The Time Value of Money and Interest Rates

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The Time Value of Money

Everyone is familiar with the saying "time is money" and in finance there is a precise mathematical meaning attached to this saying. Namely, $1 today is worth more than $1 tomorrow or in a year's time, for that matter. This is because having $1 now instead of later, gives you the opportunity to make even more money by investing it wisely. For example, if you invest your $1 with a bank for one year, you expect to be compensated for not having this $1 during that time by demanding interest to be paid to you by the bank, on a regular basis, in addition to getting back your $1 in a year's time.

Interest Rates

In real life, there are many different interest rates for different users based on credit ratings and they are function of the time period for which they apply. So, in general an interest rate can be represented by a variable \( r_T \) where \( T \) represents the time period for which this rate applies, for example \( T = 3M, 6M, 1Y, 20Y \) etc. This dependence of interest rates on time is called term structure of interest rates. Although rates vary with time, for now we are going to assume that rates are constant and independent of time. Let's denote an interest rate by \( r \) and keep in mind that interest rates are typically given as percentage rates per annum. There is simple and compound interest. Simple interest is based only on the initial principal amount while in the case of compound interest, you also get interest on the interest.

Example of Simple Interest

Under the simple interest rule, interest is computed based only on the original principal amount and is proportional to the time period the investment is held. The proportionality applies not only to whole but also fractional years. In general, if an amount \( A \) is deposited in an account at a simple interest rate of \( r \) (expressed as a decimal), the total value after \( t \) years (\( t \) need not be a whole number) is

\[
V(t) = A + A \cdot r \cdot t = A \cdot (1 + r \cdot t)
\]

So, the deposit grows linearly with time. For example, if \( A = 1000 \), \( r = 5 \% \) and \( t = 2.5 \) years, then the original principal grows to

\[
1000 \cdot (1 + 0.05 \times 2.5) = 1125.000
\]

If a bank tells you they will give you 10% annual interest rate for your one year deposit, then this information is incomplete because we also need to know the compounding frequency.

Example of Compound Interest

Suppose that an initial amount of deposit \( A \) is invested for \( n \) years (\( n \) is an integer) with annual interest rate \( r \) such that interest is compounded \( m \) times per year. Then the simple interest rate for each of the \( m \) compounding periods in one year is \( \frac{r}{m} \) so that the interest accrued over the first period is \( \frac{A \cdot r}{m} \) and is
added to the principal $A$ to get a total value at the end of the first period $A + \frac{A \cdot r}{m} = A \left(1 + \frac{r}{m}\right)$. Next, since interest also gets interest, this total amount is used as the new principal to compute the interest at the end of the second period and the total amount in the deposit grows to

$$A \cdot \left(1 + \frac{r}{m}\right) + A \cdot \left(1 + \frac{r}{m}\right) \cdot \frac{r}{m} = A \cdot \left(1 + \frac{r}{m}\right) \cdot \left(1 + \frac{r}{m}\right) = A \cdot \left(1 + \frac{r}{m}\right)^2$$

Similarly, at the end of the first year, that is, at the end of the $n$th period, the total amount in the deposit grows to $A \cdot \left(1 + \frac{r}{m}\right)^m$ and since $n$ (whole) years have $n \cdot m$ compounding periods, then after $n$ years the initial deposit grows to a terminal value of $V(n)$:

$$V(n) = A \cdot \left(1 + \frac{r}{m}\right)^{m \cdot n}$$

When the number of years is not a whole number but a decimal number, say $t = 1.6$ years, we can make the following approximation for the terminal value $V(t)$:

$$V(t) \approx A \cdot \left(1 + \frac{r}{m}\right)^{m \cdot t}, \text{ } t \text{ in years is decimal}$$

We can illustrate this approximation by considering the case $t = 1.6$ years and $m = 4$ (quarterly compounding). Since we compound interest every 3 months there is a whole number of periods up to 1.5 years, which is 6 periods and we can use the exact formula for a whole number of periods to get

$$V(1.5) = A \cdot \left(1 + \frac{r}{4}\right)^6$$

the total amount after 1.5 years. However, there is 0.1 year left and for this time we accrue interest under the simple rule, proportionally to time, namely the total amount after 1.6 years is

$$V(1.5) + V(1.5) \times r \times 0.1 = V(1.5) \left(1 + \frac{0.4r}{4}\right) \approx V(1.5) \left(1 + \frac{r}{4}\right)^{0.4} = A \cdot \left(1 + \frac{r}{4}\right)^{6.4}$$

Notice that $m \cdot t = 4 \times 1.6 = 6.4$ and the approximation that we used comes from $(1 + x)^t \approx 1 + t \cdot x$ for small $x$, where in our case $t = 0.4$ and $x = \frac{r}{4}$ is small indeed.

**Nominal vs. Effective Interest Rate**

The effect of compounding on yearly growth is highlighted by stating an effective annual interest rate, which is the equivalent annual interest rate that would produce the same result after 1 year without compounding. For example, an annual rate of 8% compounded quarterly will produce a growth factor for one year $\left(1 + \frac{0.08}{4}\right)^4 = 1.082432160$; by definition, the simple effective rate $r_{eff}$ should produce the same result: $1 + r_{eff} = 1.0824$ thus $r_{eff} = 8.24 \%$. The compounded rate of 8% is called the nominal rate.
Continuous Compounding

As the compounding frequency $m$ tends to infinity, the limit is known as continuous compounding. When we have an integer number of years, that is $n \in \mathbb{N}$, we can use the famous limit from Calculus:

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{m \cdot n} = e^{r \cdot n},$$

where $e = 2.71828 \ldots$ is the irrational (infinitely many non-repeating decimal digits) Euler number. For an arbitrary, fractional number of years $t$, the approximate formula that we derived above becomes an exact formula as the number of periods $m$ tends to infinity. The reason is that we can approximate better and better the given fractional time by a whole number of periods. Therefore, even in the more general case of a fractional number of years $t$, we get the exact formula for the growth factor with continuous compounding:

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{m \cdot t} = e^{r \cdot t}.$$

Therefore, an amount $A$ invested for $t$ years ($t \in \mathbb{R}$) at the continuously compounded rate $r$ grows to $V(t) = A \cdot e^{r \cdot t}$. For most practical purposes, continuous compounding can be thought of as being equivalent to daily compounding, as one can see from the table below. Compounding money at a continuously compounded rate $r$ for $T$ years amounts to multiplying by a growth factor of $e^{r \cdot T}$. Discounting money at a continuously compounded rate $r$ for $T$ years involves multiplying by a discount factor of $e^{-r \cdot T}$.

We can always convert a discretely compounded rate $R_m$ into the equivalent continuously compounded rate $r$ and vice versa; based on 1 year time and $1$ deposit:

$$e^r = \left(1 + \frac{R_m}{m}\right)^m \rightarrow r = m \cdot \ln \left(1 + \frac{R_m}{m}\right) \quad \text{and} \quad R_m = m \left(e^{\frac{r}{m}} - 1\right).$$

From the first equation above, we get the first formula for $r$ by taking the natural logarithm on both sides and the second formula we get by taking both sides to the power of $\frac{1}{m}$ or simply taking the $m$th root. Remember that $y = e^x$ is equivalent to $x = \ln(y)$.

Example

An interest rate is quoted as 10% per annum with semi-annual compounding. Find the equivalent rate with continuous compounding. Let $r$ be the wanted rate with continuous compounding. On the one hand, $1$ will grow to $\left(1 + \frac{0.1}{2}\right)^2$ after $t = 1$ year, under the discrete (semi-annual) compounding and on the other hand this $1$ under the equivalent continuous compounding at rate $r$ should grow to the same final amount:

$$\left(1 + \frac{0.1}{2}\right)^2 = e^{r \cdot 1}, \quad r = 2 \cdot \ln \left(1 + \frac{0.1}{2}\right) = r = 0.09758032834 \quad \text{or} \quad r = 9.76 \%.$$
Here is a table showing the effect of compounding frequency for 10% annual rate on the value of $10,000 deposit at the end of 1 year ($n = 1$)

<table>
<thead>
<tr>
<th>Compounding frequency</th>
<th>Value of $10,000 at the end of 1 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually ($m = 1$)</td>
<td>$10000 \cdot \left(1 + \frac{0.1}{1}\right) = 11000.00$</td>
</tr>
<tr>
<td>Semiannually ($m = 2$)</td>
<td>$10000 \cdot \left(1 + \frac{0.1}{2}\right)^2 = 11025.00$</td>
</tr>
<tr>
<td>Quarterly ($m = 4$)</td>
<td>$10000 \cdot \left(1 + \frac{0.1}{4}\right)^4 = 11038.13$</td>
</tr>
<tr>
<td>Monthly ($m = 12$)</td>
<td>$10000 \cdot \left(1 + \frac{0.1}{12}\right)^{12} = 11047.13$</td>
</tr>
<tr>
<td>Weekly ($m = 52$)</td>
<td>$10000 \cdot \left(1 + \frac{0.1}{52}\right)^{52} = 11050.65$</td>
</tr>
<tr>
<td>Daily ($m = 365$)</td>
<td>$10000 \cdot \left(1 + \frac{0.1}{365}\right)^{365} = 11051.56$</td>
</tr>
<tr>
<td>Continuous Compounding</td>
<td>$10000 \cdot e^{0.1} = 11051.71$</td>
</tr>
</tbody>
</table>

**Reading Assignments:**

- Read Chapter 4, sections 4.1 and 4.2 from *Options, Futures and Other Derivatives* 6th ed. by Hull.

**2nd Part of Homework 1, Due Thursday February 11, 2010**

**Time Value of Money**

1. An investor receives $1,100 in one year in return for an investment of $1,000 now. Calculate the rate of return per annum with a) Annual compounding b) Semi-annual compounding c) Monthly compounding and d) Continuous compounding.

2. An interest rate is quoted as 5% per annum with semi-annual compounding. What is the equivalent rate with a) annual compounding b) monthly compounding and c) continuous compounding.

3. **(Loan Calculation)** Suppose you have borrowed $10,000 from a bank. The terms of the loan are that the annual interest is 12% compounded monthly. You are to make equal monthly payments as to repay (amortize) this loan over 5 years. How much are your monthly payments? How much is the total interest you will pay for this loan?

   **Hint:** Either work period by period to find the fixed monthly payment that would give a net value of zero after 60 months or think about discounting the periodic fixed monthly payments so that the total present value is equal to the loan of $10,000.
The Mathematics of Bond Pricing

In finance, a **bond is a debt security**, such that the issuer of the bond owes the holder of the bond a debt and is obliged to pay interest (called **the coupon**) and to repay the principal (called **the face value**) at maturity. A bond is thus a formal contract to repay borrowed money with interest at fixed time intervals, specified in the contract.

Thus a bond is like a loan: the issuer of the bond is the borrower, the holder of the bond is the lender, and the coupon is the interest. Bonds provide the issuer with external funds to finance long-term investments or in the case of government bonds, to finance current government expenditure. Bonds and stocks are both securities, but the major difference between the two is that stockholders have an equity stake in the company (i.e., they own a piece of the company), whereas bondholders have a creditor stake in the company (i.e., they are lenders to the company). Another difference is that bonds usually have a defined term, or maturity, after which the bond is redeemed, whereas stocks may be outstanding indefinitely.

Debt securities with a term of less than one year are generally designated money market instruments rather than bonds. **Certificates of deposit** (CDs) offered by commercial banks are considered to be **money market instruments** and not bonds. Most bonds have a term of up to thirty years. In the market for U.S. Treasury securities, there are three groups of bond maturities:

- **Short term (Treasury Bills):** maturities up to one year;
- **Medium term (Treasury Notes):** maturities between one and ten years;
- **Long term (Treasury Bonds):** maturities greater than ten years.

The bonds issued by the US government are considered risk-free since the probability that the US government will default is very low, partly because the federal government can always print more money if required. Corporate bonds are considered risky and their probabilities of default (that is the inability of the corporations to meet their financial obligations) are captured by the credit ratings provided by credit rating agencies such as Standard & Poor's, Moody's, etc. Bonds have a long history:

**Zero-Coupon Bonds**

A zero-coupon bond (also called a **discount bond**) is a bond bought at a price lower than its face value, with the face value repaid at the time of maturity. It does not make periodic interest payments, the so-called "coupons," hence the term zero-coupon bond. Examples of zero-coupon bonds include U.S. Treasury bills and U.S. savings bonds.

**Zero Rates**

The **n-year zero-coupon rate** is the rate of interest earned on an investment that starts today and lasts for n years. There is only one payment at the end of n years. There are no intermediate payments. The n-year zero coupon rate is also referred to as the **n-year spot rate**, the **n-year zero rate** or simply the **n-year zero**. Treasury zero rates are determined by coupon-bearing Treasury bonds by the **bootstrap method** (homework #3 is a simple example of this approach).

**Example**
Let the 10-year zero rate with continuous compounding be quoted as 7% per annum. This means that $1000, if invested for 10 years grows to 
$$1000 \cdot e^{0.07 \cdot 10} = 2013.752707$$.

Treasury zero rates are implied from market prices of Treasury coupon bonds. The market prices of Treasury bonds are determined at auctions.

**Example**

Consider a zero-coupon bond with a face value of $100 and maturity T years from today (time 0). If the T-year zero rate with continuous compounding is known to be \( r_T \) then the price \( P_0 \) of this bond at time 0 is

\[
P_0 = 100 \cdot e^{-r_T \cdot T}
\]

We say that we simply discount back to time 0, the time T cash flow of $100 using the T-zero rate and call the **discount factor** associated with the T-spot rate. We'll always use continuous compounding from now on. Keep in mind that the time 0 value of this bond, that is \( P_0 \), should grow by a factor of \( e^{r_T \cdot T} \) after time T to \( P_0 \cdot e^{r_T \cdot T} = 100 \cdot e^{-r_T \cdot T} \cdot e^{r_T \cdot T} = 100 \) as expected.

<table>
<thead>
<tr>
<th>Discounting</th>
<th>( P_0 = 100 \cdot e^{-r_T \cdot T} )</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
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<tr>
<td>( T )</td>
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**Bond Features**

- Bond's **principal** also known as **face value** or **par value** is the amount on which interest is paid periodically in the form of coupons and the principal is received by the bond holder at the end of the bond's life.

- Most bonds provide **coupons** on a semi-annual basis and the coupon is given as a fixed percentage rate of the principal, called the **coupon rate**.

**Example of Bond Pricing**

Consider a 2-year Treasury bond with a principal of $100 paying coupons semi-annually with a coupon rate of 6% per annum. To calculate the present value of all the cash flows given in the diagram below, we need the spot rates for all payment times. The theoretical price of a bond can then be calculated as the present value of all the cash flows that will be received by the owner of the bond. To do this correctly, we have to use the zero rates from the table above to discount the cash flows for the different payment times. For example, the first coupon will be paid in 6 months, so we need to use the 0.5Y-zero rate of 5% for discounting. But what is the exact coupon? The coupons are calculated based on the principal and using the simple interest rule. Since the coupon rate is 6% per annum, for 6 months the coupon rate is then 6%\times 0.5 = 3\%$. Remember that the simple rule gives us linear dependence on time and if the annual rate is \( R \), then the **simple rate** for time \( t \) years is \( R \cdot t \). In our case \( t = 0.5 \) years. Therefore, the coupon is \$100 \cdot 3 \% = \$3 and is paid twice a year. Also, remember that the principal is received by the bond holder at the end of the
bond's life and so it has to be included in calculating the present value of the bond cash flows. So, at time \( t = 2 \) years, we have a coupon of $3 plus $100 principal and the total amount $103 has to be discounted at the 2Y-zero rate of 6.8% (see the diagram below).

The table below provides these spot rates:

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Treasury Zero rate (%) cont. compounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.0</td>
</tr>
<tr>
<td>1.0</td>
<td>5.8</td>
</tr>
<tr>
<td>1.5</td>
<td>6.4</td>
</tr>
<tr>
<td>2.0</td>
<td>6.8</td>
</tr>
</tbody>
</table>

Therefore, the theoretical price of the bond, as the present value of all the cash flows, is $98.39. Note that this price is very close but less than the face value of $100. We say it's priced at discount:

\[
V = 3 \cdot e^{-0.05 \cdot 0.5} + 3 \cdot e^{-0.058 \cdot 1} + 3 \cdot e^{-0.064 \cdot 1.5} + 103 \cdot e^{-0.068 \cdot 2} = 98.38506278
\]

Reading Assignments:
- Read Chapter 4, sections 4.3 and 4.4 from *Options, Futures and Other Derivatives* 6th ed. by Hull.

3rd Part of Homework 1, Due Thursday February 11, 2010

1. Suppose that 6-month, 12-month, 18-month, 24-month and 30-month zero rates are 4%, 4.2%, 4.4%, 4.6%, 4.8% per annum with continuous compounding respectively. Estimate the price of a bond with a face value of 100 that will mature in 30 months and pays a coupon of 4% per annum semiannually.

2. A 10-year, 8% coupon bond currently sells for $90. A 10-year, 4% coupon bond currently sells for $80. What is the 10-year zero rate? Hint: Consider taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds.

3. The prices of 6-month and 1-year Treasury bills are $94.0 and $89.0. A 1.5-year bond that will pay coupons of $4 every 6 months currently sells for $94.84. A 2-year bond that will pay coupons of $5 every 6 months currently sells for $97.12. Calculate the 6-month, 1-year, 1.5-year and 2-year zero rates.