A mathematician’s Divina Commedia
his travels through the three realms: the physical, the virtual, and the platonic

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Scholar-on-Campus
Outline

1. Size matters

... and so does shape

Symmetry and singularities

Ultraworld

Disclaimer

None of what I say is potentially nearly as false as it sounds plausibly true.
Outline

1. Size matters

2. ... and so does shape
Outline

1. Size matters
2. ... and so does shape
3. The beauty and the beast: symmetry and singularities
Outline

1. Size matters
2. ... and so does shape
3. The beauty and the beast: symmetry and singularities
4. A new utopia? Ultraworld
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Three scientific paradigms

First paradigm

The physical (inanimate) world can be modeled by mathematics.
Three scientific paradigms

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Second paradigm
Mathematics is developed rigorously through logical reasoning and hence generates irrefutable truths.
Three scientific paradigms

First paradigm
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Second paradigm
Mathematics is developed rigorously through logical reasoning and hence generates irrefutable truths.

Third paradigm
Computers can emulate this modeling, albeit only a finite portion of it.
Three scientific paradigms, and a sentimental one

First paradigm
The physical (inanimate) world can be modeled by mathematics.

Second paradigm
Mathematics is developed rigorously through logical reasoning and hence generates irrefutable truths.

Third paradigm
Computers can emulate this modeling, albeit only a finite portion of it.

Sentimental Paradigm
Mathematics is most beautiful and most perfect.
1. Size matters

2. ... and so does shape

3. The beauty and the beast: symmetry and singularities

4. A new utopia? Ultraworld
Definition (Galileo 1564–1642)

A collection is infinite if it fits inside a smaller part of itself.
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Example (Hilbert’s Grand Hotel)
Imagine a hotel with a room for each number.

1 2 3 4 5 ...
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Example (Hilbert’s Grand Hotel)
Imagine a hotel with a room for each number. Even if the hotel is booked full, a new guest can be accommodated by having everyone change to the next room.

1 → 2 → 3 → 4 → 5 → ...
Definition (Galileo 1564–1642)
A collection is infinite if it fits inside a smaller part of itself.

Example (Hilbert’s Grand Hotel)
Imagine a hotel with a room for each number. Even if the hotel is booked full, a new guest can be accommodated by having everyone change to the next room.

Corollary
There are infinitely many numbers.
Definition (Galileo 1564–1642)
A collection is infinite if it fits inside a smaller part of itself.

Example (Euclid’s line segment)
Points on a line segment.
Definition (Galileo 1564–1642)
A collection is infinite if it fits inside a smaller part of itself.

Example (Euclid’s line segment)
Points on a line segment can be squeezed together.
Finiteness: the Pigeon-Hole Principle

Theorem

*If there are more pigeons than holes, at least two will have to share a hole.*
Finiteness: the Pigeon-Hole Principle

Theorem

If there are more pigeons than holes, at least two will have to share a hole.

Proof.
Theorem

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Proof.
Schools of thought

Philosophers and scientists about

Physical world

Metaphysical world
Schools of thought

Physical world

Plato: shadows

Metaphysical world

ideas
Schools of thought

Physical world

Plato: shadows

Aristotle: \( \alpha\pi\varepsilon\iota\rho\omicron\nu = \text{potential infinity.} \)

Metaphysical world

ideas

no actual infinity
Schools of thought

**Physical world**

- **Plato:** shadows
- **Aristotle:** ἀπειρόν = potential infinity.
- **Zoroaster:** मदयम (gaēθa)

**Metaphysical world**

- ideas
- no actual infinity

(about 1500 BC)
Schools of thought

Physical world
- Plato: shadows
- Aristotle: \( \alpha \pi \epsilon \iota \rho \omicron \omicron = \) potential infinity.
- Zoroaster: finite time

Metaphysical world
- ideas
- no actual infinity
- endless time
Schools of thought

Physical world

Plato: shadows

Aristotle: $\alpha\pi\epsilon\iota\rho\omicron\upsilon = $ potential infinity.

Zoroaster: finite time

Newton: infinite Euclidean space

Metaphysical world

ideas

no actual infinity

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mechanism
Schools of thought

Physical world

Plato: shadows

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Zoroaster: finite time

Newton: infinite Euclidean space

Einstein: universe is bounded,

Metaphysical world

ideas

no actual infinity

endless time

mechanism

relativity
Schools of thought

(1858–1947)

Physical world

Plato: shadows

Aristotle: $\alpha\pi\varepsilon\iota\rho\omicron\nu = \text{potential infinity.}$

Zoroaster: finite time

Newton: infinite Euclidean space

Einstein: universe is bounded, and discrete,

Metaphysical world

ideas

no actual infinity

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mechanism

relativity

uncertainty
Schools of thought

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Zoroaster: finite time
Newton: infinite Euclidean space
Einstein: universe is bounded,
Planck: and discrete,
Hawking: and expanding.

Metaphysical world

ideas
no actual infinity
endless time
mechanism
relativity
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singularity
Schools of thought

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Planck: and discrete,
Hawking: and expanding.
Turing: hardware = limited

Metaphysical world

ideas
no actual infinity
endless time
mechanism
relativity
uncertainty
singularity
software = unlimited
Mind versus machine

Theorem (Gödel ’31)

Some mathematical facts can never be obtained by a computer.
Mind versus machine

Theorem (Gödel ’31)

Some mathematical facts can never be obtained by a computer.

Example (Matiyasevich ’70)

No computer can check whether a random equation has a solution.
Mind versus machine

Theorem (Gödel ’31)
Some mathematical facts can never be obtained by a computer.

Anti-theorem
Computers can model facts that cannot be grasped by our human minds.
Mind versus machine

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Anti-theorem

Computers can model facts that cannot be grasped by our human minds.

Example (Knuth ’76)

Recursion can create unfathomable numbers.

\[ a \uparrow^n b := a \uparrow \ldots \uparrow b \]

\[ \underbrace{a \uparrow \ldots \uparrow b}_n \]
Mind versus machine

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Example (Knuth ’76)

Recursion can create unfathomable numbers. For instance, this is a self-referential definition of Knuth’s arrow

\[ a \uparrow^n b := \underbrace{a \uparrow^{n-1} a \uparrow^{n-1} \ldots \uparrow^{n-1} a}_{b} \]
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Recursion can create unfathomable numbers.

\[ 3 \uparrow 3 = 3^3 = 27 \]
Mind versus machine

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Example (Knuth ’76)

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\[
3 \uparrow 3 = 3^3 = 27 \\
3 \uparrow\uparrow 3 = 3^{3^3} = 7,625,597,484,987
\]
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\[ 3 \uparrow\uparrow\uparrow 3 \text{ is an immense number;} \]
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3 \uparrow 3 = 3^3 = 27 \\
3 \uparrow\uparrow 3 = 3^{3^3} = 7,625,597,484,987 \\
3 \uparrow\uparrow\uparrow 3 \text{ is an immense number;} \\
9 \uparrow^9 9 \text{ is unfathomable, and to our mind looks like infinity.}
\]
Dealing with big numbers

Adjustment

*precision*

Example (engineering)

Number of seconds in a (Julian) year:

$$3600 \times 24 \times 365.25 = 31557600$$
Dealing with big numbers

Adjustment
- precision approximation

Example (engineering)

Number of seconds in a (Julian) year:

\[ 3600 \times 24 \times 365.25 = 30000000 \approx 3 \times 10^7 \]
Dealing with big numbers

Adjustment
- precision approximation
- unrestricted

Example (cosmology)
Collision speed of two light particles:
\[ v_{\text{coll}} = v_1 + v_2 = c + c \]
Dealing with big numbers

Adjustment
- precision approximation
- unrestricted restricted

Example (cosmology)
Collision speed of two light particles:
\[ v_{\text{coll}} = v_1 + v_2 = c + c = c \]
Dealing with big numbers

Adjustment
- precision approximation
- unrestricted restricted
- accuracy

Example (computing)
In a 1K-computer, we have:

\[ 881 \times 883 = 777923 \]
Dealing with big numbers

- precision approximation
- unrestricted restricted
- accuracy modularity

Example (computing)
In a 1K-computer, we have:
\[ 881 \times 883 = 777923 \equiv 923 \mod 1000 \]
Dealing with big numbers

Adjustment
- precision approximation
- unrestricted restricted
- accuracy modularity
- entropy

Example (cryptography)
Multiplying:
881 × 883 ≡ 777923
Dealing with big numbers

**Adjustment**
- precision approximation
- unrestricted restricted
- accuracy modularity
- entropy algorithm

**Example (cryptography)**

Multiplying versus factoring:

\[
881 \times 883 \equiv 777923
\]
Outline

1. Size matters
2. ...and so does shape
3. The beauty and the beast: symmetry and singularities
4. A new utopia? Ultraworld
Dimension of a line

**Definition (Dimension)**

Dimension is the number of independent directions in which we can move.

---

\[ \cdots \quad 0 \quad \cdots \]
Definition (Dimension)

Dimension is the number of independent directions in which we can move.

- We can go **left**
Dimension of a line

Definition (Dimension)
Dimension is the number of independent directions in which we can move.

- We can go left or right
Definition (Dimension)

Dimension is the number of independent directions in which we can move.

- We can go left or right
- Going left is the same as going right backwards, so, as a direction, it is dependent:
Definition (Dimension)

Dimension is the number of **independent** directions in which we can move.

- We can go left or right
- Going left is the same as going right backwards, so, as a direction, it is **dependent**:

```plaintext
line is one-dimensional
```

... 0 ...
Dimension and curvature

- Descartes’ plane is 2-dimensional

Flat geometry
Dimension and curvature

- Descartes’ plane is 2-dimensional
- Newton’s universe is 3-dimensional
Dimension and curvature

- Descartes’ plane is 2-dimensional
- Newton’s universe is 3-dimensional
- Earth’s surface is 2-dimensional

Elliptic geometry
Dimension and curvature

- Descartes’ plane is 2-dimensional
- Newton’s universe is 3-dimensional
- Earth’s surface is 2-dimensional
- Einstein’s universe is 4-dimensional

Hyperbolic geometry
Dimension and curvature

- Descartes’ plane is 2-dimensional
- Newton’s universe is 3-dimensional
- Earth’s surface is 2-dimensional
- Einstein’s universe is 4-dimensional
- String theory:

\[ \begin{align*}
\text{Descartes’ plane is 2-dimensional} \\
\text{Newton’s universe is 3-dimensional} \\
\text{Earth’s surface is 2-dimensional} \\
\text{Einstein’s universe is 4-dimensional} \\
\text{String theory:}
\end{align*} \]
Dimension and curvature

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  - particles are little strings
Dimension and curvature

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- String theory:
  - particles are little strings
  - they can wrap around a Calabi-Yau manifold
Dimension and curvature

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**Definition**

A *manifold* looks in each point like one of these worlds.
Mirror symmetry and Enumerative Geometry

Theorem

On a Calabi-Yau manifold, the number of parametric curves of degree 5 is \( n_5 = 229305888887625 \approx 2 \times 10^{14} \).
Mirror symmetry and Enumerative Geometry

**Theorem**

*On a Calabi-Yau manifold, the number of parametric curves of degree 5 is $n_5 = 229305888887625 \approx 2 \times 10^{14}$.*

**Preliminaries.**

Mirror images are the same (up to chirality).
Mirror symmetry and Enumerative Geometry

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On a Calabi-Yau manifold, the number of parametric curves of degree 5 is \( n_5 = 229305888887625 \approx 2 \times 10^{14} \).

Preliminaries.

To cheat at Blind Man’s Bluff poker, look into a mirror.
Mirror symmetry and Enumerative Geometry

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On a Calabi-Yau manifold, the number of parametric curves of degree 5 is \( n_5 = 229305888887625 \approx 2 \times 10^{14} \).

Preliminaries.

With a 3-dimensional mirror, we could see inside a tree and determine its age by counting its rings.
Mirror symmetry and Enumerative Geometry

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String theoretic proof.

- String theory: our universe has a mirror image
Mirror symmetry and Enumerative Geometry

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- in this mirror image, we can read off the value of \( n_5 \) of its corresponding Calabi-Yau manifold
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String theoretic proof.

- String theory: *our universe has a mirror image*
- In this mirror image, we can read off the value of \( n_5 \) of its corresponding Calabi-Yau manifold
- Mathematical truth is *universe-independent*, so \( n_5 \) is the correct number in our as well as any alternate universe!
Virtual worlds have finite length, often a prime number $p$.

Example (World of length $p = 7$)
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```
```

Virtual worlds can have arbitrary high dimension $d$. Such a multi-dimensional virtual world has therefore size $p^d$. 
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Example (World of length $p = 7$)

```

0 6 1
5 2 3

“clock-wise” addition:

to add 3 to 5,
```
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Example (World of length $p = 7$)

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**Example (World of length $p = 7$)**

- “clock-wise” addition:
- to add 3 to 5,
- move 3 points up, so
- $3 + 5 = 8 \equiv 1 \mod 7$. 
Virtual worlds have finite length, often a prime number $p$.

**Example (World of length $p = 7$)**

Virtual worlds can have arbitrary high dimension $d$. Such a multi-dimensional virtual world has therefore size $p^d$. 
Outline

1. Size matters
2. ... and so does shape
3. The beauty and the beast: symmetry and singularities
4. A new utopia? Ultraworld
Symmetry

(group theory) \textit{symmetry} = transformation preserving the manifold
Symmetry

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(crystallography) many physical objects have nice symmetries
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(tessalation) we can ‘tile’ the manifold to reveal its symmetry
Symmetry

(group theory) symmetry = transformation preserving the manifold

(crystallography) many physical objects have nice symmetries

(tessellation) we can ‘tile’ the manifold to reveal its symmetry

(geometery) \text{tile} + \text{symmetry} \rightsquigarrow \text{manifold}
The tile space of a symmetry

Definition: A tile is the smallest part whose copies under the symmetries cover the manifold.

Definition: Edges of the tile get identified by the symmetry. Glueing these edges gives the tile space (or, quotient space).

... and so does shape

Symmetry and singularities

Size matters
The tile space of a symmetry

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Tile spaces are more general than manifolds

Example

Consider the rotational symmetries of a triangle.
Tile spaces are more general than manifolds

Example

Consider the rotational symmetries of a triangle. Each slice is a tile.
Tile spaces are more general than manifolds

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Consider the rotational symmetries of a triangle. Each slice is a tile. To build the tile space, we glue together its sides.
Tile spaces are more general than manifolds

Example

Consider the rotational symmetries of a triangle. Each slice is a tile. To build the tile space, we glue together its sides. The result is a cone, with a sharp vertex.
Tile spaces are more general than manifolds

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Definition
Any point where the space does no longer look like a manifold, is called a singularity.
Singularities come in many shapes and sizes...

(curve) self-intersection = node

Example
Singularities come in many shapes and sizes...

(curve) self-intersection = node
(surface) fold line = pinch
Singularities come in many shapes and sizes...

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Singularities come in many shapes and sizes...

(curve) self-intersection = node
(surface) fold line = pinch
(catastrophe theory) sharp edge = cusp
(quantum gravity) black hole = sink
(cosmology) Big Bang?
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Frobenius symmetry of virtual worlds

Double product formula (student version)

\[(x + y)^2 = x^2 + y^2\]
Frobenius symmetry of virtual worlds

**Double product formula**

\[(x + y)^2 = x^2 + 2xy + y^2\]
Frobenius symmetry of virtual worlds

Newton’s (1643–1727) Binomial Formula

\[(x + y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + y^n\]
Frobenius symmetry of virtual worlds

**Khayyām's (1048–1131) Binomial Formula**

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Theorem (Frobenius (1849–1917))

In a finite world of length \(p\), the student’s version of the binomial theorem does hold true: \((x + y)^p = x^p + y^p\)
Frobenius symmetry of virtual worlds

Khayyām’s (1048–1131) Binomial Formula

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Theorem (Frobenius (1849–1917))

In a finite world of length $p$, the student’s version of the binomial theorem does hold true: $(x + y)^p = x^p + y^p$

Corollary

The $p$-th power map is a symmetry of any virtual world of length $p$, called the Frobenius symmetry.
Definition

An ultraworld is obtained by averaging infinitely many worlds.
Ultraworlds

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Theorem (Łoś ’55)
A computation in an ultraworld is true, if it is true in a randomly chosen world.
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Theorem (Łoś ’55)
A computation in an ultraworld is true, if it is true in a randomly chosen world.

Proof.
- We construct the corresponding parallel quantum ultra-computer, with each processor working in some random world.
- If, on average, something is true in each world, then our ultra-computer will detect this with probability one.
Ultraworlds

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Some ultraworld travelers

Skolem ’38: precursor to ultraworld: the non-standard numbers
Łoś ’55: ultraworld as a concept from logic
Keisler–Tarski ’64: use ultraworlds of infinity to get even larger infinities
Robinson ’66: use the ultraworld of the real world to define non-standard analysis (=foundation for Leibnitz’ infinitesimal calculus)
Ax–Kochen ’68: use ultraworlds to study average properties of finite worlds
Schmidt–van den Dries ’84: use ultraworlds to obtain computability
S. ’00: use ultraworlds as a gateway between finite and infinite worlds
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Lefschetz world

**Definition**

Lefschetz world is the ultraworld of all finite worlds.
Lefschetz world

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**Corollary**

*Lefschetz world has ultra-Frobenius symmetry.*
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**Proof.**
Finite worlds have Frobenius symmetry (given by taking $p$-th powers).
**Lefschetz world**

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**Corollary**
*Lefschetz world has ultra-Frobenius symmetry.*

**Proof.**
Finite worlds have Frobenius symmetry (given by taking $p$-th powers). Averaging this out yields ultra-Frobenius symmetry.
Definition
Lefschetz world is the ultraworld of all finite worlds.

Corollary
*Lefschetz world has ultra-Frobenius symmetry.*

Proof.
Finite worlds have Frobenius symmetry (given by taking $p$-th powers). Averaging this out yields ultra-Frobenius symmetry.

Theorem (S. ’03, Aschenbrenner-S. ’07)
*Any infinite world embeds into Lefschetz world.*
Rationality of tile spaces

Theorem (Smith ‘06)

Tile spaces inside finite worlds have nice singularities.
Example

Consider the rotational symmetries of a triangle. Each slice is a tile. To build the tile space, we glue together its sides. The result is a cone, with a sharp vertex.
Rationality of tile spaces

Theorem (Smith ’06)

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Proof.

This follows from two conflicting properties of Frobenius symmetry:
Rationality of tile spaces

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Proof.

This follows from two conflicting properties of Frobenius symmetry:

1. it contracts subspaces,
Rationality of tile spaces

Theorem (Smith ’06)

*Tile spaces inside finite worlds have *nice* singularities.*

Proof.

This follows from two conflicting properties of Frobenius symmetry:

1. it contracts subspaces,
2. yet preserves cohomology.
Theorem (Smith ’06)

*Tile spaces inside finite worlds have nice singularities.*

Proof.
This follows from two conflicting properties of Frobenius symmetry:

1. it contracts subspaces,
2. yet preserves cohomology.

This can only happen if *cohomology* is already zero, which is what is meant with *nice* singularities.
Rationality of tile spaces

**Theorem (Smith ’06)**

*Tile spaces inside finite worlds have **nice** singularities.*

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**Main Theorem (S. ’08)**

Any tile space has **nice** singularities.
Rationality of tile spaces

Theorem (Smith ’06)

*Tile spaces inside finite worlds have nice singularities.*

Main Theorem (S. ’08)

Any tile space has nice singularities.

Proof.

- embed tile space in Lefschetz world,
Rationality of tile spaces

Theorem (Smith ’06)

*Tile spaces inside finite worlds have *nice* singularities.*

Main Theorem (S. ’08)

*Any tile space has *nice* singularities.*

**Proof.**

- embed tile space in Lefschetz world,
- redo the above argument with ultra-Frobenius symmetry.
Epilogue

Corollary

*Mathematics is heaven!*
Corollary

Mathematics is heaven!

... but hell on earth!
Corollary

*Mathematics is heaven!*

Proof.

Did you not just pay attention!

... but hell on earth!
Thank you!

Be rational

Get real.

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