$ is large enough.

More precisely, the hypothesis of the above theorem says the following. Let $z$ be a point on $F$, let $x_i$ and $y_j$ be preimages of $z$ under the map $p, q$ respectively, that is, $x_i \in p^{-1}(z), y_j \in q^{-1}(z)$, where $i, j \in \{1, \cdots, r\}$. If $x_i, x_j \in \Lambda^s$, let $L_i, L_j$ be the tangent lines of $\Lambda^s$ at $x_i, x_j$ respectively. Similarly, if $y_i, y_j \in \Lambda^u$, let $V_i, V_j$ be the tangent lines of $\Lambda^u$ at $y_i, y_j$ respectively.

- For each $i, j \in \{1, \cdots, r\}$, $D_p(L_i) \neq D_q(V_j)$
- For each $i \neq j \in \{1, \cdots, r\}$, $D_p(L_i) \neq D_p(L_j)$
- For each $i \neq j \in \{1, \cdots, r\}$, $D_q(V_i) \neq D_q(V_j)$
Outline of the proof of Theorem 3

1. Let \( \lambda > 1 \). Given \( \phi \) and \( 0 < \epsilon < 1 \), there exist \( L > 0 \) and a positive integer \( N \), for any geodesic segment \( \alpha \subset S \), if \( |\alpha| > L \) and \( \alpha \notin N_\epsilon(\Lambda^s) \), then \( |\phi^n(\alpha)| > \lambda |\alpha| \) for \( n > N \).

\( N_\epsilon(\Lambda^s) \) denote all the hyperbolic geodesic segments \( \beta \subset S \) which are in the \( \epsilon \) neighborhood of the stabel geodesic lamination \( \Lambda^s \).

Define a hyperbolic geodesic segment \( \beta \) is in the \( \epsilon\text{-neighborhood} \) of a geodesic lamination \( \Lambda \), if on a subset of \( \beta \) of length at least \( (1 - \epsilon)\text{Length}(\beta) \), the distance between the tangent line of \( \beta \) and the set \( \Lambda \), measured in \( PS \), is at most \( \epsilon \).
2. We claim that under the hypothesis of Theorem 3, for any geodesic segment $\gamma \subset F$, if $\gamma$ is long enough, then at least $2r - 1$ out of $2r$ elements:

$$\{|q(\phi^n p_{1}^{-1} (\gamma))|, \cdots, |q(\phi^n p_{r}^{-1} (\gamma))|, |p(\phi^{-n} q_{1}^{-1} (\gamma))|, \cdots, |p(\phi^{-n} q_{r}^{-1} (\gamma))|\}$$

are not less than $\lambda|\gamma|$, when $n$ is large enough.
Applications

Corollary 4. Let $G, H$ be finite subgroups of $\text{MCG}(S)$, and let $\Phi \in \text{MCG}(S)$ be a pseudo-Anosov mapping class. If the virtual centralizer of $\langle \Phi \rangle$ has trivial intersection with $G$ and $H$, then $\langle G, \Phi^n H \Phi^{-n} \rangle$ is a free product in $\text{MCG}(S)$, i.e., $\langle G, \Phi^n H \Phi^{-n} \rangle \cong G * \Phi^n H \Phi^{-n}$, and its extension group is hyperbolic, for sufficiently large $n$.

For any group homomorphism $Q \to \text{MCG}(S)$, we obtain a group $\Gamma$ and a commutative diagram of short exact sequences

\[
\begin{array}{cccccc}
1 & \to & \pi_1(S, x) & \to & \Gamma & \to & Q & \to & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \pi_1(S, x) & \to & \text{MCG}(S, x) & \to & \text{MCG}(S) & \to & 1
\end{array}
\]

(1)

$\Gamma$ is called the extension group of $Q$. 

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**Corollary 5.** Suppose the virtual centralizer of $\langle \Phi \rangle$ has trivial intersection with the deck transformation groups of $p$ and $q$. In addition, suppose there are simple closed curves $a \subset F$ and $c \subset S$, as shown in Figures 3 and 4, such that $p^{-1}(a) = c$, $c \subset q^{-1}(a)$, and $q^{-1}(a)$ is disconnected. Then $\pi_1(G_{\phi^n})$ is hyperbolic if $n$ is sufficiently large. There exists a pseudo-Anosov homeomorphism $\phi_0$ of $S$, such that $\pi_1(G_{\phi_0^n})$ is not abstractly commensurate to any surface-by-free group.
Outline of the proof of Corollary 5

- Recall that, groups $G$ and $H$ are called *abstractly commensurate*, if there exist finite index subgroups $G_1 < G$ and $H_1 < H$, so that $G_1$ is isomorphic to $H_1$. A group $G$ is called a *surface-by-free* group, if there exists a hyperbolic surface or a hyperbolic orbifold $S$, and a free group $K$, such that there exists a short exact sequence:

\[ 1 \to \pi_1(S) \to G \to K \to 1 \]

- Let $G$ denote a graph of surfaces as in Figure 5, where $S$,

\[ F, p \text{ and } q \] are as described in Corollary 5.

We claim that $\pi_1(G)$ is not abstractly commensurate to a surface-by-free group according to the following lemma.
Lemma 6. Suppose the edge group $\pi_1(S)$ of $\pi_1(G)$ contains two nested sequences of finite index normal subgroups $L_1 > L_2 > \cdots$ and $R_1 > R_2 > \cdots$ which are constructed inductively as follows:

1. $H_1 = p_*(\pi_1(S)) \cap q_*(\pi_1(S))$, $L_1 = p_*^{-1}(H_1)$, $R_1 = q_*^{-1}(H_1)$, $G_1 = L_1 \cap R_1$

2. $L_{i+1} = p_*^{-1}(H_{i+1})$, $R_{i+1} = q_*^{-1}(H_{i+1})$

3. $G_{i+1} = L_{i+1} \cap R_{i+1}$, $H_{i+1} = p_*(G_i) \cap q_*(G_i)$

IF $L_i \neq R_i$ for all $i$, then $\pi_1(G)$ is not abstractly commensurate to a surface-by-free group.

• We need to find a pseudo-Anosov mapping class which fixes $L_i$ and $R_i$ for all $i$.

First, we claim that if $\gamma$ is a simple closed curve in $S$, such that $[\gamma] \subset L_i$ for some $i$, then $(\tau_{\gamma})_*$ of the Dehn-twist $\tau_{\gamma}$ fixes $L_i$.

Second, notice that $[\alpha], [\beta] \in \cap_i (L_i \cap R_i)$.
Theorem 7 (Penner). Suppose that $C$ and $D$ are each disjointly embedded collections of essential simple closed curves (with no parallel components) in an oriented surface $F$ so that $C$ hits $D$ efficiently and $C \cup D$ fills $F$. Let $R(C^+, D^-)$ be the free semigroup generated by the Dehn twists $\{\tau_c^+: c \in C\} \cup \{\tau_d^{-1}: d \in D\}$. Each component map of the isotopy class of $\omega \in R(C^+, D^-)$ is either the identity or pseudo-Anosov, and the isotopy class of $\omega$ is itself pseudo-Anosov if each $\tau_c^+$ and $\tau_d^{-1}$ occur at least once in $\omega$.

• We need to show there exist disjointly embedded collections of essential simple closed curve $C = \alpha \cup \hat{\alpha}$, and $D = \beta \cup \hat{\beta}$, such that $C \cup D$ fills $S$. In addition, $[\alpha], [\hat{\alpha}], [\beta]$ and $[\hat{\beta}] \in \cap_i(L_i \cap R_i)$. 

Theorem 8 (Bogopolski, Kudryavtseva, Zieschang).

Let $S$ be a closed orientable surface and $g, h$ non-trivial elements of $\pi_1(S)$ both containing simple closed two-sided curves $\gamma$ and $\kappa$, resp. The group element $h$ belongs to the normal closure of $g$ if and only if $h$ is conjugate to $g^\epsilon$ or to $(gug^{-1}u^{-1})^\epsilon$, $\epsilon \in \{1, -1\}$; here $u$ is a homotopy class containing a simple closed curve $\mu$ which properly intersects $\gamma$ exactly once.

- We know that the separating curve $\alpha'$ as in Figure 3 represents an element in the normal closure of $\alpha$, called $N_\alpha$. Since $N_\alpha$ is a subgroup of $\cap_i(L_i \cap R_i)$, $[\alpha'], [\beta'] \in \cap_i(L_i \cap R_i)$. 

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• Let $\psi : S - \alpha \to S - \alpha$ be a pseudo-Anosov homeomorphism. Let $k$ be a large enough integer, such that $\psi^k(\alpha') \cup \beta$ fills $S - \alpha$. Let $\hat{\alpha} = \psi^k(\alpha')$.

• Check there exists $\phi_0$ constructed as the above, and the virtual centralizer of $\langle \Phi_0 \rangle$ has trivial intersection with the deck transformation groups of $p$ and $q$. 
References