Quantum phase transitions in a chain of fluxonium qubits

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• Introduction to superconducting qubits
  • Transmon
  • Fluxonium

• Coupled fluxonium chain
  • $\ell \gg 1$
  • $1 < \ell \lesssim 2$

• Conclusions
superconducting qubits

- Qubits made from superconducting circuits
- Rapid progress in recent years
- Testing fundamentals of quantum mechanics and measurement

Devoret & Schoelkopf, Science (2013)
superconducting circuits

Quantum LC oscillator

• Useful to describe in terms of phase $\phi$

• Conjugate to Cooper pair number $n = \frac{Q}{2e}$

• Related to flux in inductor $\frac{2\pi \phi}{\Phi_0} = LI$

\[ \mathcal{L} = \frac{\dot{\phi}^2}{2E_C} - \frac{E_L}{2} \phi^2 \quad \mathcal{H} = \frac{E_C}{2} n^2 + \frac{E_L}{2} \phi^2 \]

\[ E_C = 4e^2/C \quad E_L = \Phi_0^2/4\pi^2 L \]

Girvin, Les Houches (2011)
superconducting circuits

Quantum LC oscillator

- Ladder operators

\[ \phi = \frac{2\pi}{\Phi_0} \sqrt{\frac{\hbar}{2}} \sqrt{\frac{L}{C}} (a + a^\dagger) \]

- Quantum fluctuations in \( \phi \) suppressed by capacitance

\[ \mathcal{H} = \frac{1}{\sqrt{LC}} a^\dagger a = \sqrt{E_C E_L} a^\dagger a \]

- Need something nonlinear to make a qubit

Girvin, Les Houches (2011)
superconducting qubits

Transmon qubit

Use a Josephson junction:

\[ V = -E_J \cos \phi \approx \frac{E_J}{2} \phi^2 - \frac{E_J}{4!} \phi^4 + \ldots \]

Small fluctuations \(\to\) weakly nonlinear oscillator

\[ H \approx \sqrt{E_C E_J} a^\dagger a - E_C a^\dagger a^\dagger a a + \ldots \]

Koch et. al., PRA (2007)
superconducting qubits

Add an inductor

\[ V(\phi) = \frac{E_L}{2} \phi^2 - E_J \cos(\phi - \phi_e) \]

\[ \phi_e = 2\pi \Phi_e / \Phi_0 \]

\[ \Gamma \sim (E_J E_C)^{1/4} e^{-8 \sqrt{E_J / E_C}} \]

To have multiple minima, need \[ E_L = \frac{\hbar^2}{(2e)^2 L} \lesssim E_J \]

But large inductors introduce extra parasitic \( C \) → suppresses quantum fluctuations

Girvin, Les Houches (2011)
superconducting qubits

Fluxonium qubit

“Superinductor” made from JJs in series

Manucharyan et al., Science (2009)
Pop et al., Nature (2014)
superconducting qubits

Can couple qubits to cavities or superconducting resonators

Effective Jaynes-Cummings interaction

\[ H_{JC} = g(\sigma^+ b + \sigma^- b^\dagger) \]

Can reach strong coupling \(\rightarrow\) “circuit QED”
Sensitive probe of qubit states

Paik et. al., PRL (2011)

Blais et. al., PRA (2004)

Girvin, Les Houches (2011)
quantum simulation

• Large-scale systems being made

• Opportunity to study many-body quantum models in a new environment

• Drive and dissipation → new physics

• What kinds of models can be easily realized?

Houck et. al., Nature Physics (2012)
chain of fluxonium qubits

\[ \phi_e = \frac{2\pi \Phi_e}{\Phi_0} \]

\[ \mathcal{L} = \sum_{j=0}^{N-1} \left[ \frac{\left( \dot{\phi}_{j+1} - \dot{\phi}_j \right)^2}{2EC} - \left\{ \frac{E_L}{2} \phi_j^2 - E_J \cos \left( \phi_{j+1} - \phi_j - \phi_e \right) \right\} \right] \]
chain of fluxonium qubits

- Hard to find conjugate momenta for $\phi_j$

$$\mathcal{L} = \sum_{j=0}^{N-1} \left[ \frac{(\dot{\phi}_{j+1} - \dot{\phi}_j)^2}{2E_C} - \left\{ \frac{E_L}{2} \phi_j^2 - E_J \cos (\phi_{j+1} - \phi_j - \phi_e) \right\} \right]$$

- Introduces long range interactions for $\theta_j = \phi_j - \phi_{j-1}$
classical ground state

\[ V = \sum_{j=0}^{N-1} \left[ \frac{E_L}{2} \phi_j^2 - E_J \cos (\phi_{j+1} - \phi_j - \phi_e) \right] \]

Two classical parameters

\[ \ell = 2 \sqrt{\frac{E_J}{E_L}} \]

external flux

\[ \phi_e = 2\pi \Phi_e / \Phi_0 \]

Studies on similar models

Marchand, Hood, Caillé, PRL (1987)
Yokoi, Tang, Chou, PRB (1988)
classical ground state

\[ \frac{1}{2} \sqrt{\frac{E_L}{E_J}} = \frac{1}{\ell} \]

\[ V = \sum_{j=0}^{N-1} \left[ \frac{E_L}{2} \phi_j^2 - E_J \cos \left( \phi_{j+1} - \phi_j - \phi_e \right) \right] \]
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Two regimes:

1. \[ 1 < \ell \lesssim 2 \]

2. \[ \ell \gg 1 \]
classical ground state

\[ \frac{1}{2} \sqrt{\frac{E_L}{E_J}} = \frac{1}{\ell} \]

\[ V = \sum_{j=0}^{N-1} \left[ \frac{E_L}{2} \phi_j^2 - E_J \cos (\phi_{j+1} - \phi_j - \phi_e) \right] \]

Two regimes:

1. \( \ell \gg 1 \)
2. \( 1 < \ell \lesssim 2 \)
kinks

\[ \ell \gg 1 \quad E_J \gg E_L \]

\[ V = \sum_{j=0}^{N-1} \left[ \frac{E_L}{2} \phi_j^2 - E_J \cos (\phi_{j+1} - \phi_j - \phi_e) \right] \]

Stable states for small \( \phi_e \):

- \( \phi_j = 0 \) everywhere

- \( (\phi_{j+1} - \phi_j - \phi_e - 2\pi n) \) kept small

\( \phi_j \) may jump by \( 2\pi \), elsewhere satisfies (continuum limit):

\[ E_J \partial_x^2 \phi = E_L \phi \]
Solutions take the form of kinks

\[ \phi_j = -\pi \text{sgn}(j) \, e^{-2|j|/\ell} \]

Single kink cost:

\[ E_1 = 2\pi^2 \sqrt{E_J E_L} - 2\pi E_J \phi_e \]

\[ = 2\pi \left( \frac{\pi}{\ell} - \phi_e \right) \]

Kardar, PRB (1986)

\[ \ell = 2\sqrt{\frac{E_J}{E_L}} \]

\[ \phi_{e,c} = \frac{\pi}{\ell} \]
• Solutions take the form of kinks

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Kardar, PRB (1986)

• Single kink cost:

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\[ = 2\pi \left( \frac{\pi}{\ell} - \phi_e \right) \]

\[ \phi_{e,c} = \frac{\pi}{\ell} \]
Overlapping exponential tails

Repulsive kink-kink interaction

\[ J_{ij} = \frac{4\pi^2 E_J}{\ell} \exp \left( - \frac{2|i - j|}{\ell} \right) \]

Ground states for larger \( \phi_e \) are superpositions of many kinks

ground state at some \( \phi_e \gg \phi_{e,c} \)
Introduce pseudo-spin

\[ | \downarrow \rangle_j \text{ no kink at link } j \]

\[ | \uparrow \rangle_j \text{ kink at link } j \]

\[ (\phi_j - \phi_{j-1} \approx -2\pi + \phi_e) \]

many kinks

ground state close above \( \phi_{e,c} \)

ground state at some \( \phi_e \gg \phi_{e,c} \)
long-range Ising chain

Effective classical Hamiltonian

\[ H_{cl} = \frac{\Delta}{2} \sum_j (\sigma_j^z + 1) + \frac{1}{8} \sum_{i \neq j} J_{ij} (\sigma_i^z + 1)(\sigma_j^z + 1) \]

eff. magnetic field \( \Delta = 2\pi E_J (\phi_{e,c} - \phi_e) \)

long-range interaction \( J_{ij} = \frac{4\pi^2 E_J}{\ell} \exp\left(-\frac{2|i-j|}{\ell}\right) \) \( \ell = 2\sqrt{\frac{E_J}{E_L}} \)

Frenkel-Kontorova-type model

Bak, Bruinsma, PRL (1982)
• When kinks are sparse, continuum calculation gives

\[ \rho \approx \frac{\phi_e}{2\pi}, \quad \phi_e \gg \phi_{e,\text{c}} \]

• Kinks live on discrete lattice, can't have ideal separation
• e.g. for \( \rho = 2/9 \)
devil's staircase

Kink density vs. magnetic flux $\phi_e$

Bak & Bruinsma, PRL (1982)

Step width:

$$\delta \phi_{e,p} = \frac{2\pi}{\ell^3} \frac{p}{\sinh^2(p/\ell)}$$

$$\approx \frac{2\pi}{\ell p}$$
Adding an antenna

- Couples to phase across single link
- Absorption of photon
- Creation / annihilation of local kinks
excitation spectrum

$$\frac{\omega}{\pi^2} \sqrt{E_J E_L}$$

single-kink excitations

\[ \Delta = 2\pi E_J (\phi_{e,c} - \phi_e) \]

kink creation

kink annihilation
quantum effects

\[ L = \sum_{j=0}^{N-1} \frac{(\dot{\phi}_{j+1} - \dot{\phi}_j)^2}{2E_C} - V[\phi] \]

Quantum fluctuations

\[ \Gamma \sim (E_J^3 E_C)^{1/4} \exp(-8\sqrt{E_J/E_C}) \]

overlap (c.f. fluxonium)

Quantum long-range Ising model

\[ \sigma^x = \sigma^+ + \sigma^- \]

\[ H = \frac{\Delta}{2} \sum_j (\sigma_j^\ddagger + 1) + \frac{\Gamma}{2} \sum_j \sigma_j^x + \frac{1}{8} \sum_{i \neq j} J_{ij} (\sigma_i^\ddagger + 1)(\sigma_j^\ddagger + 1) \]
quantum corrections to single-kink excitations

Quantum band
Long-range quantum hopping

\[ \frac{\Gamma}{2} \sum_j (\sigma_j^+ + \sigma_j^-) \]

Two processes:

\[ \delta E = -\Delta \]

\[ \delta E = \Delta + J_{ij} \]

Destructively interfere if \(|i - j| \gtrsim \ell \ln \left( \frac{\ell \Delta}{4\pi^2 E_J} \right)\)
quantum corrections to single-kink excitations

Quantum band
Long-range quantum hopping

$$\frac{\Gamma^2 E_J}{\Delta^2} \ln \left( \frac{E_J}{\ell \Delta} \right)$$

Quantum effects perturbative if

$$|\phi_e - \phi_{e,c}| \gg \frac{\Gamma}{(E_J^3 E_L)^{1/4}}$$

quantum critical
quantum phase diagram

\[ \ell = \sqrt{2E_J/E_L} \gg 1 \]

\[ H = \frac{\Delta}{2} \sum_j (\sigma_j^z + 1) + \frac{\Gamma}{2} \sum_j \sigma_j^x + \frac{1}{8} \sum_{i \neq j} J_{ij}(\sigma_i^z + 1)(\sigma_j^z + 1) \]

\[ \Delta = 2\pi E_J (\phi_{e,c} - \phi_e) \]

\[ J_{ij} = \frac{4\pi^2}{\ell} E_J e^{-2|i-j|/\ell} \]
quantum phase diagram

\[ \phi_e = 0 \quad \phi_e = \pi \]

\[
\begin{array}{cccc}
0 & p & p - 1 & \ldots \\
\hline
\frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2}
\end{array}
\]

e.g.: \[
\begin{array}{cccc}
0 & p & p - 1 & \ldots \\
\hline
\frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2}
\end{array}
\]

Devil's staircase has infinite number of pinned phases, period \( p \)

\[
\delta \phi_{e,p} = \frac{2\pi}{\ell^3} \frac{p}{\sinh^2(p/\ell)} \quad \delta \Delta_p = -2\pi E_J \delta \phi_{e,p}
\]

Classical phase transition to state with one extra/fewer kink
Adding/removing kink in pinned phase $\rightarrow p$ defects

Repel each other to give new classical ground state

See also: Odintsov, PRB 54, 1228 (1996); Sela et. al., PRB 84, 085434 (2014)
quantum phase diagram

\[ \phi_e = 0 \quad \text{to} \quad \phi_e = \pi \]

Adding/removing kink in pinned phase \( \rightarrow p \) defects

Repel each other to give new classical ground state

cf. domain walls in AFM

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quantum phase diagram

\[ \phi_e = 0 \quad \text{to} \quad \phi_e = \pi \]

0 \[ \begin{array}{c} p \end{array} \quad \begin{array}{c} p - 1 \end{array} \quad \ldots \quad \begin{array}{c} p - 2 \end{array} \quad \begin{array}{c} p \end{array} \quad \begin{array}{c} 2 \end{array} \]

Adding/removing kink in pinned phase \( \rightarrow p \) defects

cf. domain walls in AFM

Repel each other to give new classical ground state

See also: Odintsov, PRB 54, 1228 (1996); Sela et. al., PRB 84, 085434 (2014)
Adding/removing kink in pinned phase $\rightarrow$ $p$ defects

Repel each other to give new classical ground state

quantum phase diagram

$\phi_e = 0$ $\rightarrow$ $\phi_e = \pi$

$0$ $p$ $p - 1$ $\ldots$ $p - 2$ $2$

See also: Odintsov, PRB 54, 1228 (1996); Sela et. al., PRB 84, 085434 (2014)
quantum phase diagram

\[ \phi_e = 0 \quad \text{to} \quad \phi_e = \pi \]

\[ H = \frac{\Delta}{2} \sum_j (\sigma_j^z + 1) + \frac{\Gamma}{2} \sum_j \sigma_j^x + \frac{1}{8} \sum_{i \neq j} J_{ij}(\sigma_i^z + 1)(\sigma_j^z + 1) \]

\( p \) defects each gain kinetic energy \( \sim \frac{\Gamma^2}{E_{\text{add/rem}}} \sim \frac{\Gamma^2}{\sqrt{E_J E_L}} \)

Pinned order destroyed if \( p \frac{\Gamma^2}{\sqrt{E_J E_L}} \geq \delta \Delta = p E_J \frac{4\pi^2}{\ell^3 \sinh^2(p/\ell)} \)

\[ E_J = 10 \, \text{GHz}, \; E_C / 8 = 3 \, \text{GHz}, \; E_L = 0.5 \, \text{GHz} : \; p \sim O(10) \]

See also: Odintsov, PRB 54, 1228 (1996); Sela et. al., PRB 84, 085434 (2014)
Mobile defects → incommensurate Luttinger liquid

Described by Luttinger parameter $K$

Gapless excitations $\rho(\omega) \sim \omega^{\frac{1}{2}}(K + \frac{1}{K} - 2)$

See also: Odintsov, PRB 54, 1228 (1996); Sela et. al., PRB 84, 085434 (2014)
At boundaries $K = \frac{1}{p^2}$

**KT transition** to gapped homogeneous phase if $K \geq \frac{1}{8}$

(effect of $\Gamma(\sigma^+_j + \sigma^-_j)$)

Ising transition for $p = 2$

See also: Odintsov, PRB 54, 1228 (1996); Sela et. al., PRB 84, 085434 (2014)
quantum phase diagram

Power law interactions: Sela et. al., PRB 84, 085434 (2014)
other regime

$1 < \ell \lesssim 2$
other regime

\[ 1 < \ell \lesssim 2 \]

\[ V = \sum_{j=0}^{N-1} \left[ \frac{E_L}{2} \phi_j^2 - E_J \cos (\phi_{j+1} - \phi_j - \phi_e) \right] \]

\[ E_L \sim E_J \]

Kink picture not valid: \( \phi_j \) small everywhere

Second order transition to 1/2 state
\[ L = \sum_{j=0}^{N-1} \left[ \frac{(\dot{\phi}_{j+1} - \dot{\phi}_j)^2}{2E_C} - \left\{ E_L \phi_j^2 - E_J \cos (\phi_{j+1} - \phi_j - \phi_e) \right\} \right] \]

\[ \phi_j \to (-1)^j \varphi_j, \quad \varphi_j \ll 1, \text{ expand potential and take continuum limit} \]

\[ \tilde{\tau} = \sqrt{E_C E_J \tau} \]

\[ S = 2 \sqrt{\frac{E_J}{E_C}} \int d\tilde{\tau} dx \left[ \dot{\varphi}^2 + \frac{1}{4} |\cos \phi_e| (\partial_x \varphi)^2 \right. \]

\[ \left. \left( \frac{1}{\ell^2} - |\cos \phi_e| \right) \varphi^2 + \frac{1}{3} |\cos \phi_e| \varphi^4 \right] \]
Ising transition

\[ S = 2 \sqrt{\frac{E_J}{E_C}} \int d\tau dx \left[ \dot{\varphi}^2 + \frac{1}{4} |\cos \phi_e|(\partial_x \varphi)^2 + \left( \frac{1}{\ell^2} - |\cos \phi_e| \right) \varphi^2 + \frac{1}{3} |\cos \phi_e| \varphi^4 \right] \]

Quantum (1+1)-dimensional Ising model

Ginzburg region:

\[ |\phi - \phi^c_e| < \frac{\sqrt{2E_C/E_J}}{24 \sin \phi^c_e} \]

Quantum critical behaviour accessible for \( N \gtrsim 4 \sqrt{\frac{E_J}{E_C}} \sqrt{|\cos \phi^c_e|} \)

\[ E_J = 10 \text{ GHz}, \ E_C/8 = 3 \text{ GHz} : \ N \gtrsim O(1) \]
Excitation spectrum

Small oscillations in $\phi_j$

$$\omega = \frac{\sqrt{E_C E_L}}{2} \sqrt{\ell^2 \cos \phi_e + \frac{1}{\sin^2 \frac{k}{2}}}$$

$$\Delta_{k=\pi} = \frac{\sqrt{E_C E_L}}{2} \sqrt{\ell^2 \cos \phi_e + 1}$$

Ising transition
Excitation spectrum

Small oscillations in $\phi_j$

$$\omega = \frac{\sqrt{E_C E_L}}{2} \sqrt{\ell^2 \cos \phi_e + \frac{1}{\sin^2 \frac{k}{2}}}$$

$$\Delta_{k=\pi} = \frac{\sqrt{E_C E_L}}{2} \sqrt{\ell^2 \cos \phi_e + 1}$$

Gap closes at transition

$$\cos \phi_e^c = -\frac{1}{\ell^2}$$
Excitation spectrum

Unit cell doubled in ordered phase

\[ \phi_j \rightarrow \bar{\phi}(-1)^j + \phi_j \]

\[ \omega_+^2 - \omega_-^2 = E_C E_L \ell^2 \sin \phi_e \sin 2\bar{\phi} \]

\[ \Delta_{k=\pi/2} = \sqrt{E_C E_L} \sqrt{1 + \frac{\ell^2}{2} \cos(2|\bar{\phi}| - \phi_e)} \]
Ising transition

Excitation spectrum

Unit cell doubled in ordered phase

$$\phi_j \rightarrow \phi_j(-1)^j + \phi_j$$

$$\ell > \sqrt{2} : \text{gap closes at some } \phi_e$$

$$\Delta_{k=\pi/2} = \sqrt{E_C E_L} \sqrt{1 + \frac{\ell^2}{2} \cos(2|\phi| - \phi_e)}$$
Ising transition to $\frac{2}{4}$ phase
• Devil's staircase not present above $\ell \sim 2$
  
Marchand et. al, PRB 37, 1898 (1988)
Yokoi et. al., PRB 37, 2173 (1988)

• First order transitions

• How do homogeneous phases join?
conclusions

- Chain of fluxonium qubits with realistic parameters
- Several classes of phase transition
- Observable in excitation spectrum

$$\ell = 2\sqrt{E_J/E_L} \gg 1$$

- C-IC
- Kosterlitz-Thouless

$1 < \ell \lesssim 2$

- Ising
- First order

- Luttinger Liquid
- Ising