MAT 3788 Lecture 4, Feb 9 2010
Bond Price, Yield, Duration and Convexity
Prof. Boyan Kostadinov, City Tech of CUNY

The Mathematics of Bond Pricing

In finance, a **bond is a debt security**, such that the issuer of the bond owes the holder of the bond a debt and is obliged to pay interest (called **the coupon**) and to repay the principal (called the **face value**) at maturity. A bond is thus a formal contract to repay borrowed money with interest at fixed time intervals, specified in the contract.

Thus a bond is like a loan: the issuer of the bond is the borrower, the holder of the bond is the lender, and the coupon is the interest. Bonds provide the issuer with external funds to finance long-term investments or in the case of government bonds, to finance current government expenditure. Bonds and stocks are both securities, but the major difference between the two is that stockholders have an equity stake in the company (i.e., they own a piece of the company), whereas bondholders have a creditor stake in the company (i.e., they are lenders to the company). Another difference is that bonds usually have a defined term, or maturity, after which the bond is redeemed, whereas stocks may be outstanding indefinitely.

Debt securities with a term of less than one year are generally designated money market instruments rather than bonds. **Certificates of deposit** (CDs) offered by commercial banks are considered to be **money market instruments** and not bonds. Most bonds have a term of up to thirty years. In the market for U.S. Treasury securities, there are three groups of bond maturities:

- **Short term (Treasury Bills):** maturities up to one year;
- **Medium term (Treasury Notes):** maturities between one and ten years;
- **Long term (Treasury Bonds):** maturities greater than ten years.

The bonds issued by the US government are considered risk-free since the probability that the US government will default is very low, partly because the federal government can always print more money if required. Corporate bonds are considered risky and their probabilities of default (that is the inability of the corporations to meet their financial obligations) are captured by the credit ratings provided by credit rating agencies such as Standard & Poor's, Moody's, etc. Bonds have a long history:

**Zero-Coupon Bonds**

A zero-coupon bond (also called a **discount bond**) is a bond bought at a price lower than its face value, with the face value repaid at the time of maturity. It does not make periodic interest payments, the so-called "coupons," hence the term zero-coupon bond. Examples of zero-coupon bonds include U.S. Treasury bills and U.S. savings bonds.

**Zero Rates**

The **n-year zero-coupon rate** is the rate of interest earned on an investment that starts today and lasts for n years. There is only one payment at the end of n years. There are no intermediate payments. The n-year
zero coupon rate is also referred to as the \textbf{n-year spot rate}, the \textbf{n-year zero rate} or simply the \textbf{n-year zero}. Treasury zero rates are determined by coupon-bearing Treasury bonds by the \textbf{bootstrap method} (homework #3 is a simple example of this approach).

\textbf{Example}

Let the 10-year zero rate with continuous compounding be quoted as 7\% per annum. This means that $1000, if invested for 10 years grows to $1000 \cdot e^{0.07 \cdot 10} = 2013.75$.

Treasury zero rates are implied from market prices of Treasury coupon bonds. The market prices of Treasury bonds are determined at auctions.

\textbf{Example}

Consider a zero-coupon bond with a face value of $100 and maturity T years from today (time 0). If the T-year zero rate with continuous compounding is known to be $r_T$ then the price $P_0$ of this bond at time 0 is

$$P_0 = 100 \cdot e^{-r_T \cdot T}$$

We say that we simply discount back to time 0, the time T cash flow of $100 using the T-zero rate and $e^{-r_T \cdot T}$ we call the \textbf{discount factor} associated with the T-spot rate. We'll always use continuous compounding from now on. Keep in mind that the time 0 value of this bond, that is $P_0$ should grow by a factor of $e^{r_T \cdot T}$ after time T to $P_0 \cdot e^{r_T \cdot T} = 100 \cdot e^{-r_T \cdot T} \cdot e^{r_T \cdot T} = 100$ as expected.

\begin{center}
\begin{tabular}{ccc}
\hline
\textbf{Discounting} & \textbf{$P_0 = 100 \cdot e^{-r_T \cdot T}$} & \textbf{$\$100$} \\
0 & $T$ & \\
\hline
\end{tabular}
\end{center}

\textbf{Bond Features}

- Bond's \textbf{principal} also known as \textbf{face value} or \textbf{par value} is the amount on which interest is paid periodically in the form of coupons and the principal is received by the bond holder at the end of the bond's life.

- Most bonds provide \textbf{coupons} on a semi-annual basis and the coupon is given as a fixed percentage rate of the principal, called the \textbf{coupon rate}.

\textbf{Example of Bond Pricing}

Consider a 2-year Treasury bond with a principal of $100 paying coupons semi-annually with a coupon rate of 6\% per annum. To calculate the present value of all the cash flows given in the diagram below, we need the spot rates for all payment times. The theoretical price of a bond can then be calculated as the present value of all the cash flows that will be received by the owner of the bond. To do this correctly, we have to use the zero rates from the table above to discount the cash flows for the different payment times.
For example, the first coupon will be paid in 6 months, so we need to use the 0.5Y-zero rate of 5% for discounting. But what is the exact coupon? The coupons are calculated based on the principal and using the simple interest rule. Since the coupon rate is 6% per annum, for 6 months the coupon rate is then 6%(0.5) = 3%. Remember that the simple rule gives us linear dependence on time and if the annual rate is \( R \), then the simple rate for time \( t \) years is \( R \cdot t \), in our case \( t = 0.5 \) years. Therefore, the coupon is $100 \cdot 3 \% = $3 and is paid twice a year. Also, remember that the principal is received by the bond holder at the end of the bond's life and so it has to be included in calculating the present value of the bond cash flows. So, at time \( t = 2 \) years, we have a coupon of $3 plus $100 principal and the total amount $103 has to be discounted at the 2Y-zero rate of 6.8% (see the diagram below).

The table below provides these spot rates:

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Treasury Zero rate (%) cont. compounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.0</td>
</tr>
<tr>
<td>1.0</td>
<td>5.8</td>
</tr>
<tr>
<td>1.5</td>
<td>6.4</td>
</tr>
<tr>
<td>2.0</td>
<td>6.8</td>
</tr>
</tbody>
</table>

Therefore, the theoretical price of the bond, as the present value of all the cash flows, is $98.39. Note that this price is very close but less than the face value of $100. We say it's priced at discount:

\[
P = 3 \cdot e^{-0.05 \cdot 0.5} + 3 \cdot e^{-0.058 \cdot 1} + 3 \cdot e^{-0.064 \cdot 1.5} + 103 \cdot e^{-0.068 \cdot 2} = 98.39
\]

General Bond Pricing Formula

- \( P \) is the principal or the face value paid at maturity; \( P = 100 \) in our example.

- \( c \) is the coupon rate (as a percentage) per annum, paid semi-annually, so the coupon value is \( \frac{c \cdot P}{2} \), paid twice a year; \( c = 6 \% \) in our example.

- \( T \) is the maturity in years and \( t_1, t_2, \ldots, t_n = T \) are the coupon payment times, again in years; For
simplicity, we assume that maturity is an integer multiple of 0.5, that is we can split the life of the bond into a whole number (\( n \)) of coupon periods; \( t_1 = 0.5, t_2 = 1, t_3 = 1.5, t_4 = 2 \) in our example.

- \( r_k \) is the \( t_k \)-year zero (spot) rate, used for discounting; see the table above for our example.

- \( V \) is price of the bond today (time 0) as the present value of all the fixed cash flows:

\[
V = \frac{cP}{2} e^{-r_1 t_1} + \frac{cP}{2} e^{-r_2 t_2} + \cdots + \left( P + \frac{cP}{2} \right) e^{-r_n t_n} = \sum_{k=1}^{n} \frac{cP}{2} e^{-r_k k} + P e^{-r_n t_n}
\]

- Read Chapter 4, sections 4.3 and 4.4 from *Options, Futures and Other Derivatives* 6th ed. by Hull.

3rd Part of Homework 1, Due Thursday February 11, 2010

1. Suppose that 6-month, 12-month, 18-month, 24-month and 30-month zero rates are 4\%, 4.2\%, 4.4\%, 4.6\%, 4.8\% per annum with continuous compounding respectively. Estimate the price of a bond with a face value of 100 that will mature in 30 months and pays a coupon of 4\% per annum semiannually.

2. A 10-year, 8\% coupon bond currently sells for $90. A 10-year, 4\% coupon bond currently sells for $80. What is the 10-year zero rate? Hint: Consider taking a long position in two of the 4\% coupon bonds and a short position in one of the 8\% coupon bonds.

3. The prices of 6-month and 1-year Treasury bills are $94.0 and $89.0. A 1.5-year bond that will pay coupons of $4 every 6 months currently sells for $94.84. A 2-year bond that will pay coupons of $5 every 6 months currently sells for $97.12. Calculate the 6-month, 1-year, 1.5-year and 2-year zero rates.