Abstract
Epistemic logics are modal systems modeling the acquisition knowledge amongst agents in a either a static or evolving environment. The natural addition of a temporal component allows for a much richer investigation. Here we contrast one traditional account of TEL [4], with relative-time modal operators for “Tomorrow” and “Yesterday”, with T-SEL in which knowledge modalities are indexed by time-stamps [3]. The latter embeds into the former in many situations, notably in monotonic cases. When augmented with the justified common knowledge modality J [1], T-SEL systems provide a straightforward format for solutions to classic epistemic puzzles such as Muddy Child and Wise Men (Women). The embedding of T-SEL yields TEL solutions lacking from the literature.

[Logical Systems]

Temporal Epistemic Logic (TEL) with Justified Knowledge

Knowledge Modalities: \( K_i \varphi \) “agent i knows \( \varphi \)”

Temporal Modalities: \( \Box \varphi \) ”tomorrow \( \varphi \)”

\( \Diamond \varphi \) ”yesterday \( \varphi \)”

TEL = propositional logic & for \( i = 1, \ldots, n \)

Axioms

- \( K_i(\varphi \rightarrow \psi) \rightarrow (K_i \varphi \rightarrow K_i \psi) \)
- \( K_i \varphi \rightarrow \varphi \)
- \( K_i \varphi \rightarrow K_i K_i \varphi \)
- above axioms for \( J \)
- connection axiom: \( J \varphi \rightarrow K_i \varphi \)
- \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)
- \( \neg \psi \rightarrow \Box \neg \varphi \)
- \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)

Rules
- modus ponens
- \( K_i, J_i, \Box, \neg \), necessitation: \( \vdash \varphi \rightarrow \vdash K_i \varphi, J_i \varphi, \Box \varphi \)

Time-stamped Knowledge (T-SEL) with Justified Knowledge

Time-Stamping:

\( K_i^t \varphi \) “i knows \( \varphi \) at time \( t \)”

\( J_i^t \varphi \) “at time \( t \), \( \varphi \) is justified”

TEL = prop. logic & for \( i = 1, \ldots, n, t \in \mathbb{N} \)

Axioms

- \( K_i^t(\varphi \rightarrow \psi) \rightarrow (K_i^{t+1} \varphi \rightarrow K_i^{t+1} \psi) \), for all \( u \geq t \)
- \( K_i^t \varphi \rightarrow K_i^{t+1} \varphi \)
- \( K_i^t \varphi \rightarrow K_i^{t+1} \varphi \)
- \( \vdash \varphi \rightarrow \vdash K_i^t \varphi \)
- above axioms for \( J_i^t \)
- connection axiom: \( J_i^t \varphi \rightarrow K_i^t \varphi \)

Rules
- modus ponens
- \( K_i^t \) & \( J_i^t \) necessitation: \( \vdash \varphi \rightarrow \vdash K_i^t \varphi, J_i^t \varphi \)

Observations

Theorem 1. TEL is sound and complete for a class of discrete linear temporal models.

Our interpretation of \( \Box \) and \( \neg \) forces us to choose a discrete ordering for Time. For ease of comparison, we model time in both TEL and T-SEL as isomorphic to \( \mathbb{N} \).

Theorem 2. T-SEL is sound and complete for a class of Kripke models.

Proposition 1. Each Kripke model of T-SEL can be recast as a temporal model.

Lemma 1. In T-SEL knowledge is monotonic, i.e. TEL \( \vdash K_i^t \varphi \rightarrow K_i^{u+t} \varphi \) for all \( u \geq t \).

This means that the environment is stable over time. The only thing which changes over time is knowledge, which cannot decrease. Not all models of TEL are monotonic so for comparison with T-SEL we restrict to those which are by adding an axiom:

\( \varphi \rightarrow \Box \varphi \).

Lemma 2. In T-SEL there is perfect recall, i.e. TEL \( \vdash K_i^t \varphi \rightarrow K_i^s \varphi \) for all \( u, s \geq t \).

Definition 1. Let \( * : L_T \rightarrow L_{TEL} \) be a one-to-one translation of languages. (details omitted)

Theorem 3 (Embed). When TEL is monotonic TEL \( \vdash \varphi \rightarrow \text{TEL} \vdash \varphi^{*} \) and TEL \( \vdash \varphi \rightarrow \text{TEL} \vdash \varphi^{*} \).

Of course this theorem does not hold if TEL is not monotonic as it has a much more expressive language, i.e. there is no analogous way of expressing \( \Box \varphi \) within T-SEL.

[Epistemic Puzzles]

Muddy Children

Assume there are \( n > 1 \) children, some of them have muddy faces. Each child can see whether the others have mud on their foreheads, but no one has mentioned it. The father announces for all to hear, “at least one of you has a muddy forehead.” He asks, “Do you know whether your own forehead is dirty?” The kids answer simultaneously. The father then repeats his question.

How many rounds will it take till each child answers, “Yes, I know?”

WISE WOMEN

The queen puts a hat on three of her wise women while their eyes are closed. It is common knowledge that there are 3 red hats but only 2 white hats. The queen asks each of them in turn if they know the color of their own hat. The first says she doesn’t know; the second says says she also doesn’t know; then the third wise woman says that she knows.

a) What color is the third wise women’s hat?
b) If the third wise woman is blind, but it is common knowledge that the first two can see, can the third still determine the color of her hat?

T-SEL Solutions

For both puzzles there is a solution consisting of a formal derivation in T-SEL(KT) (weaker than T-SEL) from a specification which describes the initial state of the world.

For Muddy Children, the initial state at the father’s first declaration is given by

\[ (110) \land J_0(\text{K.A.O.}) \land J_0(\text{-000}) \]

For this example, three kids: \( a \) & \( b \) are dirty (“1”), \( c \) is clean (“0”). ‘K.A.O.’ is Knowing About Others — they can see the other’s foreheads, which we add as an axiom. Let \( p_i \) be the proposition that \( i \) is muddy.

If after \( t \) rounds of questioning kid \( b \) doesn’t know whether she’s dirty, her announcement “I don’t know” becomes common knowledge for the next round: \( J_i^{t+1}(\neg K_i^t p_b \land \neg K_i^t \neg p_b) \). In general, if there are \( m \) muddy kids, they will all know it by round \( m \) and the clean ones, on the following round.

For Wise Women, the initial situation is represented by \( J_0(\text{-000}) \land K.A.O. \) where \( K.A.O. \) is as above. Let \( K_i^t r_j \) be short for “at time \( t \), \( i \) knows whether her hat is red.” After the first and second wise women reply that they don’t know their own color, this situation is

\[ J_0(\text{-000} \land K.A.O.) \land J_1(\neg K_0^0 p_a) + J_2(\neg K_1^1 p_b) \]

from which the color of third wise woman’s hat can be derived. Inspecting the derivation shows that the answer to part b) of the puzzle is Yes.

Conclusions

TEL is well-known, with distinct modalities for time and knowledge while in T-SEL they are joined. Using time-stamped knowledge modalities is very well suited to modeling situations with stable ground facts and provides, via its embedding, solutions in TEL to classic problems of epistemic logic overlooked in temporal logic. The non-monotonic nature of TEL in general is part of it power to model diverse situations. In addition, the fact that TELf (\( K_i^t \varphi \rightarrow \varphi \)) leads to speculation that it would be especially suited to model Belief as well as knowledge.

References: