RESEARCH STATEMENT
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My interests lie broadly in the field of Riemannian geometry and geometric analysis, especially applications of the theory of optimal transport to problems in these areas. As a graduate student at the CUNY, Graduate Center, my thesis examined the effect of nonnegative Ricci curvature and volume growth on the topology of Riemannian manifolds [M1]. Later, I extended these results to limit spaces of Riemannian manifolds [M2]. As an NSF postdoc at the University of Warwick, I continued to study Ricci curvature, focusing on recent developments defining Ricci curvature for non-smooth spaces. I also examined questions regarding isoperimetric inequalities in connection with optimal transport techniques. These projects are described in more detail below.

Ricci Curvature on Riemannian Manifolds.

An important aspect of Riemannian geometry is understanding the relationship between the topology of a Riemannian manifold and its curvature. Ricci curvature plays an important role in this endeavor and much work has been done to better understand the structure of manifolds with a lower bound on Ricci curvature.

Ricci Curvature and Volume Growth. In my thesis, I employed techniques developed by G. Perelman to describe the topology of manifolds with nonnegative Ricci curvature. Specifically, I determined precisely how the volume growth influences the homotopy groups of these manifolds.

From a geometric standpoint, Ricci curvature acts as a measure of volume distortion. Let \( M^n \) be a complete open Riemannian manifold with nonnegative Ricci curvature (i.e. \( \text{Ric} \geq 0 \)). The volume growth of \( M^n \) describes the limiting behavior of the ratio of the volume of balls in \( M^n \) to the volume of balls of the same radius in the model space \( \mathbb{R}^n \). Define

\[
\alpha_M := \lim_{r \to \infty} \frac{\text{vol}(B_p(r))}{\omega_n r^n}.
\]

By the Bishop-Gromov Volume Comparison Theorem, this volume ratio is a non-increasing function of \( r \) and \( \alpha_M \leq 1 \). When \( \alpha_M > 0 \), \( M^n \) is said to have large (or Euclidean) volume growth.

It was shown independently by Anderson [A] and Li [Li] that when \( M^n \) has large volume growth, the order of \( \pi_1(M^n) \) is bounded above by \( \frac{1}{\alpha_M} \). Thus, knowing the volume growth is bounded by \( 1/2 < \alpha_M \leq 1 \) implies \( \pi_1(M^n) = 0 \). More recently, in 1994, Perelman [P] showed that there exists a small constant \( \epsilon_n > 0 \), depending only on the dimension of the manifold, so that if \( 1 - \epsilon_n < \alpha_M \leq 1 \), then \( \pi_k(M^n) = 0 \) for all \( k \geq 0 \); thus, \( M^n \) is contractible. This was later improved by Cheeger-Colding [ChCo1] who showed that, with tighter restrictions on \( \epsilon_n \), \( M^n \) is actually \( C^{1,\alpha} \) diffeomorphic to \( \mathbb{R} \). Neither Perelman nor Cheeger-Colding investigated an actual value for \( \epsilon_n \). My thesis focuses on results that fit in between those of Anderson and Li and Perelman. I showed [M1] how volume growth affects intermediate \( k \)-th homotopy groups by determining precise constants \( c_{k,n} > 0 \) such that the following theorem holds.
Theorem 1. (M-, [M1]) Let $M^n$ be a $n$-dimensional, complete, open Riemannian manifold with $\text{Ric} \geq 0$. If
$$\lim_{r \to \infty} \frac{\text{vol}(B_p(r))}{\omega_n r^n} \geq c_{k,n},$$
then $\pi_k(M^n) = 0$.

The constants I have found are the strongest available using Perelman’s techniques when constructing homotopies. In [M2] I apply the theory of Gromov-Hausdorff limits of manifolds with non-negative Ricci curvature to extend the result of Theorem 1 to limit metric spaces satisfying similar bounds on their renormalized volume, as defined by Cheeger-Colding [ChCo1].

Theorem 2. (M-, [M2]) Let $(Y,p)$ be the pointed metric measure limit of a sequence of Riemannian manifolds $(M^n_i, p_i)$ with $\text{Ric}_{M^n_i} \geq 0$, and let $\nu$ denote the renormalized limit measure of $Y$. If
$$\lim_{r \to \infty} \frac{\nu(B_p(r))}{\omega r^n} > c_{k,n},$$
then $\pi_k(Y,p) = 0$.

Both Theorem 1 and Theorem 2 require two key applications of the Ricci curvature assumption: a prevalence of geodesics in the presence of large volume and the Abresch-Gromoll excess estimate [AbGr]. For Theorem 2, the Ricci curvature lower bound throughout the sequence guarantees the existence of long geodesics in the limit space, provided the volume growth of the limit satisfies the appropriate bound. After extending the excess estimate to long thin triangles formed from these limiting geodesics in the sequence, the same methods of Theorem 1 can be used to show the specific homotopy groups are trivial.

Another direction which follows naturally from these results would be to examine the other extreme of volume growth: manifolds with nonnegative Ricci curvature and minimal rather than maximal volume growth:

**Question 1:** Investigate the topology of a manifold with $\text{Ric} \geq 0$ and linear volume growth.

These spaces and their limits were studied by Sormani in [So1, So2] using Cheeger-Colding methods applied to the Busemann function. It is possible that Perelman’s techniques could also apply and control the geodesics of gradient curves of Busemann functions.

**Ricci Curvature on Non-Smooth Spaces.**

In the 1950’s Alexandrov introduced the notion of upper and lower curvature bounds for metric spaces which lack a Riemannian structure. Since then, Alexandrov spaces have become an important area of study in geometry in large part because such spaces can arise as the Gromov-Hausdorff limit of Riemannian manifolds with sectional curvature bounded below and diameter bounded above. Further research showed that many comparison results from Riemannian geometry hold for these general metric spaces with the bound on curvature taking the place of a sectional curvature bound. Due to the success and insights garnered from studying these metric spaces with generalized sectional curvature bounds, attention has also recently focused on generalized definitions of Ricci curvature bounds for spaces which also lack the structure of a Riemannian manifold. My current research focuses on questions in this area. I am interested
in understanding more clearly the effect a weak Ricci curvature bound has on the topology of the underlying metric space.

There has recently been an increased interest in definitions of Ricci curvature lower bounds for non-Riemannian spaces and many different, but related, definitions have surfaced in a short amount of time. We will begin by discussing definitions of Ricci curvature bounds for Alexandrov spaces and our results in this setting. We will also mention the work of Lott-Villani [LV] and Sturm [S1, S2] where the authors independently introduced the notion of Ricci curvature lower bound for metric measure spaces and prove a number of fundamental results. We conclude by outlining a number of questions we are currently investigating in this direction.

**Alexandrov Spaces.** In [KS] Kuwae and Shioya suggest an infinitesimal version of the Bishop-Gromov volume comparison theorem as a definition of a lower Ricci curvature bound for an Alexandrov space. Let \((X,d)\) be an Alexandrov space of dimension \(n \geq 2\). For a point \(p \in X\) and \(0 < t \leq 1\), define \(W_{p,t} \subset X\) and a map \(\Phi_{p,t} : W_{p,t} \rightarrow X\) by

\[
W_{p,t} = \{ x \in X \mid \exists y \in X \text{ s.t. } x \in py \text{ and } d(p,x) : d(p,y) = t : 1 \}
\]

and \(\Phi_{p,t}(x) = y\). Such a point \(y \in X\) is unique and thus \(\Phi_{p,t}\) is well-defined and even Lipschitz by the convexity of the underlying Alexandrov space.

**Condition BG(\(\kappa\)) at a point** \(p \in (X,d)\) is satisfied if

\[
d(\Phi_{p,t} \ast \mathcal{H}^n)(x) \geq \frac{ts_k(td(p,x))^{n-1}}{s_k(d(p,x))^{n-1}}d\mathcal{H}^n(x)
\]

for any \(x \in (X,d)\) and \(t \in (0,1]\) such that \(d(p,x) < \frac{\pi}{\sqrt{\kappa}}\) if \(\kappa > 0\). Here \(\kappa \in \mathbb{R}\) and \(\Phi_{p,t} \ast \mathcal{H}^n\) denotes the push-forward of the \(n\)-dimensional Hausdorff measure \(\mathcal{H}^n\) by \(\Phi_{p,t}\) and \(s_k(x)\) is the solution to the Jacobi equation \(s''(r) + \kappa s(r) = 0\) with initial condition \(s_k(0) = 0, s'_k(0) = 1\).

**Definition 3.** If \((X,d)\) satisfies BG(\(\kappa\)) at every point \(p \in X\), we say that \(X\) satisfies BG(\(\kappa\)).

The BG(\(\kappa\)) condition has also been referred to as the **measure contraction property** for metric measure spaces and is weaker than the curvature-dimension condition proposed by Lott-Villani and Sturm (more on the work of Lott-Villani and Sturm below). Note that an Alexandrov space of curvature \(\geq \kappa\) satisfies the BG(\(\kappa\)) condition, as would be expected from the Riemannian case. However, we do not expect that \((X,d)\) has curvature \(\geq \kappa\) in the Alexandrov sense; we only assume that the curvature \(\geq -k^2\) for some \(k\).

In the same paper [KS], the authors prove a (distributional) Laplacian comparison theorem analogous to what is expected for Riemannian manifolds with a lower Ricci curvature bound. When \(\partial X = \emptyset\), let \(X^*\) denote the set of non-\(\delta\)-singular points for \(0 < \delta << 1/n\). Any topological singularities of \(X\) are contained in \(X \setminus X^*\) and in fact there is a canonical Riemannian metric on \(X^*\) which is absolutely continuous and has at most the regularity of locally BV. Therefore, the Laplacian of a smooth function does not become a function, and is only a Radon measure in general.

Set \(\cot s_k(r) = s'_k(r)/s_k(r)\) and \(r_p(x) = d(p,x)\) for \(p, x \in X\). The distributional Laplacian \(\Delta r_p\) becomes a signed Radon measure on \(X^*\) and we have
Theorem 4. (Kuwae-Shioya, [KS]) Let \((X^n, d)\) be an Alexandrov space, \(n \geq 2\). If \(X\) satisfies \(BG(\kappa)\) at \(p \in X\), then on \(X \setminus (S_X \cup \{p\})\)
\[
d\Delta r_p \leq (n - 1) \text{cot}_{\kappa} \circ r_p \, d\mathcal{H}^n.
\]

Immediately a natural question arises,

Question 2. Does the converse to Theorem 4 hold as it does for smooth Riemannian manifolds?

Using the Laplacian comparison we can then prove an analog of the Abresch-Gromoll excess estimate \([AbGr]\) for Alexandrov spaces satisfying \(BG(\kappa)\). Recall that the excess function \(e_{p,q}(x)\) for points \(p,q \in (X,d)\) is defined by
\[
e_{p,q}(x) = d(p,x) + d(q,x) - d(p,q)
\]
and measures how much the triangle inequality in \((X,d)\) fails to be an equality. They show

Proposition 5. (Abresch-Gromoll, [AbGr]) Let \(M^n\) be a complete, Riemannian manifold \((n \geq 3)\) with \(\text{Ric} \geq 0\), then
\[
e_{p,q}(x) \leq 2 \cdot \frac{n - 1}{n - 2} \left( \frac{1}{2} \cdot C_n \cdot d(x, pq) \right)^{1/n-1},
\]
for some constant \(C_n > 0\) and where \(pq\) denotes the geodesic connecting \(p\) and \(q\).

The proof relies primarily on the Laplacian comparison afforded from the lower bound on Ricci curvature. Thus, for Alexandrov spaces \((X,d)\) satisfying \(BG(0)\) the proof follows directly that of Abresch-Gromoll with additional care taken because we have only a distributional Laplacian comparison as opposed to a strong Laplacian comparison as in the Riemannian case. We have

Proposition 6. (M-, [M3]) Let \((X^n, d)\) be an Alexandrov space \((n \geq 3)\) satisfying \(BG(0)\), then
\[
e_{p,q}(x) \leq 2 \cdot \frac{n - 1}{n - 2} \left( \frac{1}{2} \cdot C_n \cdot d(x, pq) \right)^{1/n-1},
\]
for some constant \(C_n > 0\) and where \(pq\) denotes the geodesic connecting \(p\) and \(q\).

A lower bound on the excess is one of the primary tools when studying Riemannian manifolds with a lower Ricci curvature bound and many important consequences arise from knowing such a lower bound exists. For example, as previously mentioned, the two primary applications of \(\text{Ric} \geq 0\) that we used in Theorem 1 and Theorem 2 were the Bishop-Gromov volume comparison (of course inherent in the \(BG(0)\) condition) and the excess estimate. Thus, one should expect that some analog of Perelman’s maximal volume theorem should hold as well for an Alexandrov space satisfying \(BG(0)\).

In some ways, defining Ricci curvature for an Alexandrov space provides a natural setting in which to extend classical results from Riemannian geometry stated for manifolds with, for example, nonnegative Ricci curvature and an arbitrary lower bound on sectional curvature. There are a number of such results in the literature \([E, P1, GP1, GP2]\) and, in fact, in the paper by Abresch-Gromoll introducing the excess function and its estimate \([AbGr]\) they show

Theorem 7. (Abresch-Gromoll, [AbGr]) Let \(M^n\) be a complete, Riemannian manifold with \(\text{Ric} \geq 0\). Suppose that \(M^n\) has diameter growth of order \(o(r^{1/n})\) and sectional curvature \(\geq -k^2\),

\[
\]
for some \( k \in \mathbb{R} \). Then \( M \) is homotopy equivalent to the interior of a compact manifold with boundary.

The proof of this theorem relies on a contradiction obtained on the excess of small, thin triangles in \( M^n \). The excess estimate of Proposition 5 guarantees the excess is not too big while assuming a critical point of the distance function, an arbitrary lower sectional curvature bound, and a bound on the diameter forces the excess to not be too small. Ultimately, one obtains a contradiction between these two bounds on the excess. With little work, we have a similar result for Alexandrov spaces satisfying \( BG(0) \). Namely, after extending the excess estimate of Proposition 5, we can obtain

**Theorem 8.** (M-, [M3]) Let \((X, d)\) be an \( n \)-dimensional Alexandrov space satisfying \( BG(0) \) and diameter growth of order \( o(r^{1/n}) \). Then \((X, d)\) is homotopy equivalent to the interior of a compact manifold with boundary.

The idea behind many of the similar results for compact Riemannian manifolds is the motto that “large diameter implies small excess.” The same concept can be applied to our Alexandrov space by extending the theory of critical points of distance functions to these spaces as studied by Shen and Perelman. Thus, much like the non-compact case of Abresch-Gromoll, the lower bound on the excess given by Proposition 5 forces a contradiction when the diameter is large enough. With this in mind, another result to consider is Perelman’s twisted sphere theorem:

**Theorem 9.** (Perelman, [P]) Let \( M^n \) be a closed, Riemannian manifold with \( \text{Ric} \geq n - 1 \) and sectional curvature \( \geq -k^2 \). Then there exists an \( \epsilon(n, k) > 0 \) such that if \( \text{diam}_M \geq \pi - \epsilon(n, k) \) then \( M^n \) is a twisted sphere.

**Question 3.** Can Theorem 9 be extended to and Alexandrov space \((X, d)\) satisfying \( BG(0) \)? Alternatively, can one produce counterexamples?

Examples of Anderson and Otsu [A, O] ensure that the condition on sectional curvature cannot be omitted. Thus, this is a natural result to address for Alexandrov spaces satisfying \( BG(0) \). Recall, a twisted sphere is obtained by gluing the boundaries of two disks together along their boundaries by the map \( f \). Twisted spheres are homotopy equivalent to the standard \( n \)-sphere because the map \( f \) is homotopic to the identity; thus, with the conclusion of the Poincare Conjecture, they must be homeomorphic to the \( n \)-sphere but they are not in general diffeomorphic when \( n \geq 7 \).

**Metric Measure Spaces.** Of course, some of the most important and powerful results concerning Riemannian manifolds with lower Ricci curvature bounds do not have an additional assumption on their sectional curvature. In fact, opinions are divided as to whether the conclusion of Theorem 7 still holds with the \( \text{sec} \geq -k^2 \) condition removed. For this reason, and also reasons motivated by the convergence of manifolds with Ricci curvature lower bounds to Gromov-Hausdorff limits, many researchers are also interested in definitions for Ricci curvature lower bounds for metric spaces that are not necessarily Alexandrov. Because of the nature of Ricci curvature, it makes sense to study metric spaces equipped with a measure; i.e. metric measure spaces.
Independently, Lott-Villani [LV] and Sturm [S1, S2] have recently developed definitions for lower Ricci curvature bounds in compact metric measure spaces. The definition is obtained via the theory of optimal transport. The relationship and equivalence of a lower Ricci curvature bound in Riemannian manifolds to convexity of an entropy functional through optimal transport was initiated and proven by the work of Cordero-Erausquin, McCann, and Schmuckenschlager [CMS], Otto and Villani [OV], and Sturm and von Renesse [SvR].

To fully appreciate the definition for lower Ricci bounds in compact metric measure spaces, we briefly introduce some basic concepts of optimal transport. It is also possible to formulate the theory of optimal transport. The relationship and equivalence of a lower Ricci curvature bound in Riemannian manifolds to convexity of an entropy functional through optimal transport was initiated and proven by the work of Cordero-Erausquin, McCann, and Schmuckenschlager [CMS], Otto and Villani [OV], and Sturm and von Renesse [SvR].

Let $X$ be a compact Hausdorff space and let $P(X)$ denote the set of Borel probability measures on $X$. We equip the space $P(X)$ with a metric $W_2$ called the Wasserstein distance. For measures $\mu_0, \mu_1 \in P(X)$, define

\begin{equation}
W_2(\mu_0, \mu_1)^2 = \inf_\pi \int_{X \times X} d(x_0, x_1)^2 d\pi(x_0, x_1),
\end{equation}

where the infimum is taken over all possible probability measures $\pi \in P(X \times X)$. Intuitively, this metric determines the minimal ‘cost’ of moving probability points $x_0 \in (X, \mu_0)$ to the points $x_1 \in (X, \mu_1)$. Equipped with this metric, the Wasserstein space $(P(X), W_2)$ is a contractible, compact metric space and generally has infinite Hausdorff dimension. Furthermore [LV, S1], if $(X, d)$ is a compact length space, so is $(P(X), W_2)$ and a Wasserstein geodesic is a minimizing geodesic in the length space $(P(X), W_2)$. Loosely speaking, a lower bound on Ricci curvature in a metric space can be described as a convexity of a certain entropy functional along a specific Wasserstein geodesic. We now make this notion more precise.

Let $U : [0, \infty) \to \mathbb{R}$ be a continuous convex function with $U(0) = 0$. For a given probability measure $\nu \in P(X)$, define the entropy function $U_\nu$ on $P(X)$ by

\begin{equation}
U_\nu(\mu) = \int_X U(\rho(x))d\nu + U'(\infty)\mu_s(X),
\end{equation}

where $\mu = \rho\nu + \mu_s$ is the Lebesgue decomposition of $\mu$ so that $\rho\nu$ is an absolutely continuous part and $\mu_s$ is a singular part with respect to $\nu$, and $U'(\infty) = \lim_{r \to \infty} \frac{U(r)}{r}$. This entropy function $U_\nu$ is minimized when $\mu = \nu$.

As mentioned earlier, one motivation for defining Ricci curvature on metric spaces is to better understand the singular spaces that may arise as the metric evolves. In fact, spaces might collapse to a lower dimension and one would like to preserve the notion of the higher dimension while passing through the singular space. Thus, when computing Ricci curvature on metric spaces, it is necessary to specify the effective dimension $N$ which plays a necessary role in the definition. In effect, we define a curvature-dimension condition $CD(N, K)$ for metric measure spaces. Naturally, Riemannian manifolds which satisfy $CD(N, K)$ have dimension bounded above by $N$ and Ricci curvature bounded below by $K$. To simplify the exposition we define below only $CD(N, 0)$.

For $N \in [1, \infty)$, set $DC_N = \{U \mid \lambda^N U(\lambda^{-N})$ is convex in $\lambda$ on $(0, \infty)\}$, and define

**Definition 10.** Given $N \in [1, \infty)$, a compact measured length space $(X, d, \nu)$ satisfies the curvature-dimension $CD(N, 0)$ if for all $\mu_0, \mu_1 \in P(X)$ with $\text{supp}(\mu_0) \subset \text{supp}(\nu)$ and $\text{supp}(\mu_1) \subset$
supp($\nu$), there is some Wasserstein geodesic \( \{\mu_t\}_{t \in [0,1]} \) from \( \mu_0 \) to \( \mu_1 \) so that for all \( U \in DC_N \) and all \( t \in [0,1] \),

\[
U_\nu(\mu_t) = tU_\nu(\mu_1) + (1-t)U_\nu(\mu_0).
\]

To better appreciate this definition consider the example of the round sphere \( S^2 \) with induced Riemannian metric \( d \) and \( \nu = \frac{d\operatorname{vol}_{S^2}}{\operatorname{vol}(S^2)} \). Let \( \mu_0 \) and \( \mu_1 \) be rotationally symmetric measures centered at the North and South poles, respectively. Intuitively, the entropy functional measures how much a given measure is ‘spread out’ in relation to the underlying measure. As \( \mu_0 \) is transported to \( \mu_1 \) the mass can travel along different paths. For example, traveling along a meridian the entropy remains constant. If instead the mass separates and moves as a uniform ring down the meridians to the South pole (so that at \( t = 1/2 \) the mass is uniformly distributed as a ring around the equator) the entropy will increase along this process. Either way, the entropy functional is convex along the transportation and thus \( S^2 \) satisfies \( CD(2,0) \); and, in fact, with an adjusted definition even \( CD(2,1) \).

The books by Cedric Villani [V1, V2] provide an excellent resource on the theory of optimal transportation and its applications to generalizing Ricci curvature in non-smooth spaces. In [V2], Villani describes spaces with lower Ricci curvature in terms of a lazy gas experiment. Given a space and a mass of gas moving from one distribution to another, the lazy gas will move along a path of least resistance. If along these paths the entropy of the gas is convex, meaning it is more diffusive at intermediate steps of the transport plan, then we say the underlying space has a lower bound on its Ricci curvature.

Using this definition, Lott and Villani [LV] and Sturm [S1, S2] independently show that many results for Riemannian manifolds with nonnegative Ricci curvature have analogs in the more general setting of compact length spaces satisfying \( CD(N,0) \). One should note that Sturm’s definition differs only slightly from that of Lott-Villani’s but the definitions are equivalent for most metric measure spaces. For example, they prove an analog of the Bishop-Gromov Volume Comparison Theorem which states that for a compact length space \((X,d,\nu)\) with \( CD(N,0) \) and \( x \in X \); if \( 0 < r_1 < r_2 \), we have

\[
\frac{\nu(B_x(r_2))}{\nu(B_x(r_1))} \leq \left(\frac{r_2}{r_1}\right)^N.
\]

Furthermore, they also prove many fundamental theorems from classical Riemannian geometry including an analog of Gromov’s Precompactness Theorem: a sequence \( \{X_i,d_i,\nu_i\}_{i=1}^\infty \) of measured length spaces satisfying \( CD(N,0) \) converge in the measured Gromov-Hausdorff topology to a limit space \((X,d,\nu)\) satisfying \( CD(N,0) \).

It is also possible to prove analogs of results describing various analytic properties such as eigenvalue inequalities, Sobolev inequalities and local Poincaré inequalities for compact length spaces with \( CD(N,0) \). These analytic properties provide useful tools when studying the structure of this class of length spaces.

There remain a number of important open questions concerning the structure of metric measure spaces with \( CD(N,K) \) which I aim to explore. As mentioned before, the excess estimate is an essential tool in studying Riemannian manifolds with lower Ricci curvature bounds. However,
no such estimate has been extended to metric measure spaces satisfying a curvature-dimension condition.

**Question 4.** Can one prove an excess estimate for metric measure spaces satisfying $CD(N,K)$? What additional geometric restrictions (if any) are necessary?

It is expected that some additional geometric assumption would be required since it is known that the excess estimate above does not hold, for example, even for $\mathbb{R}^n$ with the taxi-cab metric. This is due to the fact that such a metric is branching.

**Definition 11.** A metric space $(X,d)$ is called **non-branching** if and only if for each quadruple of points $z,x_0,x_1,x_2$, with $z$ being the midpoint of $x_0$ and $x_1$ as well as the midpoint of $x_0$ and $x_2$, it follows that $x_1 = x_2$.

It is important to note that Alexandrov spaces are necessarily non-branching as a consequence of convexity. Perhaps one can expect an excess estimate to hold for non-branching metric measure spaces. In order to prove Theorem 2, I proved [M2] an extension of the excess estimate for triangles formed from limiting geodesics in the limit space. Note that, however, not all geodesics in $(Y,p)$ can be realized as limit geodesics. More generally, we can also prove an excess estimate for Alexandrov spaces satisfying $BG(0)$ as stated in Proposition 6. However, one would prefer not to require the underlying space to be Alexandrov since no sectional curvature assumption arises in the original statement of Abresch-Gromoll.

Another, and closely related question, concerns the Splitting Theorem. It is known that the Splitting Theorem of Cheeger-Gromoll [ChGr] does not hold for an arbitrary metric measure space satisfying $CD(N,0)$ and containing a line. Counterexamples can be found by changing the norm on $\mathbb{R}^n$. However, these counterexamples are branched.

**Question 5.** Can one prove an analog of the Cheeger-Gromoll Splitting Theorem for metric measure spaces with $CD(N,0)$?

In [KS], Kuwae and Shioya prove a topological splitting theorem but perhaps with an additional geometric assumption on the underlying space one can expect more rigid structure and stronger result.

I also aim to explore how much the rigidity and stability results for Riemannian manifolds with lower Ricci curvature bounds extend to metric measure spaces satisfying $CD(N,K)$. For example, it has been shown that for a metric measure space $(X,d,\nu)$ with $CD(N,N-1)$ the well-known Bonnet-Myers diameter bound holds; i.e. $\text{diam(supp(\nu))} \leq \sqrt{\frac{N-1}{K}} \pi$.

**Question 6.** Does a rigidity result similar to Cheng’s maximal diameter sphere theorem hold for metric measure spaces $(X,d,\nu)$ with $CD(N,N-1)$?

Of course, as mentioned before, the examples of Anderson-Otsu show that Riemannian manifolds with $\text{Ric} \geq n - 1$ are not stable under diameter bounds without additional geometric conditions imposed on the manifold.

**Question 7.** What other (if any) stability or rigidity results for volume, radius, eigenvalue estimates from Riemannian geometry also hold for metric measure spaces satisfying $CD(N,K)$?

It is clear that there remains a number of important issues surrounding the structure of non-smooth spaces satisfying some assumption of a lower Ricci curvature bound. A large part of my
research plan aims to focus on these questions while at the same time employing the powerful techniques of optimal transport to approach more classical open questions in geometric analysis.

REFERENCES


