MAT 3788 Lecture 8
Forwards for assets with dividends
Currency forwards
Prof. Boyan Kostadinov, City Tech of CUNY

The Value of a Forward Contract as Time Passes

Recall from last time that the forward price of an asset, that does not pay any dividends during the life of the contract, is specified at time 0 and for maturity \( T \) is

\[
F_{0, T} = S_0 \cdot e^{r \cdot T} \tag{1}
\]

where \( S_0 \) is the time 0 price of the asset and \( r \) is the risk-free interest rate with continuous compounding. If we want to compute the forward price at any time \( t \) after time 0 and before maturity \( T \), \( 0 < t \leq T \) the same argument as before gives us the more general formula for the forward price based on the current asset price \( S_t \)

\[
F_{t, T} = S_t \cdot e^{r \cdot (T - t)} \tag{2}
\]

**Question:** What is \( F_{t, T} \)?

**Example:** A one-year long forward contract on a non-dividend paying stock is entered into when the stock price is $40 and the risk-free interest rate is 10% per annum with continuous compounding. What are the forward price and the initial value of the forward contract?

**Solution**

The forward price is given by:

\[
F_{0, T} = 40 \cdot e^{0.1 \cdot 1} \quad \text{at 5 digits} \rightarrow 44.21
\]

but the initial value of the forward contract is 0 because it costs nothing to enter a forward.

**Question:** Assume that 6 months later, the price of the stock is $45 and the risk-free interest rate is still 10%. What are the current forward price for the same maturity and the value of the forward contract?

**Answer:** Formula (2) answers the first part of this question with \( t = 0.5 \) and \( T = 1 \) years, \( S_t = 45 \), \( r = 0.1 \):

\[
F_{0.5, 1} = 45 \cdot e^{0.1 \cdot 0.5} \quad \text{at 5 digits} \rightarrow 47.31
\]

**Question:** But what is the fair value of the forward contract after 6 months? Is it still zero?

**Answer:** Let's compare our position in a long forward with a newly initiated long forward at time \( t = 0.5 \) years. Using the numbers from our example we get that the original long forward obligates us to buy the asset at maturity for \( F_{0, T} = 44.207 \) but someone who enters a long forward at time \( t \) for the same maturity \( T \) is obligated to buy the asset at time \( T \) for more, namely \( F_{t, T} = 47.307 \), so the holder of the original
long forward would buy the asset cheaper by 47.307 − 44.207 = 3.10 compared to the current (time \( t \)) forward deal for the same delivery date \( T \).

The difference \( F_{t, T} - F_{0, T} \) represents the value of the original forward at time \( T \) because this is exactly how much a time \( t \)-buyer would have to pay for the existing long forward so as to be equivalent to entering a new long forward with a higher forward price. The value of the forward contract at time \( t \) we get by discounting its T-value:

\[
V_t = (F_{t, T} - F_{0, T}) \cdot e^{-r \cdot (T - t)} = (47.307 - 44.207) \cdot e^{-0.1 \cdot 0.5} = 2.95
\]

Another useful way to see the same result is the following argument:

**Question:** What happens if we also enter a short forward at time \( t \) with maturity \( T \)?

**Question:** Adding a short forward position at time \( t \), with the same maturity as the original long position from time 0, is equivalent to...?

**Answer:** ... to closing our original long forward position! Why? Just remember the definitions.

**Question:** What is the total payoff at maturity \( T \) of this portfolio of long and short forward?

**Answer:** We know from before the payoff at maturity \( T \) of a long forward with delivery price \( F_{0, T} \) and a short forward with delivery price \( F_{t, T} \)

\[
\text{Total Payoff at time } T = S_T - F_{0, T} + F_{t, T} - S_T = F_{t, T} - F_{0, T}
\]

This total payoff that we end up with if we close our original long forward position is again the value of the forward at maturity \( T \). Again, the time t-value we get by discounting. On the plot we depict the dynamics of the forward price relative to the price of the asset:

![Diagram of forward price dynamics](image)

**To summarize:** The t-value of a long forward with maturity \( T \) is given by

\[
V_t = (F_{t, T} - F_{0, T}) \cdot e^{-r \cdot (T - t)} = (S_t \cdot e^{r \cdot (T - t)} - S_0 \cdot e^{r \cdot T}) \cdot e^{-r \cdot (T - t)} = S_t - S_0 \cdot e^{r \cdot t}
\]
Check: $45 - 40 \cdot e^{0.1 \cdot 0.5}$

**Question:** What are the time 0-value and the time T-value of a long forward?

### Forwards on assets with continuously paid dividends

**Notation:**
- $S_t$ is the price of an asset at time $t$
- $r$ is the risk-free interest rate, assumed constant
- $q$ is the continuous divide rate for this asset

**Definition:** An asset pays dividends at a continuous rate $q$ if for the time interval $[t, t + dt]$ the dividend paid by the asset at time $t$ is:

$$\text{dividend}_t = q \, dt \cdot S_t$$

**Question:** What happens if we **reinvest** the dividends immediately back into the asset?

If we have 1 unit of the asset at time $t$, how many more units of the asset can we buy with the dividend?

We can buy $q \, dt$ more units of the asset at time $t$ using the dividend.

**Question:** Now, if we have $N_t$ units of the asset at time $t$, how many more units of the asset can we buy with the dividends?

If the dividend from 1 unit buys us $q \, dt$ extra units then the dividend from $N_t$ units will buy us $q \, dt \cdot N_t$ extra units. If $dN_t$ represents the infinitesimal change in the total number of units held, then it must hold:

$$dN_t = q \, dt \cdot N_t \quad \iff \quad \frac{dN_t}{dt} = q \cdot N_t$$

**Question:** Can you solve this differential equation written in a differential form?

The easiest way to solve it is by separating the variables and the integrating both sides from time 0 to $T$:

$$\frac{dN_t}{N_t} = q \, dt \quad \rightarrow \quad \int_0^T \frac{dN_t}{N_t} = \int_0^T q \, dt$$

On the one hand, we get the LHS using the Calculus result:

$$\int_a^b \frac{dx}{x} = \ln(b) - \ln(a) = \ln\left(\frac{b}{a}\right);$$
\[
\int_0^T \frac{dN_T}{N_T} = \ln\left(\frac{N_T}{N_0}\right) - \ln\left(\frac{N_0}{N_0}\right) = \ln\left(\frac{N_T}{N_0}\right)
\]

On the other hand, we get the RHS: \(\int_0^T q \, dt = QT\) and by equating the final results for both sides we get:

\[
\ln\left(\frac{N_T}{N_0}\right) = qT \rightarrow \frac{N_T}{N_0} = e^{qT} \rightarrow N_T = N_0 e^{qT}
\]

So, if we reinvest the dividends back into the asset, the number of units is growing according to the exponential law: \(N_T = N_0 e^{qT}\) which gives us the total number of units at time \(T\) if the number of units at time 0 is \(N_0\).

**Question:** How many units of the asset do we need to hold at time 0, in order to get exactly 1 unit at time \(T\)?

We need to solve the equation above for \(N_0\) with \(N_T = 1\), that is \(1 = N_0 e^{qT} \rightarrow N_0 = e^{-qT}\)

![Graph showing number of units for div rate 4%](image)

**The Forward Price for continuous dividends**

Now, let's find the forward price in the case of continuous dividends but first, let's recall the way we derived formula (1) in the previous lecture. The strategy is to buy 1 unit of asset at time 0 for \(S_0\) and enter into a short forward to sell this asset at time \(T\) for the forward price \(F_0, T\), which is specified at time 0, so it is a known cash received at time \(T\). To summarize, at time 0 we have a cost of \(S_0\) (for getting 1 unit of the
asset) because it costs nothing to enter a forward contract and at time $T$ we receive certain cash $F_{0,T}$. This strategy does not lead to any arbitrage only if the cost today is equal to the present value of the known cash $F_{0,T}$ received at time $T$:

\[ S_0 = \text{PV}(F_{0,T}) = F_{0,T}e^{-rT} \rightarrow F_{0,T} = S_0e^{rT} \]

**Question:** Can we use the same strategy to find the forward price when the asset pays dividends continuously?

Yes, if we start not with one unit but $e^{-qT}$ units, which will grow to 1 unit after time $T$. Again, buy now $e^{-qT}$ units of the asset for a total cost of $S_0e^{-qT}$ and enter a short forward to sell 1 unit of the asset at time $T$ for a price of $F_{0,T}$. To avoid arbitrage, the cost today must be equal to the present value of the known cash $F_{0,T}$ received at time $T$:

\[ S_0e^{-qT} = \text{PV}(F_{0,T}) = F_{0,T}e^{-rT} \rightarrow F_{0,T} = S_0e^{(r-q)T} - \text{multiply both sides by } e^{rT} \]

The forward price $F_{0,T}$ for maturity $T$ based on an asset with continuous dividend rate $q$ and time 0 price of $S_0$ is given by $(r$ is the risk-free interest rate):

\[ F_{0,T} = S_0e^{(r-q)T} \]

**Forward for asset with known dividend yield**

**Example:** Consider a 6-month forward contract on an asset that is expected to provide income equal to 2% of the asset price once every 6 months. The risk-free rate of interest is 10% per annum with continuous compounding. The asset price is $25. Find the 6-month forward price.

**Solution:** Here, $S_0 = 25$, $r = 0.1$, $T = 0.5$ years. The yield is 4% per annum with semi-annual compounding. We can convert it into a continuous dividend rate:

\[ (1 + 0.02)^2 = e^{q} \rightarrow q = 2 \cdot \ln(1.02) = 0.04 \]

This gives us the annual rate $q = 3.96\%$ with continuous compounding, so that the forward price is given
by:

\[ 25 \cdot e^{(0.1 - 0.0396) \cdot 0.5} = 25.77 \]

Homework Problem #1

Derive the formula for \( F_{t, T} \) the same way we derived the formula for \( F_{0, T} \).

Homework Problem #2

What is the forward price \( F_{0, T} \) of an asset whose price at time 0 is \( S_0 \) and which pays known dividends \( D_1, D_2 \) and \( D_3 \) at times \( t_1, t_2 \) and \( t_3 \) with a constant risk-free interest rate \( r \) and maturity \( T \)?

**Hint:** Remember that we have to adjust the spot price of the asset \( S_0 \) by the present value of the future known dividends because paying dividends reduces the value of the stock by transforming some of the equity into cash and by buying forward we don't get the dividends the asset pays before the delivery date of the forward.

Homework Problem #3 (Forward on a currency exchange rate)

*The Interest Rate Parity*