ON THE DEPTH OF THE ASSOCIATED GRADED RING
OF AN IDEAL

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Abstract. Let $R$ be a local Cohen-Macaulay ring, $I$ an $R$-ideal and $G$ the associated graded ring of $I$. We give an estimate for the depth of $G$ when $G$ fails to be Cohen-Macaulay. We assume that $I$ has small reduction number, sufficiently good residual intersection properties, and satisfies local conditions on the depth of some powers. The main theorem unifies and generalizes several known results. We also give conditions that imply the Serre properties of the blow-up rings.

Key words. Depth, associated graded ring, Rees algebra, analytic spread, reduction number, residual intersection.

1. Introduction

Let $R$ be a Noetherian local ring with infinite residue field $k$, and let $I$ be an $R$-ideal. The Rees algebra $\mathcal{R} = R[It] \cong \oplus_{t \geq 0} I^t$ and the associated graded ring $G = gr_I(R) = \mathcal{R} \otimes_R R/I \cong \oplus_{t \geq 0} I^t/I^{t+1}$ are two graded algebras that encode various algebraic and geometric properties of the ideal $I$. There have been several results in the literature, (see for example [15] or [10]), giving sufficient conditions for these rings to be Cohen-Macaulay.

The purpose of this paper is to estimate the depth of $G$ (and consequently of $R$) when it fails to be Cohen-Macaulay, weakening the assumptions of [15, Theorem 3.1] and of [10, Theorem 1.1]. Recently, Cortadellas and Zarzuela have come up with such formulas in [5], but only in very special cases. Theorem 2.1 unifies and generalizes all these results.

When investigating $R$ or $G$ one tries to simplify $I$ by passing to a reduction, with the reduction number measuring how closely the two ideals are related. An ideal $J \subset I$ is called a reduction of $I$ if $I^{r+1} = JJ^r$ for some $r \geq 0$ ([19]). The least such $r$ is denoted by $r_J(I)$. A reduction is minimal if it is minimal with respect to inclusion, and the reduction number $r(I)$ is defined as $\min \{ r_J(I) \mid J \text{ a minimal reduction of } I \}$. The analytic spread $\ell(I)$ of $I$ is the dimension of the ring $\mathcal{R} \otimes_R k \cong G \otimes_R k$, or equivalently, the minimal number of generators $\mu(J)$ of any minimal reduction $J$ of $I$ ([19]). For further details see [25].

Now we recall some useful definitions.

Definition. Let $R$ be a local Cohen–Macaulay ring, let $I$ be an $R$-ideal of grade $g$, and let $s \geq g$ be an integer.

1. We say that $I$ satisfies property $G_s$, if $\mu(I_p) \leq \dim R_p$ for any prime ideal $p \in V(I)$ with $\dim R_p \leq s-1$, and $I$ satisfies $G_\infty$ if $G_s$ holds for every $s$.

2. An $s$-residual intersection of $I$ is a proper $R$-ideal $K = \mathfrak{a} \cap I$ where $\mathfrak{a} \subset I$ with $\mu(\mathfrak{a}) \leq s \leq \text{ht}(K)$.

3. An $s$-residual intersection $K$ of $I$ is called a geometric $s$-residual intersection if $\text{ht}(I + K) \geq s + 1$.

4. We say that $I$ satisfies $AN_s$, if for every $g \leq i \leq s$ and every $i$-residual intersection $K$ of $I$, $R/K$ is Cohen-Macaulay.
(5) We say that $I$ satisfies $AN_i^-$ if for every $g \leq i \leq s$ and every geometric $i$-residual intersection $K$ of $I$, $R/K$ is Cohen-Macaulay.

The main result of this paper can be stated as follows.

**Theorem 2.1** Let $R$ be a local Cohen-Macaulay ring of dimension $d$ with infinite residue field, let $I$ be an $R$-ideal of grade $g$, analytic spread $\ell$, and reduction number $r$, let $k \geq 0$ be an integer with $r \leq k$, assume that $I$ satisfies $G_\ell$, $AN_{\ell-k-1}$ and that for every $p \in V(I)$, $\text{depth}(R/I)_p \geq \min\{\text{dim } R_p - \ell + j, j \in \mathbb{Z}, 1 \leq j \leq k\}$. Then $\text{depth} G \geq \min\{d, \text{depth } R/I + \ell\}.$

In particular, if the reduction number is small, we have a formula for depth $G$.

**Corollary 2.12** Let $R$ be a local Cohen-Macaulay ring of dimension $d$ with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell$, and reduction number $r \leq 1$. Assume that $I$ satisfies $G_\ell$ and $AN_{\ell-2}$. Then depth $G = \min\{d, \text{depth } R/I + \ell\}.$

**Corollary 2.13** Let $R$ be a local Cohen-Macaulay ring of dimension $d$ with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell \geq g + 1$, and reduction number $r \leq 2$. Further assume that $I$ satisfies $G_\ell$, $AN_{\ell-3}$ and that $R/I$ is Cohen-Macaulay. Then depth $G = \min\{d, \text{depth } R/I^2 + \ell\}.$

Section 2 is devoted to the proof of Theorem 2.1. The main tool is a combination of the techniques used by Johnson and Ulrich in [15] and by Goto, Nakamura and Nishida in [10]. Notice that Theorem 2.1 gives an estimate also for depth $\mathcal{R}$, since if $G$ is not Cohen-Macaulay, we have that depth $\mathcal{R} = \text{depth } G + 1$ ([12, 3.10]).

In Section 3 we give examples of classes of ideals to which Theorem 2.1 can be applied in order to compute the depth of the associated graded ring.

From Theorem 2.1 we obtain results on the Serre conditions for $\mathcal{R}$ and $G$. These are studied in Section 4 (see Theorem 4.1, Theorem 4.6 and Theorem 4.8). The $S_1$ property for $G$ is particularly interesting because it leads to criteria for when $I^n = I^{(n)}$, where $I^{(n)}$ is the $n$-th symbolic power of $I$. We recall that $I^{(n)}$ is the intersection over all isolated primary components of the ordinary power $I^n$. From Theorem 4.1 we obtain that $G$ is $S_1$ in the following cases.

**Corollary 4.2** Let $R$ be a local Cohen-Macaulay ring with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell$, and reduction number $r \leq 1$. Further assume that $I$ satisfies $G_\ell$, $AN_{\ell-2}$, and that $R/I$ has no associated primes of height $\geq \ell + 1$. Then $G$ is $S_1$.

**Corollary 4.3** Let $R$ be a local Cohen-Macaulay ring with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell \geq g + 1$, and reduction number $r \leq 2$. Further assume that $I$ satisfies $G_\ell$, $AN_{\ell-3}$, that $R/I$ is Cohen-Macaulay, and that $R/I^2$ has no associated primes of height $\geq \ell + 1$. Then $G$ is $S_1$.

In Theorem 4.7 we analyze the relation between the Serre properties for $\mathcal{R}$ and $G$, generalizing a theorem by Brumatti, Simis and Vasconcelos ([2, 1.5]).
We conclude by applying our results to the defining ideal of monomial varieties of codimension two.

2. The Main Theorem

Theorem 2.1. Let $R$ be a local Cohen-Macaulay ring of dimension $d$ with infinite residue field, let $I$ be an $R$-ideal with grade $g$, let $J$ be a reduction of $I$ generated by $s$ elements with $ht\, J \geq s$, $r = r_J(I)$, and let $k \geq 0$ be an integer with $r \leq k$. Assume that $I$ satisfies $G_s$, $AN_{-k-1}^-$, and that for every $p \in V(I)$, $\text{depth}(R/I^p) \geq \min\{\text{dim}\, R_p - s + k - j, k - j\}$ whenever $1 \leq j \leq k - 1$. Then

$$\text{depth} \, G \geq \min\{\{d\} \cup \{\text{depth} \, R/I^j + s - k + j \mid 1 \leq j \leq k\}\}.$$ 

First we summarize some technical results (Lemmas 2.2, 2.3, 2.4 and 2.5) that will play a crucial role for the proof of the theorem. Next we show that we can reduce the problem to the case where $g = 0$. Then we prove the theorem in the case where $g = 0$ and $s \leq k$, which is the main step. For this purpose we need some preliminary results (Lemmas 2.6 and 2.7, Corollary 2.8 and Lemma 2.9). The general case of the theorem then follows rather quickly.

Lemma 2.2. [15, Lemma 2.3]. Let $R$ be a local Cohen-Macaulay ring with infinite residue field, let $a \subset I$ be (not necessarily distinct) $R$-ideals with $\mu(a) \leq s \leq \text{ht} \, a : I$ and assume that $I$ satisfies $G_s$.

(a) There exists a generating sequence $a_1, \ldots, a_s$ of $a$ such that for every $0 \leq i \leq s - 1$ and for every subset $\{v_1, \ldots, v_i\}$ of $\{1, \ldots, s\}$, $ht\, (a_{v_1}, \ldots, a_{v_i}) : I \geq i$ and $ht\, I + (a_{v_1}, \ldots, a_{v_i}) : I \geq i + 1$.

(b) Assume that $I$ satisfies $AN_i^-$ for some $t \leq s - 1$ and that $a \neq I$, write $a_i = (a_1, \ldots, a_i)$, $K_i = a_i : I$ and let ‘$\cdot$’ denote images in $R/K_i$. Then for $0 \leq i \leq t + 1$:

(i) $K_i = a_i : (a_{i+1})$ and $a_i = I \cap K_i$, if $i \leq s - 1$.

(ii) $\text{depth} \, R/a_i = d - i$.

(iii) $K_i$ is unmixed of height $i$.

(iv) $ht\, \bar{I} = 1$, if $i \leq s - 1$.

Lemma 2.3. Let $R$ be a local Cohen-Macaulay ring of dimension $d$ with infinite residue field, let $a \subset I$ be $R$-ideals with $\mu(a) \leq s \leq \text{ht} \, a : I$, let $t$ be an integer with $t \leq s - 1$. Assume that $I$ satisfies $G_s$ and $AN_i^-$ and that $[a_i : (a_{i+1})] \cap I^j = a_i I^{-1}$ for $0 \leq i \leq s - 1$, $j \geq \max\{1, i - t\}$, where $a_1, \ldots, a_s$ and $a_i$ are as defined in Lemma 2.2 (a). Then $\text{depth} \, R/a_i I^j \geq \min\{\{d - i\} \cup \{\text{depth} \, R/I^j - n \mid 0 \leq n \leq i - 1\}\}$, for $0 \leq i \leq s$ and $j \geq \max\{0, i - t - 1\}$.

Proof. We use induction on $i$. The assertion being trivial for $i = 0$, we may assume that $0 \leq i \leq s - 1$. We need to show that the inequality holds for $i + 1$. 

But for $j = 0$ (which can only occur if $i + 1 \leq t + 1$), our assertion follows from Lemma 2.2 (b) (ii). Thus we may suppose that $j \geq 1$. But then by assumption, $a_i I^j \cap a_{i+1} I^j = a_{i+1}[(a_i I^j : (a_{i+1})) \cap I^j] \subset a_{i+1}[(a_i : (a_{i+1})) \cap I^j] = a_{i+1}a_i I^{j-1} \subset a_i I^j \cap a_{i+1} I^j$. Hence we obtain an exact sequence

$$0 \to a_{i+1} I^{j-1} \to a_i I^j \oplus a_{i+1} I^j \to a_i I^j \to 0. \quad (1)$$

On the other hand, by the assumption for $i = 0$, $[0 : (a_{i+1})] \cap a_i I^{j-1} \subset [0 : (a_{i+1})] \cap I^j = 0$, and therefore $a_{i+1}a_i I^{j-1} \cong a_i I^{j-1}$, $a_{i+1}I_j \cong I_j$. The conclusion follows from (1) and the induction hypothesis.

\begin{proof}

First we show (a), (b) and (c) by induction on $j$. Notice that (a) holds for $j = 0$ by Lemma 2.2 (b) (ii), (b) holds for $j = 1$ by Lemma 2.2 (b) (i) and (c) trivially holds for $j = 1$.

Claim 1. If (b) holds for $j$, then so does (a).

Proof. This follows from Lemma 2.3, with $t = s - k - 1$.

Now we can assume that (a), (b) and (c) hold for $j \geq 1$ and we show that they hold for $j + 1$.

Claim 2. If (a) holds for $j$, then (b) and (c) hold for $j + 1$ and the maximal value of $i$; namely $i = s - k + j$.

Proof. If $j \geq k$, our claim is clear since there is nothing to show in (b) and $I^{j+1} = J I^j = a_i I^j$. Let $j \leq k - 1$. In order to show (b), let $p \in \text{Ass}(R/a_i I^j)$. By (a), $\min\{\dim R_p - s + k - j, k - j\} \leq 0$; hence $\dim R_p = s - k + j = i$. It follows from Lemma 2.2 (a) that $(a_{i+1})_p = R_p$ if $I_p = R_p$, and that $I_p = (a_i)_p$ if $I_p \neq R_p$. In any case we get the desired equality of (b). Since $a_i \cap I^{j+1} \subset [a_i : (a_{i+1})] \cap I^{j+1} = a_i I^j$, (c) holds.

Claim 3. If (b) and (c) hold for $j + 1$ and maximal $i$, then (b) and (c) hold for $j + 1$ and any $i$.

\end{proof}
Proof. Take \( i < s - k + 1 \). By decreasing induction on \( i \), \( a_{i+1} \cap I^{j+1} = a_{i+1}I^j \). Hence \( a_i \cap I^{j+1} = a_i \cap a_{i+1} \cap I^{j+1} = a_i \cap a_{i+1}I^j = a_i \cap (a_iI^j + a_{i+1}I^j) = a_iI^j + a_i \cap a_{i+1}I^j \). We have that for every \( p \), \( \langle a_i : (a_{i+1}) \rangle \cap I^j = a_iI^j + a_{i+1}a_iI^{j-1} = a_iI^j \). This shows (c).

Since \( [a_i : (a_{i+1})] \cap I^{j+1} \subset [a_i : (a_{i+1})] \cap I^j \subset a_i[a_i : (a_{i+1})] \cap I^{j+1} \subset a_i \cap I^{j+1} = a_iI^j \); hence (b) holds.

The first claim of (d) follows from [24, Corollary 2.7] and part (c). Now let \( u \in [(a'_1, \ldots, a'_i) : a'_{i+1}]_j \). Picking an element \( x \in I^j \) with \( x + I^{j+1} = u \), we have \( a_{i+1}x \in a_i + I^{j+2} \), and so by part (c), \( a_{i+1}x \in a_i + (a_i + I^{j+2}) = a_i + a_{i+1}I^{j+2} = a_i + a_{i+1}I^{j+1} = a_i + a_{i+1}I^j \). Thus \( a_{i+1}(x - y) \in a_i \) for some \( y \in I^{j+1} \). Since \( x - y + I^{j+1} = x + I^{j+1} = u \), we may replace \( x \) by \( x - y \) to assume that \( a_{i+1}x \in a_i \).

But then \( x \in [a_i : (a_{i+1})] \cap I^j = a_iI^{j-1} \) by (b), which implies \( u \in (a'_1, \ldots, a'_i) \). □

Lemma 2.5. Let \( s \) and \( t \) be integers. Assume \([(a'_1, \ldots, a'_i) : a'_{i+1}]_j = [(a'_1, \ldots, a'_i)]_j \) whenever \( 0 \leq i \leq s - 1 \) and \( j \geq i - t \), where \( a'_1, \ldots, a'_s \) are defined as in Lemma 2.4. Then \( \text{depth } [G/(a'_1, \ldots, a'_i)]_j \geq \min \{ \text{depth } R/I^n + n - j - 1 \mid j - i + 1 \leq n \leq j + 1 \} \cup \{ \text{depth } R/I^{j-1} - i + 1 \} \), whenever \( 0 \leq i \leq s \) and \( j \geq i - t \).

Proof. We show by induction on \( i \) that \( \text{depth } [G/(a'_1, \ldots, a'_i)]_j \geq \min \{ \text{depth } R/I^n / I^{n+1} + n - j - 1 \mid j - i + 1 \leq n \leq j + 1 \} \), whenever \( 0 \leq i \leq s \) and \( j \geq i - t \). The assertion being trivial for \( i = 0 \), we may assume that \( 0 \leq i \leq s - 1 \). We need to show that the inequality holds for \( i + 1 \).

By assumption we have an exact sequence

\[
0 \to [G/(a'_1, \ldots, a'_i)]_j \to [G/(a'_1, \ldots, a'_i)]_{j+1} \to [G/(a'_1, \ldots, a'_{i+1})]_{j+1} \to 0 \quad (2)
\]

whenever \( 0 \leq i \leq s - 1 \) and \( j \geq i - t \).

The conclusion follows from (2) and the induction hypothesis. □

Proof of the Theorem.

Note that from the assumption on the depth of the powers we have \( k \leq s - g + 1 \).

Let \( J = (a_1, \ldots, a_s) \), where \( a_1, \ldots, a_s \) are defined as in Lemma 2.2 (a), and let \( a'_1, \ldots, a'_s \) be their images in \([G]_1\).

**STEP 1. Reduction to the case \( g = 0 \).**

We show that we may assume that \( g = 0 \). Let us denote by \( i^* \) images in \( R/a_0 \) and write \( G^* = G(I^*) \). We have that \( G^* = G/(a'_1, \ldots, a'_s) \) and depth \( G^* + g \) by Lemma 2.4 (d). The ring \( R^* \) is a local Cohen-Macaulay ring of dimension \( d - g \), \( I^* \) an \( R^* \)-ideal of grade 0, \( J^* \) a reduction of \( I^* \) generated by \( s - g \) elements, \( \text{ht } J^* : I^* \geq s - g \), and \( k \) may be taken unchanged.

The ideal \( I^* \) satisfies \( G_{s-g} \), since by Lemma 2.2 (a), \( I_p = (a_1, \ldots, a_i)_p \) for all \( p \in V(I) \) with \( \dim R_p \leq i \leq s - 1 \). Also \( I^* \) satisfies \( AN_{s-g-k-1} \).

Now we show that the condition on the depth of the powers is preserved, i.e. that for every \( p^* \in V(I^*) \), \( \text{depth}(R^*/I^{j*})_{p^*} \geq \min \{ \dim R_{p^*} - (s - g) + k - j, k - j \} \) whenever \( 1 \leq j \leq k - 1 \).
Since $R^*/I^j \cong R/a_g + I^j$, by Lemma 2.4 (c) for every $j \geq 1$ we have an exact sequence

$$0 \rightarrow R/a_g I^{j-1} \rightarrow R/a_g \oplus R/I^j \rightarrow R^*/I^j \rightarrow 0 \quad (3)$$

By (3), Lemma 2.4 (b) and Lemma 2.3 with $t = s-k-1$ we have that $\text{depth}(R^*/I^j)_p \geq \min\{\text{depth}(R/I^{j-n})_p - n \mid 0 \leq n \leq g\}$ whenever $j > 1$. Since the inequality holds also when $j = 1$, we get the desired condition on the depth of the powers.

Finally if depth $G^* \geq \min\{\{d-g\} \cup \{\text{depth } R^*/I^j + s-g-k+j \mid 1 \leq j \leq k\}\}$, again since depth$(R^*/I^s) \geq \min\{\text{depth } R/I^{j-n} - n \mid 0 \leq n \leq g\}$ for $j \geq 1$, we conclude that depth $G \geq \min\{\{d\} \cup \{\text{depth } R/I^j + s-k+j \mid 1 \leq j \leq k\}\}$. This finishes the reduction to the case $g = 0$.

**STEP 2. The proof in the case $s \leq k$.**

We now assume $g = 0$ and $s \leq k$. We are going to prove the theorem in this special case. Since $k \leq s+1$, either $k = s$ or $k = s+1$. If $s = 0$, then $I$ is nilpotent and $k \leq 1$; so we have either $G = R$ or $G = R/I \oplus I$ and depth $G = \text{depth } R/I$. Hence we can assume $s > 0$.

For $0 \leq i \leq s$ consider the graded $G$-modules $M(i) = [G/(a_1', \ldots, a_i')]_{-i+s+k+1} = G_+^{-i+s+k+1}/(a_1', \ldots, a_i')G_+^{-i+s+k}$ and $N(i) = G_+^{-i+s+k}/(a_1', \ldots, a_{i-1}')G_+^{-i+s+k-1} + a_i'G_+^{-i+s+k}$.

Notice that $[M(i)]_{-i+s+k+1} = M(i)$ and $[N(i)]_{-i+s+k} = [G/(a_1', \ldots, a_{i-1}')]_{-i+s+k}$, which yields exact sequences

$$0 \rightarrow M(i) \rightarrow N(i) \rightarrow [G/(a_1', \ldots, a_{i-1}')]_{-i+s+k} \rightarrow 0. \quad (4)$$

On the other hand, if $0 \leq i \leq s-1$, then $N(i+1) = M(i)/a_{i+1}'M(i)$ and by Lemma 2.4 (d) we have that $0 : M(i) (a_{i+1}') = 0$. Thus, in the range $0 \leq i \leq s-1$ we have exact sequences

$$0 \rightarrow M(i)(-1) \rightarrow M(i) \rightarrow N(i+1) \rightarrow 0. \quad (5)$$

Furthermore $M(s) = 0$, since $I^{k+1} = J I^k$.

Now we need a lemma, that we are going to prove in a general context.

Let $S$ be a homogeneous Noetherian ring with $S_0$ local and homogeneous maximal ideal $\mathfrak{m}$, let $H^*(\cdot)$ denote local cohomology with support in $\mathfrak{m}$.

For a graded $S$-module $N$ and an integer $j$ we put $a_j(N) = \max\{n \mid [H^j(N)]_n \neq 0\}$.

**Lemma 2.6.** Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of graded $S$-modules, let $n$ and $j$ be integers.

(a) If $a_j(A) \leq n$ and $a_j(C) \leq n$, then $a_j(B) \leq n$.

(b) (i) If $H^j(A) = 0$, then $a_j(C) \geq a_j(B)$.

(ii) If $H^j(B) = 0$, then $a_{j+1}(A) \geq a_j(C)$.

(iii) If $H^j(C) = 0$, then $a_{j+1}(B) \geq a_{j+1}(A)$.

**Proof.** The assertions follow from the long exact sequence of local cohomology

$$\cdots \rightarrow H^j(A) \rightarrow H^j(B) \rightarrow H^j(C) \rightarrow H^{j+1}(A) \rightarrow H^{j+1}(B) \rightarrow \cdots$$
and the definition of \( a_j(\cdot) \).

Let \( \lambda = \min\{\{d\} \cup \{\text{depth } R/I^j + s - k + j \mid 1 \leq j \leq k\}\} \). Recall that we want to show that depth \( G \geq \lambda \).

**Lemma 2.7.** In addition to the assumptions of Theorem 2.1, assume that \( g = 0 \) and \( s \leq k \). Let \( M_{(i)} \) be defined as above. Then:

(a) \( a_j(M_{(i)}) \leq i - s + k \) for any integer \( j \) and \( 0 \leq i \leq s \).

(b) depth \( M_{(i)} \geq \lambda - i - 1 \), and if depth \( M_{(i)} = \lambda - i - 1 \) then \( a_{\lambda - i - 1}(M_{(i)}) = i - s + k \).

**Proof.** We prove the claim by decreasing induction on \( i \). If \( i = s \), the assertion is obvious since \( M_{(s)} = 0 \). Suppose \( 0 \leq i \leq s - 1 \) and assume \( a_j(M_{(i+1)}) \leq i+1-s+k \) for any integer \( j \). By [9, Lemma 2.2] and Lemma 2.6 (a) applied to the sequence

\[
0 \to M_{(i+1)} \to N_{(i+1)} \to [G/(a'_1, \ldots, a'_i)]_{i+1-s+k} \to 0 \tag{6}
\]

we have that \( a_j(N_{(i+1)}) \leq i+1-s+k \) for any \( j \). Applying the local cohomology functor to the sequence

\[
0 \to M_{(i)}(-1) \to M_{(i)} \to N_{(i+1)} \to 0 \tag{7}
\]

it follows that for any \( j \), \([H^j(M_{(i)}))]_{n} = 0 \) whenever \( n > i-s+k \). Hence \( a_j(M_{(i)}) \leq i-s+k \) and the proof of (a) is completed.

Assume now that depth \( M_{(i+1)} \geq \lambda - i - 2 \) and that if depth \( M_{(i+1)} = \lambda - i - 2 \), then \( a_{\lambda - i - 2}(M_{(i+1)}) = i + 1 - s + k \). By Lemma 2.4 (d) and Lemma 2.5, depth \([G/(a'_1, \ldots, a'_i)]_{i+1-s+k} \geq \min\{\text{depth } R/I^j + s - k - 2 \mid k - s + 2 \leq j \leq i - s + k + 2\} \cup \{\text{depth } R/I^{k-s+1} - i + 1\} \). If \( 0 \leq i \leq s-2 \), then \( j \leq k \) and so depth \([G/(a'_1, \ldots, a'_i)]_{i+1-s+k} \geq \min\{\text{depth } R/I^j + s - k - 2 \mid 1 \leq j \leq k\} \geq \lambda - i - 2 \). If \( i = s - 1 \), since depth \( R/I^{k+1} \) is regular, \( R/I^k \geq \lambda - s \) by Lemma 2.4 (b) and Lemma 2.3, we have that depth \([G/(a'_1, \ldots, a'_{s-1})]_{k} \geq \lambda - s - 1 \). Hence in any case it follows that depth \([G/(a'_1, \ldots, a'_{s-1})]_{i+1-s+k} \geq \lambda - i - 2 \), and so by (6) we have that depth \( N_{(i+1)} \geq \lambda - i - 2 \). Since \( N_{(i+1)} = M_{(i)}/a'_{i+1}M_{(i)} \) and \( a'_{i+1} \) is \( M_{(i)} \)-regular, we conclude that depth \( M_{(i)} \geq \lambda - i - 1 \).

If depth \( M_{(i)} = \lambda - i - 1 \), then depth \( N_{(i+1)} = \lambda - i - 2 \) and so \( H^{\lambda - i - 2}(N_{(i+1)}) \neq 0 \). Applying the local cohomology functor to the sequence (6) we get the exact sequence

\[
\cdots \to H^{\lambda - i - 2}(M_{(i+1)}) \to H^{\lambda - i - 2}(N_{(i+1)}) \to H^{\lambda - i - 2}[G/(a'_1, \ldots, a'_i)]_{i+1-s+k} \to \cdots
\]

If depth \( M_{(i+1)} > \lambda - i - 2 \), then \( H^{\lambda - i - 2}(N_{(i+1)}) \equiv [H^{\lambda - i - 2}(N_{(i+1)})]_{i+1-s+k} \) and so \( a_{\lambda - i - 2}(N_{(i+1)}) = i + 1 - s + k \). If depth \( M_{(i+1)} = \lambda - i - 2 \), then by assumption \( a_{\lambda - i - 2}(M_{(i+1)}) = i + 1 - s + k \). Since depth \([G/(a'_1, \ldots, a'_i)]_{i+1-s+k} \geq \lambda - i - 2 \), applying Lemma 2.6 (b) (iii) to (6) we get that \( a_{\lambda - i - 2}(N_{(i+1)}) \geq i+1-s+k \). In any case \( a_{\lambda - i - 2}(N_{(i+1)}) \geq i+1-s+k \). Finally, since depth \( M_{(i)} = \lambda - i - 1 \), applying Lemma 2.6 (b) (ii) to the sequence (7) we conclude that \( a_{\lambda - i - 1}(M_{(i)}) \geq i - s + k \).
Corollary 2.8. With the assumptions of Lemma 2.7, we have that $a_j(G) \leq k - s$ for any integer $j$.

Proof. We have an exact sequence $0 \to M(0) \to G \to C \to 0$, where

$$C = \begin{cases} R/I & \text{if } k = s \\ R/I \oplus I/I^2 & \text{if } k = s + 1. \end{cases}$$

Since for any integer $j$, $a_j(M(0)) \leq k - s$ by Lemma 2.7 and $a_j(C) \leq k - s$, by Lemma 2.6 (a) we conclude that $a_j(G) \leq k - s$ for any integer $j$.

Lemma 2.9. If depth $G = t < d$, then $a_t(G) < \max\{0, a_{t+1}(G)\}$. In particular, in the assumptions of Corollary 2.8, one has $a_t(G) < k - s$.

Proof. Suppose $t < d$. Then we have depth $R = t + 1$ by [12, 3.10]. Hence, applying the local cohomology functor to

$$0 \to R_+(1) \to R \to G \to 0$$

we get the exact sequence

$$0 \to H^t(R) \to H^{t+1}(R_+)(1) \to H^{t+1}(R) \to H^{t+1}(G). \quad (8)$$

Let $m = \max\{0, a_{t+1}(G)\}$. If $n > m$, then $[H^{t+1}(G)]_n = 0$ and so $[H^{t+1}(R_+)]_{n+1}$ maps onto $[H^{t+1}(R)]_n$. Applying $H^*(-)$ to

$$0 \to R_+ \to R \to G \to 0$$

we get $[H^{t+1}(R_+)]_n \cong [H^{t+1}(R)]_{a_t(G)}$, since, for every integer $j$, $H^j(R) = [H^j(R)]_0 \cong H^j_m(R)$ by [9, Lemma 2.2]. Hence $[H^{t+1}(R_+)]_n = 0$ for any $n > m$. From (8) we conclude that $a_t(G) < m$.

Conclusion of the case $s \leq k$.

Consider the exact sequence $0 \to M(0) \to G \to C \to 0$.

Notice that depth $R C = \text{depth}_R C \geq \lambda - 1$, and that depth $R/I \geq \lambda - 1$ if $k = s + 1$. Since depth $M(0) \geq \lambda - 1$ by Lemma 2.7, it follows that depth $G \geq \lambda - 1$.

If depth $G = \lambda - 1$, then $a_{\lambda-1}(G) < k - s$ by Lemma 2.9. The sequence above yields

$$H^{\lambda-1}(M(0)) \to H^{\lambda-1}(G) \to H^{\lambda-1}(C). \quad (9)$$

If depth $M(0) \geq \lambda - 1$, we get a contradiction since $[H^{\lambda-1}(C)]_{a_{\lambda-1}(G)} = 0$. If depth $M(0) = \lambda - 1$, then $a_{\lambda-1}(M(0)) = k - s$ by Lemma 2.7 and hence by Lemma 2.6 (b) $a_{\lambda-1}(G) \geq k - s$, a contradiction. Hence depth $G \geq \lambda$ and this proves our result in the case $g = 0$ and $s \leq k$.

STEP 3. Proof of the general case.

Let $\delta = \delta(I) = s - g + 1 - k$ and recall that $\delta \geq 0$. We are going to induct on $\delta$. We can assume that $g = 0$; hence $\delta = s + 1 - k$. Since the theorem holds if $\delta = 0$ or $\delta = 1$ (i.e., $k = s$ or $k = s + 1$), we can assume that $\delta \geq 2$ and so $s - k - 1 \geq 0$. Hence $I$ satisfies $AN_0^-$. Write $K = 0 : I$ and let `¨'denote images in $\bar{R} = R/K$. We will show that our assumptions are preserved, and that $\delta$ decreases when passing from $R$ to $\bar{R}$. Now $\bar{R}$ is Cohen-Macaulay since $I$ satisfies $AN_0^-$, and
by Lemma 2.2 (b) and [15, Lemma 2.4 (b)], \( I \cap K = 0 \), grade \( \bar{I} = 1 \) and \( \bar{I} \) is \( AN_{\bar{s},k-1}^- \). Furthermore \( \dim \bar{R} = \dim R = d \), \( \bar{J} = (\bar{a}_1, \ldots, \bar{a}_s) \) is a reduction of \( \bar{I} \) with \( \ht \bar{J} : \bar{I} \geq s \) and thus \( k \) may be taken to remain unchanged. Clearly \( \bar{I} \) satisfies \( G_s \) since \( \bar{R} \) is equidimensional of the same dimension as \( R \). Again since \( I \cap K = 0 \) we have an exact sequence

\[
0 \to K \to G \to gr_{\bar{I}}(\bar{R}) \to 0 \quad (10)
\]

where depth \( K = d \) since depth \( \bar{R} = d \).

Now by (10), depth \( \bar{R}/\bar{I} \geq \min\{d-1, \text{depth } R/I \} \) and \( \bar{I}^j/\bar{I}^{j+1} \cong I^j/I^{j+1} \) for \( j \geq 1 \). It follows, by induction on \( j \), that whenever \( j \geq 1 \), depth \( \bar{R}/\bar{I}^j \geq \min\{\{d-1\} \cup \{\text{depth } R/I^t | 1 \leq t \leq j\}\} \). Applying this in the ring \( R = R_p \), we see that the condition on the depth of the powers is preserved, namely for every \( \bar{p} \in V(\bar{I}) \) we have that depth \( (\bar{R}/\bar{I}^j)_{\bar{p}} \geq \min\{\text{dim } R_{\bar{p}} - s + k - j, k - j\} \) whenever \( 1 \leq j \leq k - 1 \).

Now \( \delta(\bar{I}) = s - \text{grade } \bar{I} + 1 - k < s + 1 - k = \delta(I) \), thus we may use our induction hypothesis to conclude that depth \( gr_1(\bar{R}) \geq \min\{\{d\} \cup \{\text{depth } \bar{R}/\bar{I}^j + s - k + j | 1 \leq j \leq k\}\} \). Then by (10) we have that depth \( G \geq \min\{\{d\} \cup \{\text{depth } R/I^j + s - k + j | 1 \leq j \leq k\}\} \) and the proof of the theorem is complete. \( \square \)

**Remark 2.10.** The theorem still holds if the condition “\( I \) satisfies \( G_s \)” is replaced by “\( r_i \leq \max\{0, i - s + k\} \) for all \( g \leq i < s \)”, where \( r_i = \max\{r_{a_i}Q(I_Q) | Q \in V(I) \text{ and } \ht Q = i\} \) for \( g \leq i \leq s \). Here \( a_i = (a_1, \ldots, a_i) \), where \( a_1, \ldots, a_s \) are defined as in [10, Lemma 2.1]. Notice that Lemma 2.3 and Lemma 2.4 are still satisfied by [10, 2.7, 3.1, 3.2, 3.3 and their proof]. Also, Lemma 2.5 still holds. Furthermore the condition \( r_i \leq \max\{0, i - s + k\} \) for all \( g \leq i < s \) still holds when we factor out \( a_s \) to assume \( g = 0 \), and when we factor out \( K = 0 : I \). Since in the rest of the proof the \( G_s \) property is needed only to be able to use Lemma 2.3, Lemma 2.4 and Lemma 2.5, it follows that the result still holds.

In the next corollaries we need the following.

**Remark 2.11.** Let \( R \) be a Noetherian local ring, let \( I \) be an \( R \)-ideal with analytic spread \( \ell \). Then depth \( G \leq \inf\{\text{depth } R/I^j | j \geq 1\} + \ell \).

**Proof.** Since \( G \) is a Noetherian ring, we have that depth\( m_G G = \inf\{\text{depth } I^j/I^{j+1} | j \geq 1\} = \inf\{\text{depth } R/I^j | j \geq 1\} \). But depth\( m_G G \geq \text{depth } G - \dim G/mG = \text{depth } G - \ell \).

The next corollaries are special cases of Theorem 2.1, for small reduction number.

**Corollary 2.12.** Let \( R \) be a local Cohen-Macaulay ring of dimension \( d \) with infinite residue field, let \( I \) be an \( R \)-ideal with grade \( g \), analytic spread \( \ell \), and reduction number \( r \leq 1 \). Assume that \( I \) satisfies \( G_{\ell} \) and \( AN_{\ell-2}^- \). Then depth \( G = \min\{d, \text{depth } R/I + \ell\} \).
Example 3.1. We have that depth $G/J$ is Cohen-Macaulay.

Corollary 2.13. Let $R$ be a local Cohen-Macaulay ring of dimension $d$ with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell \geq g + 1$, and reduction number $r \leq 2$. Further assume that $I$ satisfies $G_{\ell}, \ AN_{\ell-3}$ and that $R/I$ is Cohen-Macaulay. Then depth $G = \min\{d, \text{depth } R/I^2 + \ell\}$.

Proof. The assertion follows from Theorem 2.1 with $s = \ell, \ J$ a minimal reduction of $I$ such that $\text{ht } J : I \geq \ell$ and $r_J(I) = r, \ k = 1$, and from Remark 2.11.

Corollary 2.14. Let $R$ be a local Cohen-Macaulay ring of dimension $d$ with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell \geq g + 2$, and reduction number $r \leq 3$. Further assume that $I$ satisfies $G_{\ell}, \ AN_{\ell-4}$, that $R/I$ is Cohen-Macaulay, and that $R/I^2$ has no associated primes of height $\geq \ell$. Then $\min\{d, \text{depth } R/I^2 + \ell - 1, \text{depth } R/I^3 + \ell\} \leq \text{depth } G \leq \min\{\text{depth } R/I^2 + \ell, \text{depth } R/I^3 + \ell\}$.

Proof. The assertion follows from Theorem 2.1 with $s = \ell, \ J$ a minimal reduction of $I$ such that $\text{ht } J : I \geq \ell$ and $r_J(I) = r, \ k = 2$, and from Remark 2.11.

Notice that Theorem 2.1 generalizes [15, Theorem 3.1] and, by Remark 2.10, it recovers also [10, Theorem 1.1] and [5, Theorem 4.1 and Theorem 5.8]. The last authors gave formulas for depth $G$ when the reduction number is at most 2. If the reduction number is 3 we get the following.

3. Examples

In this section we give examples of classes of ideals to which Theorem 2.1 can be applied in order to compute the depth of the associated graded ring.

Example 3.1. Let $R$ be a local Cohen-Macaulay ring with infinite residue field, let $I$ be an $R$-ideal with analytic spread $\ell$, satisfying $G_{\ell}$ and $AN_{\ell-2}$, and let $J$ be a minimal reduction of $I$. Since $I$ satisfies $G_{\ell}$ and $AN_{\ell-2}$, by [23, Proposition 1.11] we have that $\text{ht } J : I \geq \ell$, and therefore by [23, Remark 1.12] $J$ satisfies $G_{\ell}$ and $AN_{\ell-2}$. Clearly $r(J) = 0$. Let $G(J)$ be the associated graded ring of $J$. Then by Corollary 2.12, we have that depth $G(J) = \min\{d, \text{depth } R/J + \ell\}$.

Now we present a class of ideals whose associated graded ring is not Cohen-Macaulay and we can use our results to compute its depth.

Example 3.2. Let $R$ be a local Gorenstein ring with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell \leq g + 2$ and reduction number $r \neq 0$. Assume that $I$ satisfies $G_{\ell+1}$ and that $R/I$ is Cohen-Macaulay. Let $J$ be a minimal reduction of $I$. By [23, Proposition 1.11] we have that $\text{ht } J : I \geq \ell + 1$. As $J : I \neq R$ it follows that some associated prime of $R/J$ has height at least $\ell + 1$. Therefore $\text{depth } R/J \leq d - \ell - 1$. Let $G(J)$ be the associated graded ring of $J$. By Example 3.1, we have that depth $G(J) = \min\{d, \text{depth } R/J + \ell\} \leq d - 1$. 

Proof. The assertion follows from Theorem 2.1 with $s = \ell, \ J$ a minimal reduction of $I$ such that $\text{ht } J : I \geq \ell$ and $r_J(I) = r, \ k = 1$, and from Remark 2.11. 

Corollary 2.12. Let $R$ be a local Cohen-Macaulay ring of dimension $d$ with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell \geq g + 1$, and reduction number $r \leq 2$. Further assume that $I$ satisfies $G_{\ell}, \ AN_{\ell-3}$ and that $R/I$ is Cohen-Macaulay. Then depth $G = \min\{d, \text{depth } R/I^2 + \ell\}$.
Example 3.3. Let \( R = k[[x_1, \ldots, x_8]] \), where \( k \) is an infinite field. Let
\[
\phi = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{pmatrix}
\]
and let \( I \) be the ideal generated by the 2 by 2 minors of \( \phi \),
\[
I = (x_1 x_6 - x_2 x_5, x_1 x_7 - x_3 x_5, x_1 x_8 - x_4 x_5, x_2 x_7 - x_3 x_6, x_2 x_8 - x_4 x_6, x_3 x_8 - x_4 x_7).
\]
The ideal \( I \) has grade 3 and analytic spread 5 and \( R/I \) is Cohen-Macaulay. Since \( I \) is a complete intersection on the punctured spectrum of \( R \), \( I \) satisfies \( G_8 \). The ideal
\[
J = (x_1 x_6 - x_2 x_5, x_1 x_7 - x_3 x_5, x_1 x_8 - x_4 x_5 + x_2 x_7 - x_3 x_6, x_2 x_8 - x_4 x_6, x_3 x_8 - x_4 x_7)
\]
is a minimal reduction of \( I \). Since \( I \) and \( J \) coincide on the punctured spectrum of \( R \), we have that depth \( R/J = 0 \). By Example 3.2, we have that depth \( G(J) = 5 \).

Now we present a class of equimultiple ideals of reduction number one whose associated graded ring is not Cohen-Macaulay.

Example 3.4. Let \( R \) be a local Gorenstein ring with infinite residue field, let \( p \) be a prime ideal of height \( g \geq 2 \) such that \( R_p \) is regular, and let \( t \geq 1 \) be an integer. Let \( \alpha_1, \ldots, \alpha_g \) be a regular sequence contained in \( p(t) \), where \( p(t) \) denotes the \( t \)-th symbolic power of \( p \). Write \( J = (\alpha_1, \ldots, \alpha_g) \) and set \( I = J : p(t) = J : p^t \). If either \( g = 2 \) or \( t = 1 \), assume that at least 2 of the \( \alpha_i \)'s are contained in \( p(t+1) \). By [21, Corollary 4.2] we have that \( I^2 = JI \). Hence \( I \) is equimultiple of reduction number one. Assume that \( R/p(t) \) is not Cohen-Macaulay. Since \( I \) is linked to \( p(t) \), it follows that \( R/I \) is not Cohen-Macaulay. Let \( G \) be the associated graded ring of \( I \). By Corollary 2.12, we have that depth \( G = \min\{d, \text{depth } R/I + g\} = \text{depth } R/I + g < d \).

Example 3.5. Let \( R = k[[x_1, x_2, x_3, x_4]] \), where \( k \) is an infinite field, and let \( p \) be the defining ideal of \( k[[t^4, t^3 s, t s^3, s^4]] \),
\[
p = (x_1 x_4 - x_2 x_3, x_1^2 x_3 - x_2^2, x_1 x_3^2 - x_4 x_2^3, x_3^3 - x_4^2 x_2).
\]
The ideal \( p \) is prime of grade 2 and \( R/p \) is not Cohen-Macaulay. Let
\[
J = ((x_1 x_4 - x_2 x_3)^2, (x_3^3 - x_4^2 x_2)^2) \subset p^2
\]
and let
\[
I = J : p =
((x_1 x_4 - x_2 x_3)^2, (x_3^3 - x_4^2 x_2)^2, x_1 x_2 x_3^2 - x_1 x_2^3 - x_4^2 x_2 x_3^3 + x_3^3,
\]
\[
x_1 x_4 x_2 x_3 - x_1 x_2^3 x_3 - x_1 x_4^2 x_3^2 + x_1 x_2 x_3^3, x_1 x_2 x_3 x_4^2 - x_1 x_2 x_3^3 - x_1 x_4 x_3^3 + x_2 x_3^2).
\]
The ideal \( I \) is equimultiple of reduction number one. By Example 3.4, we have that depth \( G(I) = 3 \).
4. Serre properties of $\mathcal{R}$ and $G$.

Now we see how assumptions similar to those of Theorem 2.1 imply the Serre properties for $\mathcal{R}$ and $G$.

**Theorem 4.1.** Let $R$ be a local Cohen-Macaulay ring with infinite residue field, let $I$ be an $R$-ideal with grade $g$, let $J$ be a reduction of $I$ generated by $s$ elements with $\text{ht } J : I \geq s$, $r = r_J(I)$, and let $k \geq 0$ be an integer with $r \leq k$. Furthermore assume that $I$ satisfies $G_s$, $AN_{-k-1}^-$, and that for some integer $t \geq 1$, depth $(R/I^j)_p \geq \min\{\text{dim } R_p - s + k - j, k - j + t\}$ whenever $p \in V(I)$ and $1 \leq j \leq k$. Then $G$ is $S_t$.

**Proof.** We need to show that for every $\mathcal{P} \in \text{Spec}(G)$, depth $G_\mathcal{P} \geq \min\{t, \text{dim } G_\mathcal{P}\}$. Let $q$ denote the contraction of $\mathcal{P}$ to $R$. By Theorem 2.1 we have that depth $G_q \geq \min(\{\text{ht } q\} \cup \{\text{depth } R_q/I_q^j + s - k + j \mid 1 \leq j \leq k\}) \geq \min\{\text{ht } q, t + s\}$. If $\text{ht } q \leq t + s$, then $G_q$ is Cohen-Macaulay and so $G_\mathcal{P}$ is Cohen-Macaulay. Hence we may assume that $\text{ht } q > t + s$. Since $\text{dim } G_\mathcal{P} - \text{depth } G_\mathcal{P} \leq \text{dim } G_q - \text{depth } G_q$, and $\text{dim } G_q/\mathcal{P}G_q \leq \text{dim } G_q/qG_q = \ell(I_q) \leq \ell \leq s$, and $\mathcal{P}$ is equidimensional and catenary, we have that depth $G_\mathcal{P} \geq \text{depth } G_q - \text{dim } G_q/\mathcal{P}G_q \geq t + s - s = t$. \hfill $\Box$

In particular, if the reduction number is small, we have simpler assumptions that imply the $S_1$ property for $G$.

**Corollary 4.2.** Let $R$ be a local Cohen-Macaulay ring with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell$, and reduction number $r \leq 1$. Further assume that $I$ satisfies $G_\ell$, $AN_{-2}^-$, and that $R/I$ has no associated primes of height $\geq \ell + 1$. Then $G$ is $S_1$.

**Proof.** The assertion follows from Theorem 4.1 with $s = \ell$, $J$ a minimal reduction of $I$ such that $\text{ht } J : I \geq \ell$ and $r_J(I) = r$, $k = 1$ and $t = 1$. \hfill $\Box$

**Corollary 4.3.** Let $R$ be a local Cohen-Macaulay ring with infinite residue field, let $I$ be an $R$-ideal with grade $g$, analytic spread $\ell \geq g + 1$, and reduction number $r \leq 2$. Further assume that $I$ satisfies $G_\ell$, $AN_{-3}^-$, that $R/I$ is Cohen-Macaulay, and that $R/I^2$ has no associated primes of height $\geq \ell + 1$. Then $G$ is $S_1$.

**Proof.** The assertion follows from Theorem 4.1 with $s = \ell$, $J$ a minimal reduction of $I$ such that $\text{ht } J : I \geq \ell$ and $r_J(I) = r$, $k = 2$ and $t = 1$. \hfill $\Box$

Let $R$ be a Cohen-Macaulay ring and $I$ an $R$-ideal. We recall that if $p \in V(I)$ is the contraction of a minimal prime of $G$, then $\ell(I_p) = \text{ht } p$. Hence if $G$ is $S_1$ and $\ell(I_p) < \text{ht } p$ for every non-minimal prime $p$ in $V(I)$, then $I^n = I^{(n)}$ for all $n \geq 1$ and $I$ is normally torsion free. Hence we get the following corollary.

**Corollary 4.4.** If in addition to the assumptions of Theorem 4.1, or of Corollary 4.2, or of Corollary 4.3, we have that $\ell(I_p) < \text{ht } p$ for every non-minimal prime $p$ in $V(I)$, then $I^n = I^{(n)}$ for all $n \geq 1$ and $I$ is normally torsion free.
Before studying the Serre properties of the Rees algebra we need the following definition.

**Definition 4.5.** Let $R$ be a local Cohen–Macaulay ring, let $I$ be an $R$-ideal of grade $g$, and let $s \geq g$ be an integer. We say that $I$ is $s$-residually $S_2$ if for every $g \leq i \leq s$ and every $i$-residual intersection $K$ of $I$, $R/K$ satisfies Serre’s condition $S_2$.

**Theorem 4.6.** Let $R$ be a local Cohen–Macaulay ring with infinite residue field, let $I$ be an $R$-ideal with grade $g \geq 2$, analytic spread $\ell$, and reduction number $r$, let $k \geq 0$ be an integer with $r \leq k$. Assume that $I$ satisfies $G_{\ell}$, $AN_{\ell-k-1}^r$ and that for some integer $t \geq 1$, $\dim (R/I)_p \geq \min \{ \dim R_p - \ell + k - j, k - j + t \}$ whenever $p \in V(I)$ and $1 \leq j \leq k$. Furthermore assume that $I$ is $\ell - 2$-residually $S_2$ locally up to height $\ell + t - 1$. Then $R$ is $S_1$.

**Proof.** We need to show that for every $P \in \text{Spec}(R)$, $\dim R_P \geq \min \{ t, \dim R_P \}$. Denote by $q$ the contraction of $P$ to $R$. By Theorem 2.1 we have that depth $G_q \geq \min \{ \text{ht } q, t + \ell \}$. If $\text{ht } q \geq t + \ell$, then depth $R_q \geq t + \ell$. Since $\dim R_{P'} \geq \dim R_q - \dim R_{q'}$, and $\dim R_{q'/P} \leq \dim R_{q}/qR_{q} = \ell(I_q) \leq \ell$, it follows that depth $R_{P'} \geq \dim R_{q'} - \dim R_{q}/P \geq t$.

If $\text{ht } q \leq t + \ell - 1$, then $G_q$ is Cohen–Macaulay. We claim that also $R_q$ is Cohen–Macaulay, which implies the Cohen–Macaulayness of $R_{P'}$. By [22, Corollary 3.6] to prove our claim we only need to show that $r(I_q) < \ell(I_q)$. If $\ell(I_q) = \ell$, then by assumption $r(I_q) \leq r \leq \ell - g + 1 \leq \ell - 1$, and so we are done. If $\ell(I_q) < \ell$, then $r(I_q) = 0 < \ell(I_q)$ by [4, 2.1(g)].

Now we analyze the relationship between the Serre properties for $R$ and for $G$. The following result generalizes [2, Theorem 1.5].

**Theorem 4.7.** Let $R$ be an equidimensional and universally catenary Noetherian ring satisfying $S_1$, and let $I$ be an $R$-ideal of positive height. The following two conditions are equivalent:

1. $R$ satisfies $S_1$.
2. (i) $G$ satisfies $S_{\ell-1}$, and
   (ii) If $q \in V(I)$ and $\ell(I_q) = \text{ht } q \leq \ell - 1$, then $r(I_q) < \ell(I_q)$.

Furthermore, if $I$ is $G_{\ell}$, then (2)(ii) can be replaced by

2. (ii') If $q \in V(I)$ and $\ell = \ell(I_q) = \text{ht } q \leq \ell - 1$, then $r(I_q) < \ell(I_q)$.

**Proof.** (1) $\Rightarrow$ (2). Let $P$ be a prime ideal of $G$ and denote by $P'$ its inverse image in $R$. Localize $R$ at $p = R \cap P$ and denote the resulting local ring by $R$. Since $R$ is equidimensional and universally catenary and $\text{ht } I > 0$, we have that $\text{ht } P = \text{ht } P + 1$. Furthermore $\text{ht } P \leq \text{ht } P + 1$. Hence, if $\dim G_P \geq t - 1$, then $\dim R_{P'} \geq t$ and so depth $R_{P'} \geq t$ by (1). Also, depth $R_{P'} \geq t - 1$ since $R$ satisfies $S_1$. From the exact sequences

$$0 \rightarrow (It)R \rightarrow R \rightarrow R \rightarrow 0$$
and
\[ 0 \to IR \to R \to G \to 0 \]
it follows that depth \( G_P \geq t - 1 \). If \( \dim G_P < t - 1 \), then \( R_P \) is Cohen-Macaulay by (1), so \( G_P \) is Cohen-Macaulay and the proof of (2)(i) is complete.

In the assumptions of (2)(ii) we have that \( \dim R_q \leq t \) and so \( R_q \) is Cohen-Macaulay. The conclusion follows from [17, Theorem 2.3].

(2) \( \Rightarrow \) (1). We may assume \( t \geq 0 \). Let \( P \) be a prime ideal of \( R \) and let \( p = R \cap P \).

If \( I \not\subset p \) we have \( R_p = R_p[t] \) which satisfies \( S_1 \) since \( R_p \) does, and so \( R_p \) satisfies \( S_1 \). Hence we may assume that \( I \subset p \). If \( I t \not\subset P \), by the usual prime avoidance argument there exists a nonzero divisor \( x \in I \) such that \( xt \not\subset P \). We have that \( x \) is a regular element of \( R \) and localizing \( R \) at \( xt \) we get that \( (R/xtR)_{xt} = (G)_{xt} \). So the assumption (2)(i) implies that depth \( R_p \geq \min \{ \dim R_p, t \} \).

If \( I t \subset P \), then \( P_p \) is the irrelevant maximal ideal of \( R_p \). If \( \dim R_p > t \), (2)(i) and [12, 3.10] imply that depth \( R_p \geq t \), hence depth \( R_p \geq t \). If \( \dim R_p \leq t \), then \( R_p \) and \( G_p \) are Cohen-Macaulay. In this case \( R_p \) is Cohen-Macaulay by (2)(ii) and [17, Theorem 2.3]. If \( I = G \) (2)(ii'), [17, Theorem 2.3] and [22, Theorem 2.4] imply that \( R_p \) is Cohen-Macaulay and the proof is complete.

Notice that condition (ii) is empty if \( t \leq \text{ht} I \), and, if \( I \) is \( G \ell \), condition (ii') is empty if \( t \leq \ell \). In these cases we have that \( R \) satisfies \( S_t \) if and only if \( G \) satisfies \( S_{t-1} \).

Combining Theorem 4.1 with Theorem 4.7 we obtain the following result (compare with Theorem 4.6).

**Theorem 4.8.** Let \( R \) be a local Cohen-Macaulay ring with infinite residue field, let \( I \) be an \( R \)-ideal with grade \( g > 0 \), analytic spread \( \ell \), reduction number \( r \), and let \( k \geq 0 \) be an integer with \( r \leq k \). Furthermore assume that \( I \) satisfies \( G_\ell \), \( AN_{\ell-k-1} \) and that for some integer \( t \geq 1 \), depth \( (R/I^j)_p \geq \min \{ \dim R_p - k - j + t - 1 \} \) whenever \( p \in V(I) \) and \( 1 \leq j \leq k \). Finally suppose that if \( q \in V(I) \) and \( \ell = \ell(I_q) = \text{ht} q \leq t - 1 \), then \( r(I_q) < \ell(I_q) \). Then \( R \) satisfies \( S_t \).

**Remark 4.9.** ([2, 2.3]). Let \( R \) be a polynomial ring in \( n \) variables (localized at the maximal irrelevant ideal) over an infinite perfect field, let \( I \) be an \( R \)-ideal with grade \( g \geq 2 \), analytic spread \( \ell \), reduction number \( r \), and let \( k \geq 0 \) be an integer with \( r \leq k \). Further assume that \( I \) satisfies \( G_\ell \), \( AN_{\ell-k-1} \), that \( I \) is \( \ell - 2 \) residually \( S_2 \) locally up to height \( \ell + 1 \), and that depth \( (R/I^j)_p \geq \min \{ \dim R_p - k - j + t - 2 \} \) for every \( p \in V(I) \) and whenever \( 1 \leq j \leq k \). Let \( J \) be the ideal of the presentation of \( R \); i.e., \( R \cong R[T_1, \ldots, T_m]/J \). Let \( h_1, \ldots, h_s \) be a set of generators of \( J \) and consider the Jacobian matrix \( M = \partial(h_1, \ldots, h_s)/\partial(x_1, \ldots, x_n, T_1, \ldots, T_m) \). Let \( N \) be the ideal generated by all \( (m - 1) \times (m - 1) \) minors of \( M \). If \( \text{ht}(J, N) \geq m + 1 \), then \( R \) is normal.
We call the value \( \inf \{ \text{depth } R/I^j \mid j \geq 1 \} \) the Burch number of \( I \) and we denote it by \( B(I) \).

**Corollary 4.10.** With the assumptions of Theorem 2.1 we have that
\[
\min(\{d - \ell\} \cup \{\text{depth } R/I^j - k + j \mid 1 \leq j \leq k\}) \leq B(I) \leq d - \ell.
\]

**Proof.** By Theorem 2.1 and Remark 2.11 we have that \( \min(\{d\} \cup \{\text{depth } R/I^j - \ell - k + j \mid 1 \leq j \leq k\}) \leq \text{depth } G \leq B(I) + \ell \). The conclusion follows since \( B(I) = \text{depth}_{mG} G \leq \dim G - \dim G/mG \).

**Lemma 4.11.** Let \( R \) be a local Cohen-Macaulay ring of dimension \( d \) and let \( I \) be an \( R \)-ideal with analytic spread \( \ell \). If \( G \) satisfies \( S_t \) for some positive integer \( t \leq d - \ell \), then \( B(I) \geq t \).

**Proof.** Since \( G \) is equidimensional and catenary, we have that \( \text{ht } mG = d - \ell \); hence \( \text{depth}_{mG} G = \min\{\text{depth } G_P \mid P \in \text{V}(mG)\} \geq t \). It follows that \( B(I) \geq t \).

From Theorem 4.1 and Lemma 4.11 we get the following result.

**Corollary 4.12.** Let \( R \) be a local Cohen-Macaulay ring with infinite residue field, let \( I \) be an \( R \)-ideal with grade \( g \), analytic spread \( \ell \), reduction number \( r \), and let \( k \geq 0 \) be an integer with \( r \leq k \). Furthermore assume that \( I \) satisfies \( G_\ell \), \( AN_{\ell-1} \), and that for some integer \( t \) with \( 1 \leq t \leq d - \ell \), depth \( (R/I^j)_p \geq \min\{\dim R_p - s + k - j, k - j + t\} \) whenever \( p \in V(I) \) and \( 1 \leq j \leq k \). Then \( B(I) \geq t \).

Now we present another result on the Cohen-Macaulayness of \( G \) (compare with [15, Theorem 3.1]).

**Theorem 4.13.** Let \( R \) be a local Gorenstein ring of dimension \( d \) with infinite residue field, let \( I \) be an \( R \)-ideal with grade \( g \), analytic spread \( \ell \), reduction number \( r \), and let \( k \geq 0 \) be an integer with \( r \leq k \). Further assume that \( I \) is unmixed, generically a complete intersection, that \( I \) satisfies \( G_\ell \), \( AN_{\ell-1} \), that depth \( (R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, 1/2(\dim R_p - \ell + 1) + k - j\} \) whenever \( p \in V(I) \) and \( 1 \leq j \leq k \), and that \( \ell(I_p) < \text{ht } p \) for every non-minimal prime \( p \) of \( I \). Then \( G \) is Cohen-Macaulay.

**Proof.** First notice that our assumption on the depth of the powers implies that for every \( p \in V(I) \), depth \( (R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j + 1\} \) whenever \( 1 \leq j \leq k \). Hence \( I \) is normally torsion free by Corollary 4.4, and so \( R[I, t^{-1}] \) is quasi Gorenstein by [16, the proof of Theorem 3.2].

Next we show that depth \( R[I, t^{-1}]_P \geq 1 + 1/2 \dim R[I, t^{-1}]_P \) for every \( P \in \text{Spec}(R[I, t^{-1}]) \) with \( \text{ht } P \geq 2 \). Since the inequality is trivially satisfied if \( R[I, t^{-1}]_P \) is Cohen-Macaulay, we can assume that \( t^{-1} \in P \) and so \( R[I, t^{-1}]_P/(t^{-1})_P \cong G_P \not\equiv 0 \). Let \( q \) denote the contraction of \( P \) to \( R \), and notice that \( I \subset q \).

By Theorem 2.1 we have that depth \( G_q \geq \min(\{\text{ht } q\} \cup \{1/2(\text{ht } q + \ell + 1) \mid 1 \leq j \leq k\}) \). If \( \text{ht } q \leq \ell + 1 \), then \( R[I, t^{-1}]_q \) is Cohen-Macaulay. If \( \text{ht } q > \ell + 1 \), then the inequality \( \dim R[I, t^{-1}]_P - \text{depth } R[I, t^{-1}]_P \leq \dim R[I, t^{-1}]_q - \text{depth } R[I, t^{-1}]_q \) shows that depth \( R[I, t^{-1}]_P \geq \dim G_P + 1 - \text{ht } q + \text{depth } G_q \geq \).
\[ \dim G_P + 3/2 - 1/2 \, \text{ht} \, q + 1/2 \, \ell = 1/2(\dim G_P + 1) + 1 + 1/2(\dim G_P - \text{ht} \, q + \ell) \geq 1/2 \dim R[It, t^{-1}]_{P} + 1. \]

It follows from [14, Lemma 5.8] that \( R[It, t^{-1}] \) is Cohen-Macaulay; hence \( G \) is Cohen-Macaulay.

Notice that the depth assumptions of Theorem 4.13 are weaker than those of [15, Theorem 3.1], since in the above theorem we assume depth(\( R/I^j \)) \geq 1/2(\dim R/p - \ell + 1) + k - j for 1 \leq j \leq k, if \( \text{ht} \, p \geq \ell + 1 \).

Now we want to apply our results to the defining ideals of monomial varieties of codimension two. We can prove that \( G \) is Cohen-Macaulay. Furthermore \( I \) is normally torsion free, if and only if \( I \) is a complete intersection locally in codimension 3. We recall the definition, as in [8].

Let \( k[u_1, \ldots, u_n] \) be a polynomial ring over an infinite field \( k \). Consider the semigroup ring \( k[u_1^{a_1}, u_2^{a_2}, \ldots, u_n^{a_n}, u_1^{b_1} \ldots u_n^{b_n}] \subset k[u_1, \ldots, u_n] \), where \( a_j, b_j, c_j \in \mathbb{N}_0 \), \( a_j > 0 \), \( (b_j, c_j) \neq (0,0) \) for \( 1 \leq j \leq n \), and further \( (b_1, \ldots, b_n) \neq (0, \ldots, 0) \) and \( (c_1, \ldots, c_n) \neq (0, \ldots, 0) \). Let \( I \subset R = k[x_1, \ldots, x_n, y, z] \) denote the defining ideal of this semigroup ring.

The ideal \( I \) is prime of height two, and the variety in question is affine of codimension two. Furthermore, the analytic spread of \( I \) is equal to two if \( I \) is a complete intersection and equal to three in all the remaining cases ([7, Theorem 4.2]). Also, \( I \) has reduction number one ([1, Corollary 3.4]) and depth \( R/I \geq n - 1 \) ([20, Theorem 2.3]). Notice that \( I \) satisfies \( G_{\ell} \). It follows from Corollary 2.12 that \( G \) is Cohen-Macaulay.

If \( I \) is normally torsion free, then by [13, Theorem 2.2] we have that \( \ell(I_p) < \text{ht} \, p \) for every non-minimal prime \( p \) in \( V(I) \). It follows that if \( p \in V(I) \) and \( \text{ht} \, p = 3 \), then \( \ell(I_p) = 2 \). Hence by [6] \( I_p \) is a complete intersection.

If \( I \) is a complete intersection locally in codimension 3, then \( \ell(I_p) < \text{ht} \, p \) for every non-minimal prime \( p \) in \( V(I) \) and it follows from Corollary 4.4 that \( I \) is normally torsion free and that \( I^n = I^{(n)} \) for every \( n \geq 1 \). Also, by [11, Proposition], \( G \) is Gorenstein.

For example, if \( I \subset k[x, y, z, w] \) is the homogeneous ideal of a monomial curve in \( \mathbb{P}^3 \) lying on the quadric surface \( xy - wz \), then \( I \) is a complete intersection on the punctured spectrum of \( R \) and so \( I \) is normally torsion free, which recovers [18, Proposition 2.3]. In general a monomial curve in \( \mathbb{P}^3 \) need not be normally torsion free. For example the ideal \( I \subset k[x, y, z, w] \) defining \( k[x^3, t^4u, t^3u^2, u^5] \) is not normally torsion free, indeed \( I^2 \neq I^{(2)} \).

Now let \( R := k[x, y, z, w] \) and let \( I \subset R \) be the homogeneous ideal of a projective monomial curve defined by \( x = u_1^{a_1}, y = u_2^{a_1}, z = u_1^{b_1}u_2^{a_1-b_1}, w = u_1^{c_1}u_2^{a_1-c_1}(a_1 > b_1 > c_1) \). We want to find necessary and sufficient conditions for \( I \) to be normally torsion free, in terms of the exponents \( a_1, b_1 \) and \( c_1 \).
Since $I$ is normally torsion free if and only if $I$ is a complete intersection locally in codimension 3, take $p \in V(I)$ with $\text{ht } p = 3$ and denote by $\overline{p}$ its image in $R/I$. Since $\text{ht } \overline{p} = 1$, either $u_1^{a_1} \not\in \overline{p}$ or $u_2^{a_1} \not\in \overline{p}$. If $u_1^{a_1} \not\in \overline{p}$, then $k[u_1^{a_1}, u_2^{a_1}, u_1^{a_1}u_2] = k[x_1^{a_1}, x_1^{a_1} - b_1, x_1^{a_1} - c_1]_{\overline{p}}$, if $u_2^{a_1} \not\in \overline{p}$, then $k[u_1^{a_1}, u_2^{a_1}, u_1^{a_1}u_2] = k[x_1^{a_1}, x_1^{a_1} - b_1, x_1^{a_1} - c_1]_{\overline{p}}$, where we denote by $\overline{p}$ the dehomogenization of the ideal $\overline{p}$. The ideal $(I^n)^{\overline{p}}$ is a complete intersection if and only if $k[x_1^{a_1}, x_1^{a_1} - b_1, x_1^{a_1} - c_1]$ and $k[x_1^{a_1}, x_1^{b_1}, x_1^{c_1}]$ are Gorenstein.

Following [3, page 178] we denote by $<v_1, \ldots, v_n>$ the semigroup $S$ generated by the integers $v_1, \ldots, v_n$. The conductor $c = c(S)$ of $S$ is defined by $c = \max\{a \in \mathbb{N} | a - 1 \not\in S\}$. We say that the semigroup $S$ is symmetric if, for all $i$ with $0 \leq i \leq c - 1$, one has $i \in S$ if and only if $c - i - 1 \not\in S$. By [3, Theorem 4.4.8], $S$ is symmetric if and only if $k[t^{v_1}, \ldots, t^{v_n}]$ is Gorenstein.

Hence we have that $I$ is normally torsion free if and only if $<a_1, b_1, c_1>$ and $<a_1, a_1 - b_1, a_1 - c_1>$ are symmetric.

Notice that if $I$ is the above mentioned ideal defining $k[t^5, t^4u, t^3u^2, u^5]$, then $<a_1, b_1, c_1> = <5, 4, 3>$ is not symmetric. Hence $I$ is not normally torsionfree.

References