On the structure of pure $O$-sequences

ABSTRACT: An order ideal is a collection $X$ of monomials in the variables $y_1, \ldots, y_r$ such that, whenever $M \in X$ and $N$ divides $M$, $N \in X$. If all maximal monomials of $X$ (the order being given by divisibility) have the same degree, $X$ is called pure. A pure $O$-sequence is the vector, $h = (1, h_1, h_2, \ldots, h_e)$, counting the number of monomials of $X$ in each degree. Equivalently, pure $O$-sequences can be characterized as the $f$-vectors of pure multicompleses, or, in commutative algebra, as the $h$-vectors of artinian monomial level algebras. The study of pure $O$-sequences began with the seminal work of Stanley in the Seventies, and has since played a significant role in the development of at least three different areas: the study of simplicial complexes and their $f$-vectors, the theory of level algebras, and the theory of matroids.

In a work in preparation (joint with M. Boij, J. Migliore, R. Miró-Roig, and U. Nagel), using both algebraic and combinatorial techniques, we study the structure of pure $O$-sequences. Whereas their first half behaves extremely well, in the second half, suddenly, many bad things may happen. Thus, a complete characterization of pure $O$-sequences appears to be impossible. However, we conjecture that a very strong structural result must nonetheless be true, called the Interval Conjecture for Pure $O$-sequences (ICP). (The ICP extends the Interval Conjecture I recently formulated for artinian level algebras.)

Although we have solved the ICP in a few special cases, it is still wide open in general. Our work also includes: a study of the unimodality property for pure $O$-sequences, where we actually conjecture that this may fail in the worst possible way; a characterization of the first half of pure $O$-sequences; a useful connection of the ICP with Stanley’s conjecture on the $h$-vectors of matroids; a study of pure $O$-sequences of type 2 (that is, when there are exactly 2 maximal monomials in the pure order ideal $X$); an analysis, from a commutative algebra viewpoint, of the role played by the Weak Lefschetz Property.