6. CIRCULAR MOTION; GRAVITATION.

Key words: Uniform Circular Motion, Period of rotation, Frequency, Centripetal Acceleration, Centripetal Force, Kepler’s Laws of Planetary Motion, Gravitation, Newton’s Law of Universal Gravitation, Gravitational Constant, Satellite Motion,

Now we will apply the Newton’s law of motion to the consideration of the circular motion of the objects. The circular motion is widely represented in the technological base of our civilization. It occurs also in the nature. Very important example is the motion of planets, including our own planet Earth, and satellites motion around the Earth and other planets. All these types of motion could be treated as the simple type of circular motion – the uniform circular motion.

6.1. Kinematics of the Uniform Circular Motion.

The Uniform Circular Motion is the motion of an object in a circle with constant speed. Suppose that the radius of the circle is $r$ and the speed of an object is $v$, then we can introduced some useful physical quantities that characterized the uniform circular motion. We introduce Period of rotation $T$. Period is the time of one rotation. It can be determined according to its definition. During time interval $T$, object, moving with the speed $v$, cover distance equals to circumference length $2 \pi r$. Therefore the period can be determined from the following equation:

$$T = \frac{2 \pi r}{v}$$

(6.1)

Unit in which we measure period is second (s). We also introduce physical quantity Frequency $f$. Frequency is the number of rotations per unit time. In SI system the unit of frequency is called hertz (after great German physicist H. Hertz). 1 hertz = 1 Hz = 1/s. The period and frequency are related by the equation:

$$f = \frac{1}{T}$$

(6.2)

Now we will consider the basic physical quantity of the uniform circular motion starting with velocity. We know that the speed of this motion is constant, but what about velocity? Speed is only the magnitude of velocity. Speed is the scalar, but velocity is vector quantity, it has the magnitude
(speed) and direction. Magnitude is always constant but direction of velocity all time changing. Change in velocity causes appearance of the acceleration. It does not matter what is changing during the motion magnitude of the velocity (speed) or direction of the velocity or both of them. Anyway the acceleration during uniform circular motion will not be zero. It could be shown that acceleration during the uniform circular motion always is directed to the center of the rotation. This acceleration is called **Centripetal Acceleration** $a_c$ (from seeking the center in Latin). The formula for the magnitude of the centripetal acceleration can be written as following:

$$a_c = \frac{v^2}{r} \quad (6.3)$$

It is not surprising that the $a_c$ depends on $v$ and $r$. The greater the speed $v$, the faster the velocity changes direction; the larger the radius, the slowly the velocity changes direction.

6.2. Dynamics of the Uniform Circular Motion.

Now we will apply the Newton’s Laws of motion for the consideration of the uniform circular motion. According to the 2\textsuperscript{nd} law, when nonzero net force is applied appears an acceleration directed in the same direction as the net force. From this statement, we can deduce that if there is acceleration, there is a nonzero net force that causes this acceleration appearance. This force is directed in the same direction as the acceleration. Therefore if there is the centripetal acceleration directed to the center of rotation, the net force causes this acceleration directed to the center of rotation. This force is called **Centripetal Force**. According to the 2\textsuperscript{nd} Newton’s law of motion we can write:

$$\sum F_R = m a_c \quad (6.4)$$

Combining (6.4) and (6.3), we finally get expression of the 2\textsuperscript{nd} **Newton’s law of motion for the Uniform Circular Motion**.

$$\sum F_R = m \frac{v^2}{r} \quad (6.5)$$

Centripetal force is not the new type force of nature. Actually different forces can that are directed to the center of rotation and keep an object on the circular path (the orbit) during the circular motion can play the role of the centripetal force. It could be the tension force, frictional force, and component of the normal force, gravity force. Below we will consider
examples of different forces that keep the object on its path around the center and participate in the appearance of the centripetal force. Solving corresponding problems, you should remember that problems are the problems in which Newton’s laws of motion should be solved. Therefore, we will use the same strategy that we used before in the Chapter 4: creation of the free body diagram for an object; writing the corresponding Newton’s laws of motion (in the case of the uniform circular motion this is the equation (6.5)); solution of these equations and so on.

EXAMPLE 6.1. **Revolving ball (tension force participates in the appearance of the centripetal force).**

A ball with the mass \( m = 0.300 \text{ kg} \) on the end of a string is revolved at constant speed \( v = 4.00 \text{ m/s} \) in a vertical circle with the radius \( R = 0.720 \text{ m} \). Calculate the tension in the string when the ball is (a) at the bottom of its path, (b) at the top of its path. (c) Determine the minimum speed the ball must have at the top of its arc so that the ball continues its moving in a circle.

\[
m = 0.300 \text{ kg} \\
v = 4.00 \text{ m/s} \\
r = 0.720 \text{ m}
\]

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(a) \( F_{Tb} \) -- ?
(b) \( F_{Tt} \) -- ?
(c) \( v_{\text{min}} \) -- ?
Fig. 6.1 Example 6.1.

(a). Free-body diagram for the object at the bottom of its path is shown in the bottom of the Fig. 6.1. There two forces acting on the object: gravity force mg directed down and tension force $F_{RB}$ directed up. We will choose the Y-axis directed up as positive. Then we will write the Newton’s 2nd law of motion for this case:
\[ \Sigma F_R = ma_{c} \rightarrow F_{RB} - mg = m \frac{v^2}{r} \rightarrow F_{Tb} = m \left( \frac{v^2}{r} + g \right) = 9.61 \text{ N} \]

(b). Free-body diagram for the object at the top of its path is shown in the top of the Fig. 6.1. There two forces acting on the object: gravity force \( mg \) directed down and tension force \( F_{Tb} \) directed down. We will choose the Y-axis directed down as positive. Then we will write the Newton’s 2nd law of motion for this case:

\[ \Sigma F_R = ma_R \rightarrow F_{Tt} + mg = m \frac{v^2}{r} \rightarrow F_{Tt} = m \left( \frac{v^2}{r} - g \right) \quad (6.6) \]

\[ F_{Tt} = 3.73 \text{ N} \]

(c). The ball continues its moving in a circle until the string will remain taut. It will remain taut as long as there is tension force in it that is until \( F_{Tt} > 0 \). But if the tension disappears (because \( v \) is too small) the ball will fall out of its circular path. Thus, the minimum speed will occur if \( F_{Tt} = 0 \). Substituting \( F_{Tt} \) the in relationship, we will get

\[ 0 = m \left( \frac{v_{\text{min}}^2}{r} - g \right) \rightarrow v_{\text{min}} = \sqrt{gr} = 2.66 \text{ m/s} \]

EXAMPLE 6.2. Car rounds curved part of the highway (frictional force as centripetal force). The car (\( m = 1000 \text{ kg} \)) rounds the curved part of a highway with the radius 50.0 m. What should be the maximum speed at which frictional force keeps the car at the curved path (a) at the dry conditions (\( \mu_s = 0.60 \)) and (b) at the icy condition (\( \mu_s = 0.25 \)). (c). Is the result independent of the mass of the car?

\( m = 1000 \text{ kg} \)
\( r = 50.0 \text{ m} \)

(a) \( \mu_s = 0.60, \ v_{\text{max, dry}} \) ?

(b) \( \mu_s = 0.25, \ v_{\text{max, icy}} \) ?

(c) Is the result independent of the \( m \)?
The free body diagram for this example is shown in Fig. 6.2. The following forces acting on the car are the force of gravity $mg$ is directed down, the normal force $F_N$ exerted by the road directed up and there is horizontally directed static friction force. Car is moving along the road but the static frictional force $F_f$ keep the car on the track. This force is directed to the center of rotation and there is no motion in this direction. We can choose vertical direction as the positive direction of Y-axis. The direction to the center of rotation can be chosen as the positive. The Newton’s 2\textsuperscript{nd} low of motion in components can be written as followings.

$$\Sigma F_x = m \ a \quad \rightarrow \quad \Sigma F_R = m \ a_r \rightarrow \quad F_f = m \ v_{\text{max}}/r \quad (6-6)$$
\[ \sum F_y = 0 \Rightarrow F_N - mg = 0 \Rightarrow F_N = mg \quad (6-7) \]

To diminish the number of unknowns, we can add the definition of the static frictional force and get the expression for this force

\[ F_{fs} = \mu_s F_N = \mu_s mg \quad (6-8) \]

Substituting (6-8) into (6-6) we will get \( m \frac{v^2}{r} = mg \). Mass is canceled and we will finally get expression for the maximal speed.

\[ v_{\text{max}} = \sqrt{\mu_s r g} \quad (6.9) \]

But what actually \( v_{\text{max}} \) means? If the velocity of a car will be \( v \leq v_{\text{max}} \), then the car will move along the highway. If the velocity of the car will be \( v > v_{\text{max}} \), then sufficient friction force cannot be applied and the car will skid out of circular path into a more nearly straight path. Substituting correspondent values from the condition, we will get:

(a) \( v_{\text{max dry}} = 17.1 \text{ m/s} \).
(b) \( v_{\text{max icy}} = 11.1 \text{ m/s} \).
(c) Mass is absent in the final expression (6.9), so the \( v_{\text{max}} \) is the same for all cars disregarding their mass.

Drive car carefully, pay attention to traffic signs specifying the maximal speed at the curved parts of the highways and take onto account weather conditions.

6.3. **Gravitation.**

The force of gravity attracts all objects with mass on the surface of the Earth. This force causes all objects to fall with the same acceleration – acceleration due to gravity \( g \) – at the same location on the Earth. We have seen that this motion (free fall motion) was analyzed by the Galileo. He had no real explanation of this fact. Newton was also thinking about the problem of gravity and used in the analysis his laws of motion. Since falling object is moving with acceleration, there must be the force acting on it (2\textsuperscript{nd} Newton’s Law). This is gravity force. This force is caused by the interaction with another object (3\textsuperscript{rd} Law). The force is directed in the same direction as the acceleration of the free fall – to the center of the Earth. Newton concluded that it must be the Earth exerts the gravitational force on objects on its surface. The next step made by Newton was described in the
Newton’s Apple Story.

This is the real event happened with Newton. He told his friend about what happened. Once upon a time, Isaac Newton was in a garden and noticed an apple drop from a tree. As a result of this observation, Newton has been struck with a sudden inspiration. He made an extremely important revolutionary conclusion. If gravity acts at the tops of trees, and even at the tops of mountains, then perhaps it acts all the way to the Moon.


Here, we should stress, that all scientist before Newton starting from Aristotle thought that terrestrial and celestial objects are governed by different laws of nature. Aristotle stated that Sun and planets are rotating around the Earth (geocentric model). The celestial objects are moving by rotation of crystal spheres. The motion of celestial objects is absolutely perfect, orbits are circular and speed of motion is constant. In this time, there was other idea in the ancient world. Aristarchus thought that the Earth is moving around the Sun. But this idea contradicted the observations of ancient scientists. If the Earth is moving around the Sun, The positions of the stars measured from the different points of orbit (for example, in the Summer and in the Winter) must be different. This phenomenon is called the Stellar Parallax. But, the measurements of star positions made by ancient scientist show no any measurable difference in the positions of the stars during different seasons. Stellar parallax does occur, but it is too small to detect with the unaided eye. (Actually the possibility to measure stellar parallax was achieved astronomers only in XIX century). For this reason, and because of the dominant influence of Aristotle ideas, Aristarchus model was not accepted. The ancient Greek scientist developed further Aristotle geocentric model (we can mention Hipparchus and especially Ptolemy). The basic aspects of a geocentric model explained most of the motions of the celestial objects by various geometric devices. The model was strong enough to survive for a long time – more than a thousand years. In new times, Copernicus introduced his heliocentric model in which Earth and other planets are rotating around the Sun. He supposed that orbits of the planets are circles and that the planets are moving along these circles with constant speed. We know now that heliocentric model is right model, however Copernicus predictions came out no better than those based on the Ptolemaic geocentric model. Why?
The explanations were found when German scientist Johannes Kepler processed experimental date about planetary motion of planets (especially the planet Mars) collected by Danish astronomer Tycho Brahe. Johannes Kepler derived from these data three empirical Laws.

**Kepler’s Laws of Planetary Motion.**

1st Law. The path of each planet about the Sun is an ellipse.
2nd Law. Each planets moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal periods of time.
3rd Law. The ratio of the square of the period of planet rotation around the Sun to the cube of its mean distance from the Sun is the same for all planets.

We can only imagine the titanic work done by Kepler to derive these results from the raw experimental data fixing positions of the planets.

It is clearly seen from these laws that orbits of planets are not circle, but ellipses (1st Law), and that planets speed of motion along their orbits is not constant (2nd Law). This explained why the prediction of Copernicus heliocentric model was not precise. Nobody could explain the Kepler’s Law of Planetary Motion. This work was done by Newton.

6.4. **Newton’s Law of Universal Gravitation.**

When Newton came to the revolutionary idea that the motion of the celestial object – the Moon is governed by the same laws as the terrestrial objects, he decided to apply to the Moon motion his laws of motion and compare the gravitational force acting on the Moon with the gravitational force acting on the object at the surface of the Earth. According to the 2nd law the force is proportional to the acceleration. Using astronomical data, Newton calculated the centripetal acceleration of the Moon. It turned out that acceleration of the Moon is $3600 = 60^2$ times smaller than acceleration due to gravity at the Earth surface. But the Moon is 60 times further from the center of the Earth than the object at the Earth surface. But, according to the 2nd Newton’s Law of motion, the force is proportional to the acceleration. Therefore Newton concluded that the gravitational force exerted by the Earth on any object decreases with the square of its distance r from the Earth’s center. It is understandable that the force of gravity depends not only on distance but also on the object’s mass. According to Newton’s third law, when the Earth exerts its gravitational force on the Moon, the Moon exerts an equal and opposite force on the Earth. Because of this symmetry, the magnitude of the
force of gravity must be proportional to both of the masses. Thus, Newton proposed the **Newton’s Law of Universal Gravitation:**

*Every particle in the universe attracts every other particle with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them. This force acts along the line joining the two particles.*

\[
F = G \frac{m_1 m_2}{r^2} \tag{6-10}
\]

Where \(m_1\) and \(m_2\) are the masses of the two particles, \(r\) is the distance between them, and \(G\) is a **Gravitational Constant** unknown in the time of Newton. It must be measured experimentally and has the same numerical value for all objects in the Universe. Strictly speaking the formula (6-10) could be applied to a point-like objects. When extended objects are small compared to the distance between them (as for the Earth – Sun system), little inaccuracy results from considering them as point-like particles. But how we can apply formula (6-10) to the case when objects could not be treated as point-like objects (for example the object at the surface of the Earth)? Newton showed for two uniform spheres, relationship (6-10) gives the correct force where \(r\) is the distance between their centers.

Force of gravity is comparatively small. For example, the electric force of interaction between two electron is \(10^{42}\) times smaller than force of force of the gravitational interaction between them. In 1798, over 100 years after Newton published his law, English physicist Henry Cavendish performed fine experiment that is treated as the landmark in the experimental Physics and confirmed The Newton’s law of universal gravitation. Cavendish measured \(F\), \(m_1\), \(m_2\), and \(r\) for two lead balls and determine the numerical value of the Universal Gravitational Constant. Its accepted value today is

\[
G = 6.67 \times 10^{-11} \text{ (N m}^2/\text{kg}^2) \tag{6-11}
\]

These results allowed to Cavendish to estimate the mass of the Earth. Let us consider the gravity force acting on the object with mass \(m\) at the surface of the Earth. We know that it could be written as \(F_g = mg\). But, according to the Newton’s Law of Universal gravitation (6-10), the same force can be written as following \(F_G = F = G \frac{m m_E}{(R_E)^2}\), where \(m_E\) is the mass of the Earth and \(R_E\) is the radius of the Earth. Because \(F_g = F_G\), we can write
$mg = G \left( m m_E \right) / (R_E)^2$  \hspace{1cm} (6-12)

Mass of the object is gone, and we can derive from relationship (6-12) expression for the mass of the Earth:

$$M_E = g \left( R_E \right)^2 / G$$  \hspace{1cm} (6-13)

Cavendish used this expression, value of $G$ and calculated the mass of the Earth. He published his results in the article with the title “I weighed the Earth”. It allows to calculate the average density of the Earth $\rho_E = m_E / V_E$, where $V_E$. It turned out that $\rho_E$ is much greater than the density of the rock. It was the hint that there is heavy iron core at the center of the Earth.

From equation (6-12) we can derive other important relationship:

$$g = G \left( m \right) / (R_E)^2$$  \hspace{1cm} (6-12)

We can understand now the mystery faced during study of Free Fall motion. All objects are falling with the same acceleration (acceleration due to gravity) at the same point of the Earth surface. Acceleration due to gravity $g$ on the surface of the earth has a little different value at different locations because the Earth is not a perfect sphere. The value of $G$ can vary locally because of the presents of irregularities and rocks of different densities. Geophysicists use the precise measurement of $g$ as part of their investigations into the structure of the Earth’s crust, and in mineral and oil exploration.

The equation (6-12) can be used to determine acceleration due to gravity on different celestial objects like planets and their moons. For example, if we put in (6-12) astronomical data for the Moon, we will see that the acceleration to the gravity at the Moon surface 6 times smaller than on the Earth. Now we can understand why American astronauts were able to make so high jumps at the surface of the Moon.

The **Satellite Motion**.

We apply the Newton’s Law of Universal gravitation to the satellite motion. It could be natural satellites (for example, Earth’s natural satellite the Moon) or artificial satellite. The possibility to launch artificial satellites was discussed by Newton. If it will be launch with high speed it can orbit the
Earth. But what about gravity force and Free Fall? In fact, a satellite is falling (accelerating toward the Earth), but its high tangential speed keeps it on the orbit. To describe the satellite motion we will use Newton’s 2nd Law of motion. The force at the left side of the equation is the gravitation force (6-10) acting on the satellite from the Earth. For simplicity, we consider the satellite that is moving along the circular orbit with constant speed (uniform circular motion).

\[ \Sigma F_R = m a_R \Rightarrow G \left( \frac{m m_E}{r^2} \right) = m \frac{v_{sat}^2}{r} \quad (6-13) \]

Where \( m \) is the mass of the satellite and \( v_{sat} \) is its speed, \( r \) is the radius of the orbit. From equation (6-13) we can derive equation for the satellite speed:

\[ v_{sat} = \sqrt{\frac{G m_E}{r}} \quad (6-14) \]

Analysis of the equation (6-14) shows that that we cannot choose the radius \( r \) of the orbit and the speed of the satellite \( v_{sat} \) independently. If we launch the satellite with the speed \( v_{sat} \), then the radius of the orbit \( r \) is determined. The motion of the satellite does not depend on its mass, because it does not appear in the equation (6-14).

EXAMPLE 6.3. The Hubble space telescope is orbiting at a height \( h \) of 596 km above the Earth surface and has a mass 11 000 kg. Determine: (a) the speed of the Hubble space telescope; (b) the period of its revolution around the Earth. We need some astronomical data: the mass of the Earth is \( m_E = 5.98 \times 10^{24} \) kg; the radius of the Earth is \( R_E = 6.38 \times 10^6 \) m.

\[ h = 596 \text{ km} = 5.96 \times 10^5 \text{ m} \]
\[ m_E = 5.98 \times 10^{24} \text{ kg} \]
\[ R_E = 6.38 \times 10^6 \text{ m} \]
\[ G = 6.67 \times 10^{-11} \text{ (N m}^2)/\text{(kg}^2) \]

(a) \( v_{sat} \)?
(b) \( T \)?

(a) Before using expression (6-14) we should take into account that the radius of the orbit, the height of the satellite, and the radius of the Earth are related:
\[ r = h + R_E \quad (6-15) \]

Then, substituting \( r \) into (6.14) we will get

\[ v_{\text{sat}} = \sqrt{G \frac{m_E}{(R_E + h)}} = 7.56 \times 10^3 \text{ m/s} \quad (6-16) \]

(b) We can use relationship (6-14) for determination the rotation period (time of one rotation) of the satellite combining (6-14) with (6.1).

\[ T = \frac{2 \pi r}{v} = \frac{(2 \pi r^3)^{\frac{2}{3}}}{\sqrt{G m_E}} = 5800 \text{ s} = 97 \text{ min} = 1.61 \text{ h} \quad (6-17) \]

Gravity and Construction of the Universe.

The introduction of the Law of gravitation (6.10) was not the explanation of the planetary motion yet. The Robert Hook the rival of Isaac Newton stated that he knew inverse square law for force of gravity before Newton. But Newton not only introduced the law of gravitation (6.10). He showed that this Law allows explaining Kepler’s Laws of planetary motion. To proof this he invented calculus. Therefore he was not only great physicist but also the great mathematician. (Many scientists believe that Isaac Newton was the greatest scientist ever). We could not demonstrated here how to get 1st Kepler’s Law from the Law of universal gravitation, but we will do this for 3rd Kepler’s Law. We will modify expression (6.17) for the case of the solar system. The period of rotation of the planet (natural satellite of the Sun) \( T_p \) can be found from the relationship

\[ T_p = \frac{(2 \pi r^3)^{\frac{2}{3}}}{\sqrt{G m_S}} \quad (6-18) \]

Where \( m_S \) is the mass of the Sun.

\[ T^2 / r^3 = \frac{(4 \pi^2)}{(G m_E)} \quad (6.18) \]

At the right side of the equation (6.18) are the same for all the planets of the solar system as was predicted by 3rd Kepler’s Law of planetary motion.

The Newton’s Law of Universal Gravitation explained all known at those time astronomical phenomena data. Mankind understood the construction of the universe. It was a great achievement of science. Later some discrepancy was found. The orbit of more distant from the Sun planet Uranus deviated from orbit predicted by the Newton’s Law of Universal Gravitation. The
English astronomer Adams and then French astronomer Leverrier explained these discrepancies by gravitational influence of unknown planet and predicted the position of this planet. German astronomer Galle really found this planet. It was called Neptune. A huge blue planet was discovered as said at the tip of the pen. It was a triumph of Newtonian mechanics. A little later, in 1865, the new astronomical data was collected that needed explanation: the orbit of the Mercury, the planet closest to the Sun, changes its position in space with time. It is extremely small but measurable change. But, there was any explanation of this phenomenon until 1916 when Albert Einstein introduced its General Theory of Relativity. In this theory, a new concept of gravity was suggested. According to Einstein, matter bends or curves space and thus controls the behavior of nearby bodies. Now we could not understand new phenomena discovered in the universe like black holes, neutron stars and so on without the General Relativity. This theory is both conceptually and mathematically difficult and far beyond the level of this course. However, it should be noted that in weak gravitational fields, Einstein’s theory reduces to Newton’s, so that everything that we have done so far, remains correct. It is only in strong gravitational fields, such as close to the Sun or other stars, that the differences become important.