Chapter 3

1. Describe an algorithm that takes a list of \( n \) integers \( a_1, a_2, \ldots, a_n \) and finds the number of integers each greater than five in the list.
   Ans: procedure greaterthanfive\((a_1, \ldots, a_n: \text{integers})\)
   \[
   \begin{align*}
   \text{answer} & : = 0 \\
   \text{for } i : = 1 \text{ to } n \\
   & \quad \text{if } a_i > 5 \text{ then } \text{answer} : = \text{answer} + 1.
   \end{align*}
   \]

2. Describe an algorithm that takes a list of integers \( a_1, a_2, \ldots, a_n \) \((n \geq 2)\) and finds the second-largest integer in the sequence.
   Ans: procedure secondlargest\((a_1, \ldots, a_n: \text{integers})\)
   \[
   \begin{align*}
   \text{largest} : & = a_1 \\
   \text{secondlargest} : & = a_2 \\
   & \quad \text{if } a_2 > a_1 \text{ then} \\
   & \quad \quad \text{begin} \\
   & \quad \quad \quad \text{secondlargest} : = a_1 \\
   & \quad \quad \quad \text{largest} : = a_2 \\
   & \quad \quad \end{begin} \\
   & \quad \text{if } n = 2 \text{ then} \\
   & \quad \quad \text{stop} \\
   & \quad \text{for } i : = 3 \text{ to } n \\
   & \quad \quad \text{if } a_i > \text{largest} \text{ then} \\
   & \quad \quad \quad \text{begin} \\
   & \quad \quad \quad \text{secondlargest} : = \text{largest} \\
   & \quad \quad \quad \text{largest} : = a_i \\
   & \quad \quad \end{begin} \\
   & \quad \text{if } (a_i > \text{secondlargest \textbf{and} } a_i \leq \text{largest}) \text{ then} \\
   & \quad \quad \text{secondlargest} : = a_i.
   \end{align*}
   \]

3. Describe an algorithm that takes a list of \( n \) integers \((n \geq 1)\) and finds the location of the last even integer in the list, or returns 0 if there are no even integers in the list.
   Ans: procedure lasteven\((a_1, \ldots, a_n: \text{integers})\)
   \[
   \begin{align*}
   \text{location} & : = 0 \\
   \text{for } i : = 1 \text{ to } n \\
   & \quad \text{if } 2 \mid a_i \text{ then } \text{location} : = i.
   \end{align*}
   \]
4. Describe an algorithm that takes a list of \( n \) integers \((n \geq 1)\) and finds the average of the largest and smallest integers in the list.

Ans: procedure \texttt{avgmaxmin}(a_1,\ldots,a_n: \text{integers})

\begin{align*}
\text{max: } &= a_1 \\
\text{min: } &= a_1 \\
\text{for } i &= 2 \text{ to } n \\
\text{begin} \\
\text{if } a_i > \text{max} \text{ then } \text{max: } &= a_i \\
\text{if } a_i < \text{min} \text{ then } \text{min: } &= a_i \\
\text{end} \\
\text{avg: } &= (\text{max + min})/2.
\end{align*}

5. Describe in words how the binary search works.

Ans: To search for \( x \) in an ordered list \( a_1,\ldots,a_n \), find the “midpoint” of the list and choose the appropriate half of the list. Continue until the list consists of one element. Either this element is \( x \), or else \( x \) is not in the list.

6. Show how the binary search algorithm searches for 27 in the following list: 5 6 8 12 15 21 25 31.

Ans: The consecutive choices of sublists of the original list are: 15 21 25 31,  25 31,  and 25. Since 27 \( \neq \) 25, the integer 25 is not in the list.

7. You have supplies of boards that are one foot, five feet, seven feet, and twelve feet long. You need to lay pieces end-to-end to make a molding 15 feet long and wish to do this using the fewest number of pieces possible. Explain why the greedy algorithm of taking boards of the longest length at each stage (so long as the total length of the boards selected does not exceed 15 feet) does not give the fewest number of boards possible.

Ans: The greedy algorithm first chooses a 12-foot-long board, and then three one-foot-long boards. This requires four boards. But only three boards are needed: each five feet long.

8. Use the definition of big-oh to prove that \( 1^2 + 2^2 + \ldots + n^2 \) is \( O(n^3) \).

Ans: \( 1^2 + 2^2 + \ldots + n^2 \leq n^2 + n^2 + \ldots + n^2 = n \cdot n^2 = n^3 \).

9. Use the definition of big-oh to prove that \( \frac{3n-8-4n^3}{2n-1} \) is \( O(n^2) \).

Ans: \( \frac{3n-8-4n^3}{2n-1} \leq \frac{3n^3+8n^3+4n^3}{2n-n} = \frac{15n^3}{n} = 15n^2 \) if \( n \geq 1 \).

10. Use the definition of big-oh to prove that \( 1^3 + 2^3 + \ldots + n^3 \) is \( O(n^4) \).

Ans: \( 1^3 + 2^3 + \ldots + n^3 \leq n^3 + n^3 + \ldots + n^3 = n \cdot n^3 = n^4 \).
11. Use the definition of big-oh to prove that \( \frac{6n + 4n^5 - 4}{7n^2 - 3} \) is \( O(n^3) \).

Ans: \( \frac{6n + 4n^5 - 4}{7n^2 - 3} \leq \frac{6n^5 + 4n^5}{7n^2 - n^2} = \frac{10n^5}{6n^2} = \frac{5}{3}n^3 \), if \( n \geq 2 \).

12. Use the definition of big-oh to prove that \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n - 1) \cdot n \) is \( O(n^3) \).

Ans: \( 1 \cdot 2 + 2 \cdot 3 + \ldots + (n - 1) \cdot n \leq (n - 1) \cdot n + (n - 1) \cdot n + \ldots + (n - 1) \cdot n = (n - 1)^2 \cdot n \leq n^3 \).

13. Let \( f(n) = 3n^2 + 8n + 7 \). Show that \( f(n) \) is \( O(n^2) \). Find \( C \) and \( k \) from the definition.

Ans: \( f(n) \leq 3n^2 + 8n^2 + 7n^2 = 18n^2 \) if \( n \geq 1 \); therefore \( C = 18 \) and \( k = 1 \) can be used.

Use the following to answer questions 14-19:

In the questions below find the best big-oh function for the function. Choose your answer from among the following:

- \( 1 \)
- \( \log_2 n \)
- \( n \)
- \( n \log_2 n \)
- \( n^2 \)
- \( n^3 \)
- \( \ldots \)
- \( 2^n \)
- \( n! \)

14. \( f(n) = 1 + 4 + 7 + \ldots + (3n + 1) \).

Ans: \( n^2 \).

15. \( g(n) = 1 + 3 + 5 + 7 + \ldots + (2n - 1) \).

Ans: \( n^2 \).

16. \( \frac{3 - 2n^4 - 4n}{2n^3 - 3n} \).

Ans: \( n \).

17. \( f(n) = 1 + 2 + 3 + \ldots + (n^2 - 1) + n^2 \).

Ans: \( n^4 \).

18. \( \lceil n + 2 \rceil \cdot \lceil n/3 \rceil \).

Ans: \( n^2 \).

19. \( 3n^4 + \log_2 n^8 \).

Ans: \( n^4 \).

20. Show that \( \sum_{j=1}^{n} (j^3 + j) \) is \( O(n^4) \).

Ans: \( \sum_{j=1}^{n} (j^3 + j) \leq \sum_{j=1}^{n} (n^3 + n^3) = n \cdot 2n^3 = 2n^4 \).
21. Show that \( f(x) = (x + 2)\log_2(x^2 + 1) + \log_2(x^3 + 1) \) is \( O(x\log_2 x) \).
   Ans: \( \log_2(x^2 + 1) \) and \( \log_2(x^3 + 1) \) are each \( O(\log_2 x) \). Thus each term is \( O(x\log_2 x) \), and hence so is the sum.

22. Find the best big-\( O \) function for \( n^3 + \sin n^7 \).
   Ans: \( n^3 \).

23. Find the best big-\( O \) function for \( \frac{x^3 + 7x}{3x+1} \).
   Ans: \( x^2 \).

24. Prove that \( 5x^4 + 2x^3 - 1 \) is \( \Theta(x^4) \).
   Ans: \( 5x^4 + 2x^3 - 1 \) is \( O(x^4) \) since \( |5x^4 + 2x^3 - 1| \leq |5x^4 + 2x^4| \leq 7|x^4| \) (if \( x \geq 1 \)). Also, \( x^4 \) is \( O(5x^4 + 2x^3 - 1) \) since \( |x^4| \leq |5x^4 + x^3| \leq |5x^4 + 2x^3 - 1| \) (if \( x \geq 1 \)).

25. Prove that \( \frac{x^3 + 7x^2 + 3}{2x+1} \) is \( \Theta(x^2) \).
   Ans: \( \frac{x^3 + 7x^2 + 3}{2x+1} \) is \( O(x^2) \) since \( \frac{x^3 + 7x^2 + 3}{2x+1} \leq \frac{x^3 + 7x^3 + 3x^3}{2x} = \frac{11x^3}{2x} = \frac{11}{2}x^2 \) (if \( x \geq 1 \)).
   Also, \( x^2 \) is \( O \left( \frac{x^3 + 7x^2 + 3}{2x+1} \right) \) since \( x^2 = \frac{x^3}{x} \leq \frac{x^3 + 7x}{2x} \leq \frac{x^3 + 7x + 3}{2x+1} \leq \frac{x^3 + 7x^2 + 3}{2x+1} \) (if \( x \geq 1 \)).

26. Prove that \( x^3 + 7x + 2 \) is \( \Omega(x^3) \).
   Ans: \( x^3 + 7x + 2 \geq 1 \cdot x^3 \) (if \( x \geq 1 \)).

Use the following to answer questions 27-37:

In the questions below find the “best” big-oh notation to describe the complexity of the algorithm. Choose your answers from the following:

1, \( \log_2 n \), \( n \), \( \log_2 n \), \( n \log_2 n \), \( n^2 \), \( n^3 \),..., \( 2^n \), \( n! \).

27. A binary search of \( n \) elements.
   Ans: \( \log_2 n \).

28. A linear search to find the smallest number in a list of \( n \) numbers.
   Ans: \( n \).

29. An algorithm that lists all ways to put the numbers 1,2,3,...,\( n \) in a row.
   Ans: \( n! \).

30. An algorithm that prints all bit strings of length \( n \).
   Ans: \( 2^n \).
31. The number of print statements in the following:
   
   $i := 1, j := 1$
   
   while $i \leq n$
   begin
   while $j \leq i$
   begin
   print "hello";
   $j := j + 1$
   end
   $i := i + 1$
   end.
   
   Ans: $n^2$.

32. The number of print statements in the following:
   
   while $n > 1$
   begin
   print "hello";
   $n := \lfloor n/2 \rfloor$
   end.
   
   Ans: $\log_2 n$.

33. An iterative algorithm to compute $n!$, (counting the number of multiplications).
   
   Ans: $n$.

34. An algorithm that finds the average of $n$ numbers by adding them and dividing by $n$.
   
   Ans: $n$.

35. An algorithm that prints all subsets of size three of the set \{1,2,3,..., $n$\}.
   
   Ans: $n^3$.

36. The best-case analysis of a linear search of a list of size $n$ (counting the number of comparisons).
   
   Ans: 1.

37. The worst-case analysis of a linear search of a list of size $n$ (counting the number of comparisons).
   
   Ans: $n$.

38. Prove or disprove: For all integers $a,b,c,d$, if $a \mid b$ and $c \mid d$, then $(a + c)(b + d)$.
   
   Ans: False: $a = b = c = 1, d = 2$.

39. Prove or disprove: For all integers $a,b,c$, if $a \mid b$ and $b \mid c$ then $a \mid c$.
   
   Ans: True: If $b = ak$ and $c = bl$, then $c = a(kl)$, so $a \mid c$. 
40. Prove or disprove: For all integers \(a, b, c\), if \(a \mid c\) and \(b \mid c\), then \((a + b) \mid c\).
   Ans: False: \(a = b = c = 1\).

41. Prove or disprove: For all integers \(a, b, c, d\), if \(a \mid b\) and \(c \mid d\), then \((ac) \mid (b + d)\).
   Ans: False: \(a = b = 2, c = d = 1\).

42. Prove or disprove: For all integers \(a, b\), if \(a \mid b\) and \(b \mid a\), then \(a = b\).
   Ans: False: \(a = 1, b = -1\).

43. Prove or disprove: For all integers \(a, b, c\), if \(a \mid (b + c)\), then \(a \mid b\) and \(a \mid c\).
   Ans: False: \(a = 2, b = c = 3\).

44. Prove or disprove: For all integers \(a, b, c\), if \(a \mid bc\), then \(a \mid b\) or \(a \mid c\).
   Ans: False: \(a = 4, b = 2, c = 6\).

45. Prove or disprove: For all integers \(a, b, c\), if \(a \mid c\) and \(b \mid c\), then \(ab \mid c^2\).
   Ans: True: If \(c = ak\) and \(c = bl\), then \(c^2 = ab(kl)\), so \(ab \mid c^2\).

46. Find the prime factorization of 1,024.
   Ans: \(2^{10}\).

47. Find the prime factorization of 1,025.
   Ans: \(5^2 \cdot 41\).

48. Find the prime factorization of 510,510.
   Ans: \(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17\).

49. Find the prime factorization of 8,827.
   Ans: \(7 \cdot 13 \cdot 97\).

50. Find the prime factorization of 45,617.
   Ans: \(11^2 \cdot 13 \cdot 29\).

51. Find the prime factorization of 111,111.
   Ans: \(3 \cdot 7 \cdot 11 \cdot 13 \cdot 37\).

52. List all positive integers less than 30 that are relatively prime to 20.
   Ans: 1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29.

53. Find \(\gcd(20!, 12!)\) by directly finding the largest divisor of both numbers.
   Ans: 12!.

54. Find \(\gcd(2^{89}, 2^{346})\) by directly finding the largest divisor of both numbers.
   Ans: \(2^{89}\).
55. Find $\text{lcm}(20!, 12!)$ by directly finding the smallest positive multiple of both numbers.
   Ans: $20!$.

56. Find $\text{lcm}(2^{89}, 2^{346})$ by directly finding the smallest positive multiple of both numbers.
   Ans: $2^{346}$.

57. Suppose that the lcm of two numbers is 400 and their gcd is 10. If one of the numbers is 50, find the other number.
   Ans: 80.

58. Applying the division algorithm with $a = -41$ and $d = 6$ yields what value of $r$?
   Ans: 1.

59. Find $18 \mod 7$.
   Ans: 4.

60. Find $-88 \mod 13$.
    Ans: 3.

61. Find $289 \mod 17$.
    Ans: 0.

62. Find the hexadecimal expansion of $ABC_{16} + 2F5_{16}$.
    Ans: $(DB1)_{16}$.

63. Prove or disprove: A positive integer congruent to 1 modulo 4 cannot have a prime factor congruent to 3 modulo 4.
    Ans: False: $9 = 4 \cdot 2 + 1 = 3 \cdot 3$.

64. Find $50! \mod 50$.
    Ans: 0.

65. Find $50! \mod 49!$.
    Ans: 0.

66. Prove or disprove: The sum of two primes is a prime.
    Ans: False; 3 + 5 is not prime.

67. Prove or disprove: If $p$ and $q$ are primes (> 2), then $p + q$ is composite.
    Ans: $p + q$ is even, hence composite.

68. Prove or disprove: There exist two consecutive primes, each greater than 2.
    Ans: False; one of any two consecutive integers is even, hence not prime.
69. Prove or disprove: The sum of two irrational numbers is irrational.
   Ans: False; \( \sqrt{2} + (-\sqrt{2}) = 0 \).

70. Prove or disprove: If \( a \) and \( b \) are rational numbers (not equal to zero), then \( a^b \) is rational.
   Ans: False; \( (1/2)^{1/2} = \sqrt{2}/2 \), which is not rational.

71. Prove or disprove: If \( f(n) = n^2 - n + 17 \), then \( f(n) \) is prime for all positive integers \( n \).
   Ans: False, \( f(17) \) is divisible by 17.

72. Prove or disprove: If \( p \) and \( q \) are primes (> 2), then \( pq + 1 \) is never prime.
   Ans: \( pq + 1 \) is an even number, hence not prime.

73. Find three integers \( m \) such that \( 13 \equiv 7 \pmod{m} \).
   Ans: 2, 3, 6.

74. Find the smallest positive integer \( a \) such that \( a + 1 \equiv 2a \pmod{11} \).
   Ans: 12.

75. Find four integers \( b \) (two negative and two positive) such that \( 7 \equiv b \pmod{4} \).
   Ans: 3, 7, 11, 15, ..., -1, -5, -9, ...

76. Find an integer \( a \) such that \( a \equiv 3a \pmod{7} \).
   Ans: 0, ±7, ±14, ...

77. Find integers \( a \) and \( b \) such that \( a + b \equiv a - b \pmod{5} \).
   Ans: \( b = 0, ±5, ±10, ±15, ...; \) \( a \) any integer.

Use the following to answer questions 78-84:

In the questions below determine whether each of the following “theorems” is true or false. Assume that \( a, b, c, d, \) and \( m \) are integers with \( m > 1 \).

78. If \( a \equiv b \pmod{m} \), and \( a \equiv c \pmod{m} \), then \( a \equiv b + c \pmod{m} \).
   Ans: False

79. If \( a \equiv b \pmod{m} \) and \( c \equiv d \pmod{m} \), then \( ac \equiv b + d \pmod{m} \).
   Ans: False

80. If \( a \equiv b \pmod{m} \), then \( 2a \equiv 2b \pmod{m} \).
   Ans: True

81. If \( a \equiv b \pmod{m} \), then \( 2a \equiv 2b \pmod{2m} \).
   Ans: True
82. If \( a \equiv b \pmod{m} \), then \( a \equiv b \pmod{2m} \).
   Ans: False

83. If \( a \equiv b \pmod{2m} \), then \( a \equiv b \pmod{m} \).
   Ans: True

84. If \( a \equiv b \pmod{m^2} \), then \( a \equiv b \pmod{m} \).
   Ans: True

85. Either find an integer \( x \) such that \( x \equiv 2 \pmod{6} \) and \( x \equiv 3 \pmod{9} \) are both true, or else prove that there is no such integer.
   Ans: There is no such \( x \); if there were, then there would be integers \( k \) and \( l \) such that \( x - 2 = 6k \) and \( x - 3 = 9l \). Hence \( 1 = 6k - 9l = 3(2k - 3l) \), which is not possible.

86. What sequence of pseudorandom numbers is generated using the pure multiplicative generator \( x_{n+1} = 3x_n \pmod{11} \) with seed \( x_0 = 2 \)?
   Ans: The sequence 2,6,7,10,8 repeats.

87. Encrypt the message NEED HELP by translating the letters into numbers, applying the encryption function \( f(p) = (p + 3) \pmod{26} \), and then translating the numbers back into letters.
   Ans: Encrypted form: QHHG KHOS.

88. Encrypt the message NEED HELP by translating the letters into numbers, applying the encryption function \( f(p) = (3p + 7) \pmod{26} \), and then translating the numbers back into letters.
   Ans: Encrypted form: UTTQ CTOA.

89. Suppose that a computer has only the memory locations 0,1,2,...,19. Use the hashing function \( h \) where \( h(x) = (x + 5) \pmod{20} \) to determine the memory locations in which 57, 32, and 97 are stored.
   Ans: 2,17,3.

90. A message has been encrypted using the function \( f(x) = (x + 5) \pmod{26} \). If the message in coded form is JCFHY, decode the message.
   Ans: EXACT.

91. Explain why \( f(x) = (2x + 3) \pmod{26} \) would not be a good coding function.
   Ans: \( f \) is not \( 1 - 1 \) (\( f(0) = f(13) \)), and hence \( f^{-1} \) is not a function.

92. Encode the message “stop at noon” using the function \( f(x) = (x + 6) \pmod{26} \).
   Ans: YZUV GZ TUUT.
93. Explain in words the difference between \( a \mid b \) and \( \frac{b}{a} \).

Ans: \( a \mid b \) is a statement; \( \frac{b}{a} \) represents a number.

94. Prove or disprove: if \( p \) and \( q \) are prime numbers, then \( pq + 1 \) is prime.

Ans: False; \( p = q = 3 \).

95. (a) Find two positive integers, each with exactly three positive integer factors greater than 1.
(b) Prove that there are an infinite number of positive integers, each with exactly three positive integer factors greater than 1.

Ans: (a) 8, 27. (b) Any integer of the form \( p^3 \) where \( p \) is prime.

96. Convert \((204)_{10}\) to base 2.

Ans: 1100 1100.

97. Convert \((11101)_{2}\) to base 16.

Ans: 1D.

98. Convert \((11101)_{2}\) to base 10.

Ans: 29.

99. Convert \((2AC)_{16}\) to base 10.

Ans: 684.

100. Convert \((10000)_{10}\) to base 2.

Ans: 10 0111 0001 0000.

101. Convert \((8091)_{10}\) to base 2.

Ans: 1 1111 1001 1011.

102. Convert \((BC1)_{16}\) to base 2.

Ans: 1011 1100 0001.

103. Convert \((10011010011)_{2}\) to base 16.

Ans: 4C3.

104. Take any three-digit integer, reverse its digits, and subtract. For example, \( 742 - 247 = 495 \). The difference is divisible by 9. Prove that this must happen for all three-digit numbers \( abc \).

Ans: \( abc - cba = 100a + 10b + c - (100c + 10b + a) = 99a - 99c = 9(11a - 11c) \).

Therefore \( 9 \mid abc - cba \).
105. Prove or disprove that $30!$ ends in exactly seven 0s.
   Ans: True. When the factors 5, 10, 15, 20, and 30 are multiplied by the factor 2, each contributes one zero; when the factor 25 is multiplied by two factors 2, it contributes two zeros.

106. Here is a sample proof that contains an error. Explain why the proof is not correct.
   **Theorem:** If $a \mid b$ and $b \mid c$, then $a \mid c$.
   **Proof:** Since $a \mid b$, $b = ak$.
   Since $b \mid c$, $c = bk$.
   Therefore $c = bk = (ak)k = ak^2$.
   Therefore $a \mid c$.
   Ans: The proof is not correct since there is no guarantee that the multiple $k$ will be the same in both cases.

107. Prove: if $n$ is an integer that is not a multiple of 3, then $n^2 \equiv 1 \mod 3$.
   Ans: Proof by cases. Suppose $n$ is not a multiple of 3. Then $n = 3k + 1, n = 3k + 2$ for some integer $k$.
   Case 1, $n = 3k + 1$: therefore $n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, and hence $n^2 \equiv 1 \mod 3$.
   Case 2, $n = 3k + 2$: therefore $n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, and hence $n^2 \equiv 1 \mod 3$.

108. Prove: if $n$ is an integer that is not a multiple of 4, then $n^2 \equiv 0 \mod 4$ or $n^2 \equiv 1 \mod 4$.
   Ans: Proof by cases. Suppose $n$ is not a multiple of 4. Then there is an integer $k$ such that $n = 4k + 1, n = 4k + 2, n = 4k + 3$.
   Case 1, $n = 4k + 1$: therefore $n^2 = (4k + 1)^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1$, and hence $n^2 \equiv 1 \mod 4$.
   Case 2, $n = 4k + 2$: therefore $n^2 = (4k + 2)^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1)$, and hence $n^2 \equiv 0 \mod 4$.
   Case 3, $n = 4k + 3$: therefore $n^2 = (4k + 3)^2 = 16k^2 + 24k + 9 = 4(4k^2 + 6k + 2) + 1$, and hence $n^2 \equiv 1 \mod 4$.

109. Use the Euclidean algorithm to find gcd(44,52).
   Ans: 4.

110. Use the Euclidean algorithm to find gcd(144,233).
   Ans: 1.

111. Use the Euclidean algorithm to find gcd(203,101).
   Ans: 1.

112. Use the Euclidean algorithm to find gcd(300,700).
   Ans: 100.
113. Use the Euclidean algorithm to find gcd(34, 21).
   Ans: 1.

114. Use the Euclidean Algorithm to find gcd(900, 140).
   Ans: 20.

115. Use the Euclidean Algorithm to find gcd(580, 50).
   Ans: 10.

116. Use the Euclidean Algorithm to find gcd(390, 72).
   Ans: 6.

117. Use the Euclidean Algorithm to find gcd(400, 0).
   Ans: 400.

118. Use the Euclidean Algorithm to find gcd(128, 729).
   Ans: 1.

119. Find the two's complement of 12.
   Ans: 0 1100.

120. Find the two's complement of -13.
   Ans: 1 0011.

121. Find the two's complement of 9.
   Ans: 0 1001.

122. Find a 2 x 2 matrix \( A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) such that \( A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \).
   Ans: A matrix of the form \( \begin{bmatrix} -2a & a \\ -4a & 2a \end{bmatrix} \) where \( a \neq 0 \).

123. Suppose \( A \) is a 6 x 8 matrix, \( B \) is an 8 x 5 matrix, and \( C \) is a 5 x 9 matrix. Find the number of rows, the number of columns, and the number of entries in \( A(BC) \).
   Ans: \( A(BC) \) has 6 rows, 9 columns, and 54 entries.

124. Let \( A = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \). Find \( A^n \) where \( n \) is a positive integer.
   Ans: \( A^n = \begin{bmatrix} 1 & mn \\ 0 & 1 \end{bmatrix} \).
125. Suppose \( A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \) and \( C = \begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix} \). Find a matrix \( B \) such that \( AB = C \) or prove that no such matrix exists.

Ans: \( \begin{bmatrix} 4 & -13 \\ -2 & 8 \end{bmatrix} \).

126. Suppose \( B = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \) and \( C = \begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix} \). Find a matrix \( A \) such that \( AB = C \) or prove that no such matrix exists.

Ans: \( \begin{bmatrix} 3 & -7/2 \\ -6 & 9 \end{bmatrix} \).

127. Suppose \( B = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \) and \( C = \begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix} \). Find a matrix \( A \) such that \( AB = C \) or prove that no such matrix exists.

Ans: None exists since \( \det B = 0 \) and \( \det C \neq 0 \).

Use the following to answer questions 128-134:

In the questions below determine whether the statement is true or false.

128. If \( AB = AC \), then \( B = C \).

Ans: False

129. If \( A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \), then \( A^{-1} = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix} \).

Ans: False

130. If \( A = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix} \), then \( A^2 = \begin{bmatrix} 1 & 9 \\ 25 & 4 \end{bmatrix} \).

Ans: False

131. If \( A \) is a 6 \times 4 matrix and \( B \) is a 4 \times 5 matrix, then \( AB \) has 16 entries.

Ans: False

132. If \( A \) and \( B \) are 2 \times 2 matrices such that \( AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \), then \( A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) or \( B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \).

Ans: False
133. If $A$ and $B$ are $2 \times 2$ matrices, then $A + B = B + A$.
   Ans: True

134. $AB = BA$ for all $2 \times 2$ matrices $A$ and $B$.
   Ans: False

135. What is the most efficient way to multiply the matrices $A_1, A_2, A_3$ of sizes $20 \times 5, 5 \times 50, 50 \times 5$?
   Ans: $A_1(A_2A_3)$, 1750 multiplications.

136. What is the most efficient way to multiply the matrices $A_1, A_2, A_3$ of sizes $10 \times 50, 50 \times 10, 10 \times 40$?
   Ans: $(A_1A_2)A_3$, 9000 multiplications.

137. Suppose $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Find
   (a) the join of $A$ and $B$.
   (b) the meet of $A$ and $B$.
   (c) the Boolean product of $A$ and $B$.
   Ans: (a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.
   (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.
   (c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

138. Suppose $A$ is a $2 \times 2$ matrix with real number entries such that $AB = BA$ for all $2 \times 2$ matrices. What relationships must exist among the entries of $A$?
   Ans: $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.

139. Given that $\gcd(620, 140) = 20$, write 20 as a linear combination of 620 and 140.
   Ans: $620 \cdot (-2) + 140 \cdot 9$.

140. Given that $\gcd(662, 414) = 2$, write 2 as a linear combination of 662 and 414.
   Ans: $662 \cdot (-5) + 414 \cdot 8$.

141. Express $\gcd(84, 18)$ as a linear combination of 18 and 84.
   Ans: $18 \cdot (-9) + 84 \cdot 2$.

142. Express $\gcd(450, 120)$ as a linear combination of 120 and 450.
   Ans: $120 \cdot 4 + 450 \cdot (-1)$.
143. Find an inverse of 5 modulo 12.
   Ans: 5.

144. Find an inverse of 17 modulo 19.
   Ans: 9.

145. Solve the linear congruence $2x \equiv 5 \pmod{9}$.
   Ans: $7 + 9k$.

146. Solve the linear congruence $5x \equiv 3 \pmod{11}$.
   Ans: $5 + 11k$. 