Bond's Yield to Maturity (YTM)

A bond's yield to maturity is this artificial discount rate that when applied to all cash flows of a bond, gives a bond price equal to the market price of the bond. The market price of the bond is determined by the market forces of supply and demand as well as the current level of interest rates for different maturities (the term structure). If the market expects changes in interest rates then the market forces changes in the levels of bond yields to maturity, which translates to new market bond prices. Basically, the YTM is an effective rate of return on the bond if we keep the bond until maturity. Banks build proprietary Treasury Yield Curves that represent the YTM as a function of maturity, based on the current market prices of bonds and other money market instruments. So, when interest rates for some or all maturities go up then the yields for those maturities should go up as well; this moves the entire yield curve and as a result the market prices of bonds change (fall, in this case). It is important to realize that there is a single yield curve, which applies to bonds of any coupon structure since investors expect the same yields (returns) for a given maturity no matter what the coupon structure is; different coupon structures simply produce different bond prices.

Let us consider the bond whose theoretical price of $98.39 we derived above and let's assume that this is also the current market price of the bond. If \( y \) is the yield of the bond with continuous compounding then we have

\[
3 \cdot e^{-0.5y} + 3 \cdot e^{-y} + 3 \cdot e^{-1.5y} + 103 \cdot e^{-2y} = 98.39
\]

MAPLE solved this non-linear equation and gave \( y = 6.76\% \) for the yield to maturity for this bond.

When Treasury bonds are first sold by the government at special auctions, where only big financial institutions can bid, they are sold at par, which means that the initial price is equal the face value (the principal). In this case the yield is very close to the coupon rate, when we have continuous compounding (and exactly equal for discrete compounding):

\[
3 \cdot e^{-0.5y} + 3 \cdot e^{-y} + 3 \cdot e^{-1.5y} + 103 \cdot e^{-2y} = 100
\]

Remember that in this case the coupon rate was 6% per annum and we are getting for the yield 5.91%.

The Price-Yield Curve

The Price-Yield curve for the 2Y bond from our example is given by the equation:

\[
P = 3 \cdot e^{-0.5y} + 3 \cdot e^{-y} + 3 \cdot e^{-1.5y} + 103 \cdot e^{-2y}
\]
Note that the Price-Yield curve in this example is almost a straight line but for other bonds this curve could be heavily curved, that is with much bigger curvature or convexity. This happens for bonds with longer maturities. For example, a 20Y bond has much more curvature:

\[
P = \sum_{i=1}^{40} 3 \cdot e^{-y \cdot 0.5 \cdot i} + 100 \cdot e^{-y \cdot 20} \rightarrow
\]

Note that if we price the bond at par, the yield is again 5.91% and so the par yield matches the coupon rate (approximately, for continuous compounding and exactly for discrete compounding) no matter what the maturity of the bond is. Here is the par yield for the 20Y bond:

\[
100 = \sum_{i=1}^{40} 3 \cdot e^{-y \cdot 0.5 \cdot i} + 100 \cdot e^{-y \cdot 20} \quad \text{solve for } y \quad \rightarrow 0.0591
\]
Note that the Par point on the price-yield curve stays the same even if the maturity of the bond changes but the coupon structure remains the same. We see now that the price-yield curve has some curvature (convexity). Understanding the bond convexity is very important for hedging purposes when bond traders are trying to make their bond portfolios immune to even large changes in the yield (reflecting changes in interest rates) so that they have less interest rate exposure.

**Duration**

Bond traders are interested in the sensitivity of the bond price with changes in interest rates. Since the YTM captures in a one-dimensional way the changes in the term structure of interest rates, practitioners use the sensitivity of the bond price with respect to the yield as a simple measure of this interest rate risk. This first-order measure to interest rate risk is the **duration**, which is defined as:

\[
D = -\frac{1}{P} \frac{dP}{dy} > 0
\]

where \( P(y) \) is the price-yield function and the negative is needed in order to have a positive duration for a long position in a bond. The reason is that the price-yield curve is decreasing in \( y \) so \( \frac{dP}{dy} < 0 \). If the bond provides the holder with cash flows \( c_i \), being the coupons for all times \( t_i (1 \leq i < n) \) with \( c_n \) being the sum of the last coupon and the face value, then the price \( P \), the yield \( y \) (cont. compounded) are related by

\[
P = \sum_{i=1}^{n} c_i e^{-y \cdot t_i} \rightarrow \frac{dP}{dy} = -\sum_{i=1}^{n} c_i t_i e^{-y \cdot t_i}
\]

The duration, using the definition, is then

\[
D = \frac{1}{P} \sum_{i=1}^{n} c_i t_i e^{-y \cdot t_i} = \sum_{i=1}^{n} \frac{c_i e^{-y \cdot t_i}}{P} = \sum_{i=1}^{n} t_i w_i,
\]

where \( w_i = \frac{c_i e^{-y \cdot t_i}}{P} \) and \( \sum_{i=1}^{n} w_i = \frac{1}{P} \sum_{i=1}^{n} c_i e^{-y \cdot t_i} = \frac{P}{P} = 1 \)

So, the **duration is a weighted average of all payment times**:

\[
D = \sum_{i=1}^{n} t_i w_i \text{ since the weights sum to } 1: \sum_{i=1}^{n} w_i = 1 \text{ and duration has units of time.}
\]

**Question (from job interviews):** What is the duration of a zero-coupon bond with maturity \( T \) and what are the duration bounds for coupon bonds with maturity \( T \)?

**Bond price changes for small yield changes**

When we use differentials, we know from Calculus that the following exact relation holds:
\[ dP = \frac{dP}{dy} dy \]

If we consider small changes instead of differentials, the above relation becomes approximate:

\[ \Delta P \approx \frac{dP}{dy} \Delta y \]

If we divide both sides above by the bond price \( P \) then we get: \( \frac{\Delta P}{P} \approx \frac{1}{P} \cdot \frac{dP}{dy} \Delta y = -D \Delta y \)

The important relation here is the following approximation, which is good enough for small yield changes:

\[ \frac{\Delta P}{P} \approx -D \Delta y \]

which gives us the percentage change in bond price, \( \frac{\Delta P}{P} \), in terms of the yield change \( \Delta y \) and the duration \( D \). Because of this simple but useful relation duration is still a very popular measure even though it was introduced by Macaulay in 1938 (sometimes it is called the Macaulay duration).

**Example**

Consider a 3-year 10% coupon bond with semi-annual coupons and a face value of $100. Suppose that the yield on the bond is 12% per annum with continuous compounding. Find the bond duration.

**Solution** We first need to find the bond price for the given yield, based on $5 coupons paid every 6 months for 3 years (total of 6 coupons):

\[
Bond \ Price \rightarrow P = \sum_{i=1}^{6} 5 \cdot e^{-0.12 \cdot i \cdot 0.5} + 100 \cdot e^{-0.12 \cdot 3} \at\text{5 digits} \rightarrow P = 94.2130
\]

\[
Bond \ Duration \rightarrow D = \frac{1}{94.213} \left( \sum_{i=1}^{6} 5 \cdot i \cdot 0.5 \cdot e^{-0.12 \cdot i \cdot 0.5} + 100 \cdot 3 \cdot e^{-0.12 \cdot 3} \right) \at\text{5 digits} \rightarrow D = 2.6530
\]

So, the bond price is $94.213 and the duration is 2.6530 years.

**Example**

For the bond from the previous example, find how much the bond price would change if the yield increases by 10 basis points.

**Solution** Remember that 1 basis point = \( 10^{-4} = 0.0001 = 0.01 \)%, so 10 bps = 0.1% = 0.001 = \( 10^{-3} \). We can find how much the bond price changes by two different approaches, one approximate and one precise:

- The fastest way to estimate the change in bond price \( \Delta P \) is to use the approximate relation we derived above: \( \Delta P \approx -P \cdot D \Delta y \), because we already know the price and the duration, so for \( \Delta y = +10 \) bps we get \( \Delta P \approx -94.213 \cdot 2.653 \cdot 10^{-3} \at\text{5 digits} \rightarrow \Delta P \approx (-0.2499) \). Thus, the bond price decreases to \( 94.213 + \Delta P \approx 94.213 - 0.2499 \at\text{5 digits} \rightarrow (94.2130 + \Delta P) \approx (93.9631) \) is the new bond price.
• The precise approach is to reprice the bond using the modified yield $0.12 + 10 \text{ bps} = 0.12 + 0.001 = 0.121$:

$$\text{New Bond Price} \rightarrow P = \sum_{i=1}^{6} 5 \cdot e^{-0.121 \cdot i \cdot 0.5} + 100 \cdot e^{-0.121 \cdot 3} \quad \text{at 5 digits} \quad P = 93.9630$$

If we compare the two results, we see that the new bond prices we get from the two approaches are the same up to 3 decimal digits. Keep in mind that the approximate approach gives good results only when the change in yield is small enough and for bonds with short maturities as they have less convexity. For longer maturity bonds and large moves in the yield, using the duration alone is not enough to give good results because of the large convexity.

**Convexity**

A measure of convexity is related to the second derivative of the price with respect to the yield:

$$C = \frac{1}{P} \frac{d^2P}{dy^2} = \frac{1}{P} \sum_{i=1}^{n} c_i \cdot t_i^2 \cdot e^{-y \cdot t_i} > 0$$

From **Taylor series** expansion up to second order, we obtain a more accurate approximation, given by:

$$\Delta P \approx \frac{dP}{dy} \Delta y + \frac{1}{2} \frac{d^2P}{dy^2} (\Delta y)^2$$

If we divide both sides by the bond price $P$ and use the definitions of duration and convexity above, we get:

$$\frac{\Delta P}{P} \approx \frac{1}{P} \frac{dP}{dy} \Delta y + \frac{1}{2} \frac{1}{P} \frac{d^2P}{dy^2} (\Delta y)^2$$

$$\frac{\Delta P}{P} \approx -D \Delta y + \frac{1}{2} C (\Delta y)^2$$

**Reading Assignments:**

• Read Chapter 4, sections 4.8 and 4.9 from *Options, Futures and Other Derivatives* 6th ed. by Hull.

1. **1st Part of Homework 2, Due Thursday February 18, 2010**

   1. A 5-year bond with yield of 11% (cont. compounded) pays an 8% coupon at the end of each year.

   • What is the bond price?

   • What is the bond duration?

   • Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.

   • Recalculate the bond price on the basis of a 10.8% per annum yield and verify that the result is in agreement with your answer to the previous question.