Write $0.02303$ as a fraction.

Solution: In

$$0.02303 = 0.02 + \frac{303}{10^5} + \frac{303}{10^8} + \ldots$$

the second part is a geometric series with initial term $a = 303/10^5$ and $r = 1/1000$, and so using the formula $a/(1 - r)$ for the geometric series, which is allowed since $r < 1$, we get

$$\frac{2}{100} + \frac{\frac{303}{100000}}{1 - \frac{1}{1000}} = \frac{2}{100} + \frac{303}{100000 - 100} = \frac{2 \times 999 + 303}{100 \times 999 + 99900} = \frac{1998 + 303}{99900} = \frac{2301}{99900}$$

where in the second step, we multiplied top and bottom with $100,000$ to simplify the compounded fraction.

Find a power series (give all terms up to degree 8) for

$$\cos(x^3)e^{x^2}$$

Solution: Since

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots$$

substituting $x^3$ for $x$ gives

(1) $$\cos(x^3) = 1 - \frac{1}{2}x^6 + \frac{1}{24}x^{12} + \ldots$$

Likewise, since

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$$

substituting $x^2$ for $x$ gives

(2) $$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \ldots$$

So, multiplying (1) with (2) gives

$$\left(1 - \frac{1}{2}x^6 + \frac{1}{24}x^{12} + \ldots\right)\left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \ldots\right)$$

$$= 1 + x^2 + \frac{1}{2}x^4 + \left(\frac{1}{6} - \frac{1}{2}\right)x^6 + (\frac{1}{2} + \frac{1}{24})x^8 + \ldots$$

$$= 1 + x^2 + \frac{1}{2}x^4 - \frac{1}{3}x^6 - \frac{11}{24}x^8 + \ldots$$

Write the following definite integral

$$\int_0^{1/2} \ln(1 + 4x^2) \, dx$$

as a series and show that it is convergent, using the following steps:

(1) find a power series for $\frac{1}{1+4x}$ and determine its radius of convergence;
(2) using anti-derivatives and the series from (1), find a power series for \( \ln(1 + 4x) \) and determine its radius of convergence;
(3) using substitution and the series from (2), find a power series for \( \ln(1 + 4x^2) \) and determine its radius of convergence;
(4) using anti-derivatives and the series from (3), find a power series for the indefinite integral \( \int \ln(1 + 4x^2) \, dx \) and determine its radius of convergence;
(5) use the fundamental theorem of calculus to find a series for \( \int_0^{1/2} \ln(1 + 4x^2) \, dx \), and show that it converges.

Solution:
(1) Using the geometric series \( \frac{1}{1-a} = \sum a^n \) for \( a = -4x \), we get
\[
\frac{1}{1 + 4x} = \sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-4)^n x^n.
\]
which converges for \( |a| = | -4x | < 1 \), so that \( |x| < 1/4 \), whence the radius of convergence is \( R = 1/4 \);
(2) Taking anti-derivatives of both sides (and compensating with a factor \( 1/4 \) caused by the chain rule), we get
\[
\frac{1}{4} \ln(1 + 4x) = \sum_{n=0}^{\infty} (-4)^n \frac{x^{n+1}}{n+1}
\]
which has the same radius of convergence \( R = 1/4 \);
(3) Bringing the factor to the other side and substituting \( x^2 \) for \( x \) gives
\[
\ln(1 + 4x^2) = 4 \sum_{n=0}^{\infty} (-4)^n \frac{(x^2)^{n+1}}{n+1} = 4 \sum_{n=0}^{\infty} (-4)^n \frac{x^{2n+2}}{n+1}
\]
and this converges for \( |x^2| = |x|^2 < 1/4 \), whence \( |x| < 1/2 \);
(4) Taking again the anti-derivative and then using the fundamental theorem of calculus gives
\[
\int_0^{1/2} \ln(1 + 4x^2) \, dx = 4 \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n+3}}{n+1 (2n+3)} \bigg|_0^{1/2}
\]
\[
= 4 \sum_{n=0}^{\infty} \frac{(-4)^n}{(n+1)(2n+3)} \left( \frac{1}{2} \right)^{2n+3} - 0
\]
\[
= \frac{4}{8} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(2n+3)}
\]
where we simplified in the last line by writing
\[
\left(\frac{1}{2}\right)^{2n+3} = \left(\frac{1}{2}\right)^{2n} \left(\frac{1}{2}\right)^3 = \left(\left(\frac{1}{2}\right)^2\right)^n \frac{1}{8} = \left(\frac{1}{4}\right)^n \frac{1}{8} = \frac{1}{8} \cdot 4^n.
\]

The final series converges by the alternating series test since the limit of the general term (without sign) \(\lim_{n \to \infty} \frac{1}{(n+1)(2n+3)} = \frac{1}{\infty} = 0\).