

# A NATURAL MODEL OF THE MULTIVERSE AXIOMS

VICTORIA GITMAN AND JOEL DAVID HAMKINS

ABSTRACT. If ZFC is consistent, then the collection of countable computably saturated models of ZFC satisfies all of the Multiverse Axioms of [Hama].

## 1. INTRODUCTION

The multiverse axioms that are the focus of this article arose in connection with a continuing debate in the philosophy of set theory between the *Universe view*, which holds that there is a unique absolute set-theoretical universe, serving as set-theoretic background for all mathematical activity, and the *Multiverse view*, which holds that there are many set-theoretical worlds, each instantiating its own concept of set. We refer the reader to [Hama] and to several other articles in the same special issue of the Review of Symbolic Logic for a fuller discussion of this philosophical exchange (see also [Hamb, Ham09]). The multiverse axioms express a certain degree of richness for the set-theoretic multiverse, flowing from a perspective that denies an absolute set-theoretic background.

Meanwhile, the multiverse axioms admit a purely mathematical, non-philosophical treatment, on which we shall focus here. We shall internalize the study of multiverses to set theory by treating them as mathematical objects within ZFC, allowing for a mathematized simulacrum inside  $V$  of the full philosophical multiverse (which would otherwise include universes outside  $V$ ). Specifically, in this article we define that a *multiverse* is simply a nonempty set or class of models of ZFC set theory. The multiverse axioms then correspond to the features listed in Definition 1, which such a collection may or may not exhibit.

**Definition 1** (Multiverse Axioms). Suppose that  $\mathcal{M}$  is a multiverse, a nonempty collection of models of ZFC.

- (1) The *Realizability* axiom holds for  $\mathcal{M}$  if whenever  $M$  is a universe in  $\mathcal{M}$  and  $N$  is a definable class of  $M$  satisfying ZFC from the perspective of  $M$ , then  $N$  is in  $\mathcal{M}$ .
- (2) The *Forcing Extension* axiom holds for  $\mathcal{M}$  if whenever  $M$  is a universe in  $\mathcal{M}$  and  $\mathbb{P}$  is a forcing notion in  $M$ , then  $\mathcal{M}$  has a forcing extension of  $M$  by  $\mathbb{P}$ , a model of the form  $M[G]$ , where  $G$  is an  $M$ -generic filter for  $\mathbb{P}$ .
- (3) The *Class Forcing Extension* axiom holds for  $\mathcal{M}$  if whenever  $M$  is a universe in  $\mathcal{M}$  and  $\mathbb{P}$  is a ZFC-preserving class forcing notion in  $M$ , then  $\mathcal{M}$  has a forcing extension of  $M$  by  $\mathbb{P}$ , a model of the form  $M[G]$ , where  $G$  is an  $M$ -generic filter for  $\mathbb{P}$ .

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- (4) The *Countability* axiom holds for  $\mathcal{M}$  if for every universe  $M$  in  $\mathcal{M}$  there is another universe  $N$  in  $\mathcal{M}$ , such that  $M$  is a countable set in  $N$ .
- (5) The *Wellfoundedness Mirage* axiom holds for  $\mathcal{M}$  if for every universe  $M$  in  $\mathcal{M}$ , there is  $N$  in  $\mathcal{M}$ , which thinks  $M$  is a set with an ill-founded  $\omega$ .

Although the next axioms do not appear in [Hama], we shall nevertheless consider them here. They follow a suggestion of Jonas Reitz, who proposed that whenever a universe  $M$  in the multiverse has a measurable cardinal, then it should be the internal ultrapower of another universe  $V$ , sending its critical point to that cardinal. That is, the suggestion is that we should be able to iterate large cardinal embeddings backwards. Here, we generalize the idea to other ultrapowers and to embeddings generally.

- (6) The *Reverse Ultrapower Axiom* holds for  $\mathcal{M}$  if for every universe  $M$  in  $\mathcal{M}$ , there is a universe  $N$  in  $\mathcal{M}$  such that  $M$  is the internal ultrapower of  $N$  by an ultrafilter on  $\omega$  in  $N$ .
- (7) The *Strong Reverse Ultrapower Axiom* holds for  $\mathcal{M}$  if for every universe  $M_1$  in  $\mathcal{M}$  and every ultrafilter  $U_1$  in  $M_1$  on a set  $X_1$  in  $M_1$ , there is  $M_0$  in  $\mathcal{M}$ , with an ultrafilter  $U_0$  on a set  $X_0$ , such that  $M_1$  is the internal ultrapower of  $M_0$  by  $U_0$ , sending  $U_0$  to  $U_1$ .
- (8) The *Reverse Embedding Axiom* holds for  $\mathcal{M}$ , if for every universe  $M_1$  in  $\mathcal{M}$  and every embedding  $j_1 : M_1 \rightarrow M_2$  definable in  $M_1$  from parameters and thought by  $M_1$  to be elementary, there is  $M_0$  in  $\mathcal{M}$  and similarly definable  $j_0 : M_0 \rightarrow M_1$  in  $M_0$ , such that  $j_1$  is the iterate of  $j_0$ , meaning  $j_1 = j_0(j_0)$ .

In other words, the Reverse Embedding axiom asserts that every internal elementary embedding  $j_1 : M_1 \rightarrow M_2$  arises as an iterate of an earlier embedding. The idea is that if we are living in  $M_1$  and see the embedding  $j_1$ , then for all we know, it has already been iterated an enormous number of times. To be precise, by  $j_1 = j_0(j_0)$ , we mean that if  $j_0$  is definable by  $\varphi(x, y, a)$  over  $M_0$ , then  $j_1$  is definable by  $\varphi(x, y, j_0(a))$  over  $M_1$ .

There is, of course, a certain degree of redundancy in the axioms; for example, the Class Forcing Extension axiom implies the Forcing Extension axiom, the Reverse Embedding axiom implies the Reverse Ultrapower axioms, and the Reverse Ultrapower and Countability axioms imply the Wellfoundedness Mirage. There are also a few subtler points. In several of the axioms, when it is stated that one model  $M$  is an element of another model  $N$ , what is meant is that there is an object in  $N$  that  $N$  thinks is a pair  $\langle m, E \rangle$  for which  $E$  is a binary relation on  $m$ , and externally, the set  $\{a \in N \mid N \models a \in m\}$  with the relation  $\{(a, b) \mid N \models aEb\}$  is isomorphic to  $M$  with its relation. Another subtle issue is that in the Countability axiom, although  $M$  must be a countable set in  $N$ , there is no insistence that  $N$  regard  $M$  as a model of ZFC; indeed, since  $N$  itself may be nonstandard, it may have a nonstandard version of ZFC, with nonstandard size axioms that  $M$  does not satisfy in  $N$ , even if  $M$  satisfies ZFC externally. Similarly, the Wellfoundedness Mirage axiom requires that  $M$  is seen to be ill-founded by  $N$ , but again  $N$  may not look upon  $M$  as a model of ZFC, since  $N$  may itself have nonstandard size axioms, and there is no reason to expect that  $N$  believes  $M$  to satisfy them. The *Realizability* axiom, on the other hand, only applies to the definable models of the universe that satisfy its own version of ZFC. For example, if  $M$  is a model of ZFC with nonstandard  $\omega$ , then for every natural number  $n$  in  $M$ , there will be  $V_\alpha^M$  that  $M$  believes to model the  $\Sigma_n$  theory of its ZFC, and when  $n$  is nonstandard, this includes all of

the standard ZFC; so although these  $V_\alpha^M$  are models of full ZFC, the Realizability axiom does not apply to them.

Hamkins [Hama] provided a model of the first five multiverse axioms constructed via iterated ultrapowers, and he inquired at that time whether the collection of all countable nonstandard models of set theory might already be a model of the multiverse axioms. Observation 3 and Theorem 11 below show that this is too much, since the axioms impose a requirement of computable saturation on the models. Nevertheless, in this article we prove that this is the only such obstacle, for our main theorem shows that if ZFC is consistent, then the collection of all countable computably saturated models of ZFC satisfies all of the multiverse axioms.

**Main Theorem 2.** *If ZFC is consistent, then the collection of all countable computably saturated models of ZFC satisfies all the multiverse axioms.*

We note that the main theorem will imply that it is not true that every multiverse satisfying the axioms must consist of countable models. For example, if  $N$  is a nonstandard model of  $\text{ZFC} + \text{Con}(\text{ZFC})$ , then we may let  $\mathcal{M}$  be the models that  $N$  thinks are countable computably saturated models of ZFC. By the main theorem,  $N$  thinks that this collection satisfies all the multiverse axioms, and it follows that this will really be the case also outside of  $N$ . But the models in  $\mathcal{M}$  will be at least as large as  $\omega^N$ , which could be as large in cardinality as we like.

## 2. COMPUTABLY SATURATED MODELS OF SET THEORY

In this section, we shall explain precisely why we must restrict to the computably saturated models and review the key properties of these models that are needed for the proof of the main theorem. Computable saturation was introduced in [BS76a] and is also commonly known as *recursive saturation*. A model  $M$  of a computable language  $\mathcal{L}$  is said to be *computably saturated*, if for every finite tuple  $\bar{a}$  in  $M$ , every finitely realizable computable type  $p(\bar{a}, \bar{x})$  is already realized in  $M$ . A type  $p(\bar{y}, \bar{x})$  in a computable language  $\mathcal{L}$  is computable when the set of the Gödel codes of its formulas is a computable set in the usual sense of Turing computability. In the future, we shall freely associate types with subsets of  $\mathbb{N}$  consisting of the Gödel codes of their formulas. A model of ZFC set theory is  $\omega$ -*nonstandard* if it has a nonstandard  $\omega$ . Because tuples can be viewed as a single set in models of ZFC, for these it suffices to consider only computable types of the form  $p(a, x)$ . Note that a computably saturated model of ZFC must necessarily be  $\omega$ -nonstandard since the type  $p(x) = \{\mathbf{n} < x \mid n \in \omega\} \cup \{x < \omega\}$ , where  $\mathbf{n}$  is the term  $1 + \dots + 1$  with  $n$  many 1s, is a finitely realizable computable type over any model of ZFC. For any model  $M$  of ZFC, we may consider the trace on the natural numbers of the sets that exist in  $M$ . Specifically, we say that a set  $A \subseteq \mathbb{N}$  is *coded* by  $a$  in  $M$ , if  $a$  is a set in  $M$  whose intersection with the standard natural numbers is exactly  $A$ . When  $a$  is a set of natural numbers in  $M$ , then  $A$  is also known as the *standard part* of  $a$ , and the collection of all sets  $A$  arising this way is accordingly called the *Standard System* of  $M$ , denoted  $SSy(M)$ . Note that the standard system of any model must include all computable sets, since the model will agree on the behavior of any computation that halts in the standard  $\mathbb{N}$ . Standard systems have been extensively studied in the context of models of Peano Arithmetic where they play a crucial conceptual role, but the notion was originally introduced for models of various set theories [Fri73].

**Observation 3.** *Any multiverse satisfying the Wellfoundedness Mirage axiom must consist entirely of computably saturated models of ZFC.*

*Proof.* If  $\mathcal{M}$  satisfies the Wellfoundedness Mirage axiom, it follows that every member of  $\mathcal{M}$  has a nonstandard  $\omega$ . Thus, again by the Wellfoundedness Mirage axiom, every member of  $\mathcal{M}$  is a set in a model of ZFC having a nonstandard  $\omega$ . Thus, every member of  $\mathcal{M}$  is computably saturated by Lemma 4.  $\square$

**Lemma 4.** *Every model of ZFC that is an element of an  $\omega$ -nonstandard model of ZFC is computably saturated.*

*Proof.* Suppose that  $M$  is a model of ZFC that is an element of an  $\omega$ -nonstandard model  $N$  of ZFC. In order to see that  $M$  is computably saturated, suppose that  $p(b, x)$  is a computable finitely realizable type over  $M$ . Let  $a \in N$  code  $p(y, x)$ . Since  $p(b, x)$  is finitely realizable and  $N$  has a truth predicate for  $M$ , for every  $n \in \mathbb{N}$ , the model  $N$  knows that there is  $c \in M$  such that  $M \models \varphi(b, c)$  for every formula  $\varphi(y, x)$  with Gödel code less than  $n$  in  $a$ . Because the standard  $\mathbb{N}$  is not definable in  $N$ , there must be a nonstandard natural number  $d \in N$  and an element  $e \in M$  with  $N$  satisfying that  $M \models \varphi(b, e)$  for every (possibly nonstandard) formula  $\varphi(y, x)$  with Gödel code less than  $d$  in  $a$ . Since  $d$  is nonstandard, this includes every formula in  $p(y, x)$ . By the absoluteness of satisfaction for standard formulas, it follows that  $M \models \varphi(b, e)$  for every  $\varphi(b, x)$  in  $p(b, x)$  and thus  $e$  realizes  $p(b, x)$ .  $\square$

**Lemma 5.** *If ZFC is consistent, then there are  $2^{\aleph_0}$  many pairwise non-isomorphic countable computably saturated models of ZFC. Every real is in the standard system of such a model.*

*Proof.* If ZFC consistent, then every completion of ZFC as a theory has a countable computably saturated model, because any countable model of ZFC can be extended elementarily to a computably saturated model by successively realizing types in a countable elementary chain. For any real  $x$ , one can ensure that the type expressing that  $x$  is coded is realized.  $\square$

A model  $M$  of ZFC is said to be  $SSy(M)$ -saturated if it realizes every finitely realizable type coded in  $M$ . It turns out that a model  $M$  is computably saturated if and only if it is  $SSy(M)$ -saturated. To see this, fix a type  $p(y, x)$  coded by  $a \in M$  and fix  $b \in M$  such that  $p(b, x)$  is finitely realizable. Define a new type  $q(b, a, x)$  to consist of all formulas of the form  $(\ulcorner \varphi(y, x) \urcorner \in a) \rightarrow \varphi(b, x)$ , and observe that  $q(y, z, x)$  is computable and finitely realizable, using the objects realizing the corresponding fragment of  $p(b, x)$ . Thus, there is some  $e$  in  $M$  realizing  $q(b, a, e)$ , and it follows that  $e$  realizes  $p(b, e)$ , as desired. We note also that the type of any element in a computably saturated model is in the standard system of that model: for  $a \in M$ , define  $p(a, x)$  to be the type consisting of all formulas of the form  $\ulcorner \varphi(y) \urcorner \in x \leftrightarrow \varphi(a)$  and observe that it is computable and finitely realizable; thus, the type of  $a$  is coded in  $M$ . In particular, the theory of any computably saturated model is an element of its standard system. According to [Kay91],  $SSy(M)$ -saturation was introduced by Wilmers in his unpublished 1975 thesis where he established the above equivalence.

The next lemma generalizes another fundamental result from models of PA that appears in [Smo81a] but has as well been attributed to Jensen and Ehrenfeucht [JE76], and Wilmers, among others.

**Key Lemma 6.** *Any two countable computably saturated models of ZFC with the same theory and the same standard system are isomorphic.*

*Proof.* This is a standard model-theoretic back-and-forth construction. The observations above ensure that the models are standard system-saturated, and all types of their elements are coded in the standard system. Thus, we may construct the desired isomorphism in a countable recursive procedure that maps elements of one model to elements in the other realizing the same types over what has been defined so far.  $\square$

The following lemma will be critical for our verification of the Wellfoundedness Mirage axiom in the Main Theorem. This fact may have been known some time ago. For example, Schlipf [Sch77, III.2.6] proved that every computably saturated model of ZF is an element of an  $\omega$ -nonstandard model of ZF, and Ressayre [RW87, 3.3] proved that every model of ZF is elementarily equivalent to a model of ZF containing as an element an isomorphic copy of itself. (See [Hal] for an interesting discussion.)

**Lemma 7.** *Every countable computably saturated model of ZFC contains an isomorphic copy of itself as an element, which it thinks is  $\omega$ -nonstandard. That is, if  $M$  is a countable computably saturated model of ZFC, then  $M$  has an element  $N$  which it thinks is a countable  $\omega$ -nonstandard model of a fragment of set theory, such that  $M \cong N$ .*

*Proof.* Suppose that  $M$  is a countable computably saturated model of ZFC. As we noted above,  $\text{Th}(M)$  is coded by some  $a \in M$ . By the Reflection Theorem, every finite subset of this theory is true in some rank initial segment of  $M$ , and  $M$  recognizes this for any particular such finite subset. Since the standard cut  $\mathbb{N}$  is not definable in  $M$ , there must be a nonstandard natural number  $b$  in  $M$ , such that  $M$  thinks the theory consisting of all formulas whose Gödel codes are in  $a$  and less than  $b$  is consistent. Since  $b$  is nonstandard, this includes the entire  $\text{Th}(M)$ . By the Completeness Theorem in  $M$ , therefore, we may build a model  $N$  in  $M$  satisfying this consistent fragment of  $a$ , which includes all of  $\text{Th}(M)$ , such that additionally,  $M$  thinks  $N$  is  $\omega$ -nonstandard. Since  $\omega^M$  is an initial segment of  $\omega^N$  and  $M$  is  $\omega$ -nonstandard, it follows that  $M$  and  $N$  have the same standard system. Also, since  $M$  is  $\omega$ -nonstandard, it follows by Lemma 4 that  $N$  is computably saturated. We conclude by Lemma 6 that actually  $M \cong N$ .  $\square$

By considering the situation from the perspective of the smaller copy of the model, we deduce:

**Corollary 8.** *Every countable computably saturated model of ZFC is an element of another countable computably saturated model of ZFC that thinks it is a countable  $\omega$ -nonstandard model of a (nonstandard) fragment of set theory.*

Note in Lemma 7 that although we know on the outside that  $N \models \text{ZFC}$ , since it satisfies  $\text{Th}(M)$ , it could happen that  $M \not\models "N \models \text{ZFC}"$ , since perhaps  $M$  thinks that some of the nonstandard ZFC axioms of  $M$  fail in  $N$ . Despite this, Corollary 8 suffices to verify the Countability and Well-foundedness Mirage axioms for the collection of countable computably saturated models of ZFC, since as we mentioned there was no insistence in the axioms that the larger model look upon the smaller as a model of what it thinks is full ZFC. Nevertheless, under a stronger assumption

it is possible to obtain the stronger conclusion. Surely a stronger assumption is required, since if  $N \models \text{ZFC} + "M \models \text{ZFC}"$ , then  $N \models \text{ZFC} + \text{Con}(\text{ZFC})$ , and so  $\text{Con}(\text{ZFC} + \text{Con}(\text{ZFC}))$ . And if this  $N$  is an element of a further such model, then we get  $\text{Con}(\text{Con}(\text{Con}(\text{ZFC})))$ , and so on transfinitely. The stronger assumption we shall make is that for every countable computably saturated model  $M$  of ZFC, the theory  $T_M = \text{ZFC} + \{\text{Con}(\text{ZFC} + \Gamma) \mid \Gamma \subseteq_{\text{Fin}} \text{Th}(M)\}$  is consistent.

**Theorem 9.** *If  $M$  is a computably saturated countable model of ZFC, then there is a countable computably saturated model  $N$  of ZFC containing  $M$  as an element and satisfying that  $M$  is a nonstandard model of ZFC if and only if the theory  $T_M$  is consistent.*

*Proof.* The forward implication is immediate, since any such model  $N$  will satisfy the theory  $T_M$ . For the converse implication, suppose that  $T_M$  is consistent. By Lemma 6, it suffices to show that there exists  $N$  containing a countable computably saturated model  $K$  with the same theory and standard system as  $M$  that it recognizes as a model of ZFC. It will immediately follow that  $\text{SSy}(N) = \text{SSy}(K)$  and so we shall need to ensure that  $\text{SSy}(N) = \text{SSy}(M)$ . Scott observed in [Sco62] that every standard system is a *Scott set*, that is, a Boolean algebra of subsets of natural numbers that is closed under relative computability and contain at least one branch through every element that is a binary tree. In that paper, he famously showed that given a countable Scott set  $\mathfrak{X}$  and a theory  $T \in \mathfrak{X}$  extending PA, there is a model of  $T$  whose standard system is exactly  $\mathfrak{X}$ . Wilmer in his thesis, showed that this easily generalizes to obtaining a computably saturated model. Let  $\mathfrak{X} = \text{SSy}(M)$  and observe that  $T_M \in \mathfrak{X}$ , since  $T_M$  is computable in  $\text{Th}(M)$ , which is an element of  $\mathfrak{X}$ . Summarizing, we can build a countable computably saturated model  $N$  of ZFC satisfying  $T_M$  and having same standard system as  $M$ . Since  $N$  satisfies  $T_M$ , it satisfies  $\text{Con}(\text{ZFC} + \Gamma)$  where  $\Gamma$  is a nonstandard segment containing  $\text{Th}(M)$ . So  $N$  can build a countable model  $K$  that it thinks satisfies  $\text{ZFC} + \Gamma$ .  $\square$

We have observed that the assumption that  $T_M$  is consistent transcends  $\text{Con}(\text{ZFC})$ . But the assumption is not so strong, for if  $M$  is an element of an  $\omega$ -model  $N$  of ZFC, then  $N$  satisfies  $T_M$ . In particular, if there is a transitive model  $N$  of ZFC, then it satisfies  $T_M$ , and hence also  $\text{Con}(T_M)$ , for every countable model  $M$  in  $N$ .

Let us close this section by mentioning the concept of *resplendency*, a powerful generalization of computable saturation that has unified many applications of it. Resplendency is a second-order analogue of computable saturation, in that it concerns realizing second-order types; that is, it is about interpreting a new predicate symbol on the universe. Specifically, a first order structure  $M$  is *resplendent*, if every finitely-realized computable type  $p(X, \vec{a})$  in the language of  $M$  expanded by a predicate symbol  $X$  with  $\vec{a}$  a finite list of parameters from  $M$  is realized in  $\langle M, X \rangle$  for some interpretation of  $X$ . (The type is finitely realized if all finite subsets of  $p$  are realized in such a model  $\langle M, X \rangle$ .) The concept of resplendency was introduced by Barwise and Schlipfe [BS76b], and independently by Ressayre [Res86], who proved that every countable computably saturated model is resplendent (see also [Smo81b]). Schlipfe [Sch80] proved that a countable model of set theory is computably saturated if and only if it is  $\omega$ -nonstandard and there is a club of ordinals  $\alpha$  with  $V_\alpha \prec V$ . Moschovakis and Chang (see [CK90]) proved that every saturated model is resplendent. Although we have presented our arguments in an

elementary manner appealing only to computable saturation, it appears that many of our lemmas can be fruitfully generalized, by proving them via resplendency.

### 3. PROOF OF MAIN THEOREM

Let us now complete the proof of the main theorem, which we restate here for convenience.

**Main Theorem.** *If ZFC is consistent, then the collection of all countable computably saturated models of ZFC satisfies all the multiverse axioms.*

*Proof.* We shall argue in turn that the collection  $\mathcal{M}$  of all countable computably saturated models of ZFC satisfies each of the multiverse axioms. First, since we have assumed that ZFC is consistent, Lemma 5 shows that in fact there are many countable computably saturated models of ZFC. So  $\mathcal{M}$  is nonempty.

Consider now the Realizability axiom. Suppose that  $M \in \mathcal{M}$  and  $N$  is a definable class in  $M$  and a model of ZFC. Since  $M$  is an element of some other nonstandard model  $M'$  by Corollary 8, it follows that  $N$  is also an element of  $M'$ , and so by Lemma 4, it follows that  $N$  is computably saturated. Since  $N$  is clearly also countable, as  $M$  was countable, it follows that  $N \in \mathcal{M}$ . Thus,  $\mathcal{M}$  satisfies the Realizability axiom.

For the Forcing axioms, suppose that  $M \in \mathcal{M}$  and  $\mathbb{P}$  is a forcing notion in  $M$ . Certainly we can easily produce by diagonalization an  $M$ -generic filter  $G \subseteq \mathbb{P}$  and form the forcing extension  $M[G]$ . Furthermore by Corollary 8, we can do so inside any model  $M'$  which looks upon  $M$  as countable. Thus, there is a forcing extension  $M[G]$  inside such an  $M'$ . It now follows by Lemma 4 that  $M[G]$  is computably saturated, as desired. So  $\mathcal{M}$  satisfies the Forcing and Class Forcing Extension axioms.

The difficult cases of the Wellfoundedness Mirage and Countability axioms are exactly provided for by Corollary 8.

The Reverse Ultrapower axioms follow from the Reverse Embedding axiom, so it suffices to consider that axiom. Suppose that  $M_1$  is countable and computably saturated and  $j_1 : M_1 \rightarrow M_2$  is an elementary embedding in  $M_1$ , defined in  $M_1$  from some parameter  $z$ , so that  $j_1(x) = y \iff M_1 \models \varphi(x, y, z)$ . By interpreting this definition in  $M_2$  using  $j_1(z)$  we obtain the iterate embedding  $j_2 = j_1(j_1) : M_2 \rightarrow M_3$ , defined by  $j_2(x) = y \iff M_2 \models \varphi(x, y, j_1(z))$ . Since the critical point of  $j_1$  must be at least  $\omega^{M_1}$ , which is nonstandard, it follows that  $M_1$  and  $M_2$  share a nonstandard initial segment of their natural numbers and therefore have the same standard system. Since they also have the same theory, it follows by Lemma 6 that there is an isomorphism  $\pi : M_1 \cong M_2$ . Since the type of  $z$  in  $M_1$  is the same as the type of  $j_1(z)$  in  $M_2$ , we may assume in the back-and-forth argument that  $\pi(z) = j_1(z)$ . Thus, since  $j_1$  is defined in  $M_1$  by  $\varphi(x, y, z)$ , the map  $\pi$  carries  $j_1$  to the class defined in  $M_2$  by  $\varphi(x, y, \pi(z))$ , which is  $j_2$ . In other words,  $\pi$  carries the entire map  $j_1 : M_1 \rightarrow M_2$  isomorphically to the map  $j_2 : M_2 \rightarrow M_3$ . And since  $j_2 = j_1(j_1)$  is by definition an iterate of  $j_1$ , the Reverse Embedding axiom holds in the case of  $j_2 : M_2 \rightarrow M_3$ . Since this is isomorphic via  $\pi$  to  $j_1 : M_1 \rightarrow M_2$ , it follows by replacing the objects with their image under  $\pi$  that there is  $j_0 : M_0 \rightarrow M_1$  such that  $j_1 = j_0(j_0)$ , as desired.  $\square$

Recall that a model  $M$  is said to be  $\kappa$ -saturated for a cardinal  $\kappa$  if every finitely realizable type in the language extended to include some  $< \kappa$ -many constants for

elements of the model is already realized. A model of cardinality  $\kappa$  is said to be simply *saturated* if it is  $\kappa$ -saturated. It is a basic fact that any two saturated models of the same theory and same cardinality are isomorphic. If  $M$  is a saturated model of ZFC and  $N$  is a model of ZFC that is an element of  $M$ , then  $N$  must be saturated and have the same cardinality as  $M$ . The cardinality is the same since  $\omega^M$ , by saturation, is already of the same cardinality as  $M$ . For details on saturated models, see [CK90]. Thus, it easily follows that every saturated model of ZFC of cardinality  $\kappa$  has an isomorphic copy of itself that it thinks is a countable  $\omega$ -nonstandard model of a finite fragment of ZFC. Other facts necessary for the proof of the Main Theorem follow for saturated models of ZFC of cardinality  $\kappa$  as well; in most cases they are easier to see than for computable saturation because any two elementarily equivalent saturated models of the same cardinality are isomorphic. Thus, we get the following corollary of the Main Theorem.

**Corollary 10.** *If there are saturated models of ZFC of cardinality  $\kappa$ , then the collection of these satisfies all the multiverse axioms.*

It is natural to wonder whether the collection of all models of ZFC forms a model of the multiverse axioms, or whether the collection of all countable models of ZFC does so. Unfortunately, neither does.

**Theorem 11.** *If ZFC is consistent, then the collection of all models of ZFC is not a model of the multiverse axioms. Neither is the collection of all countable models of ZFC, nor the collection of all countable nonstandard models of ZFC, nor the collection of countable  $\omega$ -nonstandard models of ZFC, nor the collection of such models restricted to a given consistent completion of ZFC.*

*Proof.* By Observation 3, all we need to do for the first part is to show that there is a model of ZFC that is not computably saturated. In fact, every consistent completion of ZFC has a countable  $\omega$ -nonstandard model that is not computably saturated (and this proves the subsequent claims). To see this, take any countable nonstandard model  $M$  of ZFC. The *definable cut* of  $M$  consists of all  $x \in M$  such that  $x \in (V_\alpha)^M$ , where  $\alpha$  is a definable ordinal in  $M$  (without parameters). If  $M_0$  is the definable cut of  $M$ , then it is relatively easy to verify the Tarski-Vaught criterion, and so  $M_0 \prec M$ . It follows that  $M_0$  has exactly the same definable ordinals as  $M$ , and these are unbounded in the ordinals of  $M_0$ . Thus,  $M_0$  omits the type  $p(x)$  asserting that whenever there is a unique ordinal satisfying  $\varphi(y)$ , then  $y < x$ . This is a computable finitely realizable type not realized in  $M_0$ , and so  $M_0$  is not computably saturated. Thus, by Observation 3, it can not be in any model of the multiverse axioms.  $\square$

Let us conclude this paper by considering the degree to which we might expect a multiverse to be upward directed. Specifically, a multiverse  $\mathcal{M}$  is *upward directed*, if for any two elements  $M, N \in \mathcal{M}$  there is an element  $W \in \mathcal{M}$  containing (isomorphic copies of)  $M$  and  $N$  as elements. The multiverse  $\mathcal{M}$  is *countably upward directed* if for any countable subcollection  $\mathcal{M}_0 \subseteq \mathcal{M}$ , there is an element  $W \in \mathcal{M}$  containing (an isomorphic copy of) every element of  $\mathcal{M}_0$ . It is easy to see that the multiverse of all countable computably saturated models of ZFC is not upward directed. This is because any two elements of an upward directed multiverse  $\mathcal{M}$  containing only  $\omega$ -nonstandard models must have the same standard system. Suppose that  $M$  and  $N$  are elements of an upward directed multiverse  $\mathcal{M}$  containing only  $\omega$ -nonstandard



models. By directedness, there is  $W \in \mathcal{M}$  with  $M$  and  $N$  both in  $W$ . Since the  $\omega^W$  is an initial segment of  $\omega^M$  and  $\omega^N$ , and is itself nonstandard, it follows that all three models  $M$ ,  $N$  and  $W$  have the same standard system. Thus, all models in  $\mathcal{M}$  have the same standard system. Since any real can be placed into the standard system of a countable computably saturated model of ZFC, it follows that not all countable computably saturated models of ZFC have the same standard system. So this multiverse is not upward directed. Nevertheless, this is the only obstacle.

**Theorem 12.** *If ZFC is consistent, the multiverse of countable computably saturated models having a fixed standard system is countably upward directed, and continues to satisfy all the multiverse axioms.*

*Proof.* Fix a given Scott set  $S$  and consider the multiverse  $\mathcal{M}_S$  of all countable computably saturated models of ZFC having standard system  $S$ . We observe first that the proof of the main theorem goes through for  $\mathcal{M}_S$ , since in each part of that argument, the desired universe had the same standard system as the original model. So it remains only to argue that  $\mathcal{M}_S$  is countably upward directed. Suppose that  $\mathcal{M}_0 = \{M_0, M_1, \dots\}$  is a countable subcollection of  $\mathcal{M}_S$ , so that every  $M_n$  is a countable computably saturated model of ZFC with standard system  $S$ . By the remarks before Lemma 6, it follows that  $\text{Th}(M_n)$  is in  $S$  for every  $n$ . Let  $M$  be any  $\omega$ -nonstandard model having standard system  $S$ . Since  $\text{Th}(M_n)$  is coded in  $M$ , by arguments of the proof of Lemma 7,  $M$  can build a model  $m_n$  satisfying the theory  $\text{Th}(M_n)$  and having  $SSy(m_n) = S$ . Therefore, by Lemma 6, it follows that  $m_n$  and  $M_n$  are isomorphic. In summary, we have proved that every model in  $\mathcal{M}_S$  serves as a witness to the countable upward directedness of  $\mathcal{M}_S$ .  $\square$

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V. GITMAN, NEW YORK CITY COLLEGE OF TECHNOLOGY OF THE CITY UNIVERSITY OF NEW YORK, MATHEMATICS, 300 JAY STREET, BROOKLYN, NY 11201  
*E-mail address:* [vgitman@nylogic.org](mailto:vgitman@nylogic.org), <http://websupport1.citytech.cuny.edu/faculty/vgitman>

J. D. HAMKINS, MATHEMATICS, THE GRADUATE CENTER OF THE CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NY 10016 & MATHEMATICS, COLLEGE OF STATEN ISLAND OF CUNY, STATEN ISLAND, NY 10314  
*E-mail address:* [jhamkins@gc.cuny.edu](mailto:jhamkins@gc.cuny.edu), <http://jdhamkins.org>