Intersecting loops on surfaces and string topology

Kate Poirier, UC Berkeley

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Question

What is the algebraic topology of a manifold?

We will discuss the *string topology* of surfaces, 3-dimensional manifolds, and higher-dimensional manifolds. Our first example of a *string topology operation* will be the *Goldman bracket*. 
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A *surface* is a real 2-dimensional manifold.

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A *surface* is something that locally looks like the Euclidean plane $\mathbb{R}^2$. 
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![Diagram of a surface](image-url)
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Are all surfaces homeomorphic to the 2-dimensional sphere? **NO!**
Surfaces

Other examples
Surfaces

Theorem (Classification of closed, orientable surfaces)

A (closed, orientable) surface is completely determined up to homeomorphism by its genus.
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The surface of the chair has genus 9.
Curves on Surfaces

Definition

A *closed curve* or *loop* on a surface is a continuous map from the circle to the surface.
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Example: every loop on the 2-dimensional sphere is freely homotopic to a constant loop.
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Exercise: the loops $ab$ and $ba$ on the torus are freely homotopic.
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**Theorem (2-dimensional Poincaré conjecture)**

*If every loop on a fixed surface is homotopic to a constant loop, then the surface is homeomorphic to the 2-dimensional sphere.*
The Goldman Bracket

Fix an oriented surface $\Sigma$. 
Consider two free homotopy classes $\alpha$ and $\beta$ of closed curves on $\Sigma$. 
Consider representative curves that intersect one another only in transverse double points $p$. 

The Goldman Bracket
Cut $\alpha$ and $\beta$ at $p$ and reconnect the strands in the other way that respects their orientation.
Let $\alpha \cdot_p \beta$ be the closed curve obtained by cutting and reconnecting.
The Goldman Bracket

Each intersection point \( p \) of \( \alpha \) and \( \beta \) gives a free homotopy class of closed curves \( \alpha \cdot_p \beta \).

Let \( H \) be the \( \mathbb{Q} \)-vector space generated by the set of free homotopy classes of closed curves on \( \Sigma \). (In general, \( H \) is infinite dimensional.)

Define

\[
[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \pm \alpha \cdot_p \beta.
\]

Signs depend on the orientation of \( \Sigma \)

\[
[\alpha, \beta] = \quad \alpha \cdot_q \beta \quad \text{and} \quad \alpha \cdot_p \beta
\]
The Goldman Bracket

Definition (Goldman Bracket)

Extend \([ , , ]\) linearly to obtain a map \([ , , ] : H \otimes H \to H.\)

Theorem (Goldman)

The bracket is well defined, is skew-symmetric \([\alpha, \beta] = -[\beta, \alpha]\) and satisfies the Jacobi identity \([[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0. That is, \((H, [ , , ])\) is a Lie algebra.

Idea of proof of Jacobi identity: terms cancel in pairs.
The Goldman Bracket

Example

Let $\Sigma$ be the 2-dimensional torus. Every closed curve on the 2-dimensional torus has the form $a^k b^\ell$ for some $k, \ell \in \mathbb{Z}$. Consider $[a^k b^\ell, a^m b^n]$.
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\[ [a^k b^\ell, a^m b^n] = \pm (kn - \ell m) a^{k+m} b^{\ell+n} \]
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There are $kn + \ell m$ intersection points $p$. Each term $(a^k b^\ell) \cdot_p (a^m b^n)$ is freely homotopic to $\pm a^{k+m} b^{\ell+n}$. Therefore

$$[a^k b^\ell, a^m b^n] = \pm (kn - \ell m) a^{k+m} b^{\ell+n}.$$
If $\alpha$ and $\beta$ have representative closed curves that are disjoint, then $[\alpha, \beta] = 0$.

**Theorem (Goldman)**

Let $\alpha$ and $\beta$ be free homotopy classes such that $\alpha$ has a representative with no self intersection and such that $[\alpha, \beta] = 0$. Then $\alpha$ and $\beta$ have disjoint representatives.

The number of terms in $[\alpha, \beta]$ (counting multiplicity) is always less than or equal to the minimal intersection number of curves representing $\alpha$ and $\beta$.

**Theorem (Chas-Gadgil)**

Let $\alpha$ and $\beta$ be free homotopy classes such that $\alpha$ has a representative with no self intersection. Then the number of terms in $[\alpha, \beta]$ is equal to the minimal intersection number of curves representing $\alpha$ and $\beta$.

That is, if $\alpha$ has a representative with no self intersection, there is no cancellation in the Goldman bracket.
The Goldman Bracket

Theorem (Gadgil)

Let $f : \Sigma \to \Sigma'$ be a homotopy equivalence. Then $f$ is homotopic to a homeomorphism if and only if $f$ respects the Goldman bracket.
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Let $f : \Sigma \rightarrow \Sigma'$ be a homotopy equivalence. Then $f$ is homotopic to a homeomorphism if and only if $f$ respects the Goldman bracket.

This theorem is uninteresting in the case of closed surfaces; every homotopy equivalence is homotopic to a homeomorphism. However, the Goldman bracket may be defined for surfaces with boundary. In this case, the theorem says that the Goldman bracket detects whether a weak equivalence of surfaces is equivalent to a homeomorphism.

Surfaces that are homotopy equivalent but not homeomorphic.
3-dimensional version

The Goldman Bracket generalizes to higher dimensional manifolds.

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A 3-dimensional manifold is something that locally looks like Euclidean 3-dimensional space $\mathbb{R}^3$. 
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**Definition**

A 3-dimensional manifold is something that locally looks like Euclidean 3-dimensional space $\mathbb{R}^3$.

The surface is replaced by a 3 manifold; a loop on the surface is replaced by a loop in the manifold or a fibered torus in the manifold.

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A closed 1-dimensional family of loops or fibered torus in a 3-dimensional manifold is a continuous map from the fibered torus to the manifold.
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We consider fibered tori in a 3-manifold $M$ up to deformation.

**Definition**

Let $H_0$ be the $\mathbb{Q}$-vector space generated by deformation classes of closed curves in $M$. Let $H_1$ be the $\mathbb{Q}$-vector space generated by deformation classes of fibered tori in $M$. 
3-dimensional version

Intersections

\[ H_0 \otimes H_1 \rightarrow H_0 \]
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The generalized string bracket is skew symmetric and satisfies the Jacobi identity. In fact, the Goldman Bracket generalizes to all dimensions.

**Theorem (Chas-Sullivan)**

Let $M$ be a closed, oriented $d$-dimensional manifold and let $LM = \text{Maps}(S^1, M)$ be its free loop space. Then the $S^1$-equivariant homology $H^*_S(LM)$ of the free loop space of $M$ is a graded Lie algebra.
Previous work of Abbaspour uses the string product (an operation related to the string bracket) to detect hyperbolic 3-dimensional manifolds.

Current work of Chas-Gadgil uses the string bracket to study decompositions of 3-dimensional manifolds.

Such decompositions are related to those guaranteed by Thurston’s celebrated Geometrization conjecture, which has been proven by Perelman and which implies the 3-dimensional Poincaré conjecture.
The construction producing the Goldman Bracket generalizes to other algebraic operations.

Theorem (Turaev)
\[ \Delta : H \rightarrow H \otimes H \] is well defined. \((H, [\ , \ ], \Delta)\) is a Lie bialgebra.

If \(\alpha\) has a representative loop with no self intersections, then \(\Delta(\alpha) = 0\).

Question (Turaev)
If \(\Delta(\alpha) = 0\) then does \(\alpha\) have a representative with no self intersections?

Answer (Chas): No! Turaev's cobracket is zero in many nontrivial examples.
Turaev’s cobraclacket for surfaces

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Again, the theorem generalizes to higher-dimensional manifolds.

**Theorem (Chas-Sullivan)**

Let \( M \) be a closed, oriented \( d \)-dimensional manifold and let \( LM \) be its free loop space. Then \( H^*_{S^1}(LM, M) \) is a graded Lie bialgebra.
Cutting and reconnecting at intersection points yields generalized operations

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String topology

Gluing of surfaces along boundary corresponds to composition of operations $H^\otimes k \to H^\otimes \ell$. For any surface with boundary, the operation corresponding to it can be given as a composition of the bracket and cobaracket.
String topology

This structure generalizes to closed, oriented $d$-dimensional manifolds. Every orientable surface with boundary gives rise to a string topology operation.

\[
H_*^k \to H_*^\ell \quad \text{where} \quad H_* = \begin{cases} 
H_* (LM) & \text{or} \\
H_*^{S^1} (LM, M) 
\end{cases}
\]

Gluing of surfaces along boundary corresponds to composition of operations.

Theorem (Chas-Sullivan, Cohen-Godin, Chataur, Godin, P.-Rounds, ...)

String topology operations describe a topological quantum field theory (TQFT) associated to $H_*$. 
Thank you!