Substituting Supercompactness by Strong Unfoldability

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This talk presents joint work with Joel D. Hamkins.

The two main results can be viewed as analogues of the following two theorems, but in the context of strong unfoldability:

**Theorem (Laver ’78)**

*If $\kappa$ is supercompact, then after suitable preparatory forcing, the supercompactness of $\kappa$ becomes indestructible by all $<\kappa$-directed closed forcing.*

**Theorem (Baumgartner ’79)**

*If there exists a supercompact cardinal in $V$, then there is a forcing extension of $V$ in which PFA holds.*
Strongly Unfoldable Cardinals
are defined via embeddings whose domain is a set, not the whole universe \( V \)

**Definition**

For an inaccessible cardinal \( \kappa \), a \( \kappa \)-model of set theory is a transitive set \( M \) of size \( \kappa \) such that \( M \models \text{ZFC}^- \), \( \kappa \in M \), and \( M^{<\kappa} \subseteq M \).

**Definition (Villaveces ’98)**

An inaccessible cardinal \( \kappa \) is strongly unfoldable if for every ordinal \( \theta \) and every \( \kappa \)-model \( M \) there is an elementary embedding \( j : M \to N \) with \( \text{cp}(j) = \kappa \), \( j(\kappa) > \theta \) and \( V_\theta \subseteq N \).

- view them as “miniature strong” cardinals
- Strong cardinals are strongly unfoldable
Theorem (Villaveces '98)

Strongly unfoldable cardinals

- are weakly compact
- are totally indescribable
- are downwards absolute to $L$

Moreover

- measurable cardinals are strongly unfoldable in $L$, but not necessarily in $V$
- same for Ramsey cardinals

In consistency strength, strongly unfoldable cardinals are

- bounded below by the indescribable cardinals
- bounded above by the subtle cardinals
- relatively low in the hierarchy of large cardinals
Strongly unfoldable cardinals can be viewed as “miniature supercompact” also!

**Theorem (Miyamoto ’98, indep. Dzamonja/Hamkins ’06)**

The following are equivalent:

- For every ordinal $\theta$ and every $\kappa$-model $M$ there is $j : M \to N$ with $\text{cp}(j) = \kappa$, $j(\kappa) > \theta$ and $V_\theta \subseteq N$.

- For every ordinal $\theta$ and every $\kappa$-model $M$ there is $j : M \to N$ with $\text{cp}(j) = \kappa$, $j(\kappa) > \theta$ and $N^\theta \subseteq N$.

This equivalence was discovered independently by Miyamoto ’98 in the context of his $H_{\kappa^+}$-reflecting cardinals, an equivalent large cardinal notion.
Indestructibility

Question (Villaveces '98)
Can we make a strongly unfoldable cardinal $\kappa$ indestructible by $\text{Add}(\kappa, 1)$? How about $\text{Add}(\kappa, \theta)$? What's the strength of a strongly unfoldable $\kappa$ where GCH fails?

Idea: Borrow lifting techniques from other large cardinals.

- Hamkins '01 used strongness methods to lift through fast function forcing, through $\text{Add}(\kappa, 1)$ and Easton support iterations that control GCH
- Dzamonja and Hamkins '06 used supercompactness methods to show that $\diamondsuit_\kappa(\text{REG})$ can fail at a strongly unfoldable cardinal $\kappa$

This hinted at a general indestructibility phenomenon.
The $\kappa$-proper posets

- recall that proper forcing is defined by considering whether the generic filter is generic over countable elementary submodels $X \prec H_\lambda$.
- $\kappa$-proper forcing generalizes this situation to those elementary submodels $X \prec H_\lambda$ of size $\kappa$.
- $\kappa^+$-c.c. forcing is $\kappa$-proper; so is $\leq \kappa$-closed forcing.
- $\kappa$-proper forcing preserves $\kappa^+$.

Idea:

- Take a large $\kappa$-proper poset $\mathbb{P}$
- Put $\mathbb{P}$ into $X \prec H_\lambda$ of size $\kappa$
- If $\pi : X \rightarrow M$ is Mostowski collapse, then $M$ is a $\kappa$-model
- $\mathbb{P}$ would never fit into $M$, but we work with $\pi(\mathbb{P})$
- Key point: The pointwise image $\pi'' G$ is an $M$-generic filter for $\pi(\mathbb{P})$, by $\kappa$-properness!
- Lift the embedding $j : M \rightarrow N$ to $j^* : M[\pi'' G] \rightarrow N^*$
Theorem (J.,’06)

If $\kappa$ is strongly unfoldable, then after suitable preparatory forcing, the strong unfoldability of $\kappa$ becomes indestructible by all $<\kappa$-closed $\kappa$-proper forcing. This includes all $<\kappa$-closed $\kappa^+$-c.c forcing and all $\leq\kappa$-closed forcing.

- proof uses supercompactness methods (as in [Laver78])
- the preparatory forcing is the lottery preparation of $\kappa$ (as in [Hamkins00])
- indestructibility by all $<\kappa$-closed forcing, not merely $<\kappa$-directed closed
- indestructibility by $\text{Add}(\kappa, 1)$, $\text{Add}(\kappa, \theta)$, and $\text{Coll}(\theta, \kappa^+)$ for $\theta \geq \kappa^+$
- finite iterations of $<\kappa$-closed $\kappa$-proper posets are $<\kappa$-closed $\kappa$-proper

Question (J.’06)

Can we make $\kappa$ indestructible by all $<\kappa$-closed $\kappa^+$-preserving forcing?
Answer: Yes!

Main Theorem (Hamkins and J.,’07)

If $\kappa$ is strongly unfoldable, then after suitable preparatory forcing, the strong unfoldability of $\kappa$ becomes indestructible by all $<\kappa$-closed $\kappa^+$-preserving forcing.

- a key technical step allows us to reduce the case of a $\kappa^+$-preserving poset to the main idea that worked with $\kappa$-proper posets
- this result is optimal within the class of $<\kappa$-closed posets!
  (If $\kappa$ is weakly compact in a $<\kappa$-closed forcing extension $V[G]$ collapsing $\kappa^+V$, then $\square_\kappa$ fails in $V$. But this is a very strong hypothesis, already infinitely many Woodin cardinals.)
- it is impossible to relax $<\kappa$-closure to $<\kappa$-strategic closure
  (the standard forcing to add a $\kappa$-Souslin tree is $<\kappa$-strategically closed, but destroys the weak compactness of $\kappa$)
Corollary

If there is a model of ZFC with a strongly unfoldable cardinal, then there is a model of ZFC with a weakly compact cardinal $\kappa$ that is indestructible by all $<\kappa$-closed $\kappa^+$ preserving forcing.

Open Question

What is the exact consistency strength of a weakly compact cardinal $\kappa$ that is indestructible by all $<\kappa$-closed $\kappa^+$ preserving forcing?

The question is also open for a weakly compact cardinal $\kappa$ indestructible by all $<\kappa$-closed $\kappa$-proper forcing, or even only $<\kappa$-closed $\kappa^+$-c.c. forcing.
The forcing axioms \( \text{PFA and } \text{PFA}(\Gamma) \text{ and } \text{PFA}_\delta \)

**Definition**

\( \text{PFA} \) is the principle asserting that for every proper poset \( Q \) and for every collection \( D \) of \( \aleph_1 \) many maximal antichains of \( Q \), there exists a \( D \)-generic filter \( G \subseteq Q \).

- If \( \Gamma \) is any class of posets, then \( \text{PFA}(\Gamma) \) is the corresponding assertion restricted to proper posets \( Q \in \Gamma \).
- If \( \delta \) is a cardinal, then \( \text{PFA}_\delta \) is the corresponding assertion where the antichains in \( D \) must have size at most \( \delta \).
The PFA lottery preparation of a cardinal $\kappa$, relative to a function $f : \kappa \to \kappa$, is the countable support $\kappa$-iteration, which forces at stages $\gamma \in \text{dom}(f)$ with the lottery sum of all proper forcing $Q$ in $V[G_{\gamma}]$ having hereditary size at most $f(\gamma)$.

The PFA lottery preparation

- modifies Hamkins’ lottery preparation [Hamkins00] in a similar way as Baumgartner’s iteration modifies Laver’s preparation [Laver78]
- works best when $f$ exhibits a certain fast-growing behavior
- is flexible tool for various large cardinal notions—no need for Laver functions
- forces $c = 2^\omega = \kappa = \aleph_2$
- of a supercompact cardinal forces PFA
- of a strongly unfoldable cardinal forces what?...
The forcing axioms $PFA_{\aleph_2}$ and $PFA_{\aleph_3}$

Answer:

Theorem (Hamkins & J. ’06)

The PFA lottery preparation of a strongly unfoldable cardinal $\kappa$ forces $PFA(\aleph_2$-proper), with $c = \aleph_2 = \kappa$.

- recall: $\aleph_2$-proper posets include all $\aleph_3$-c.c posets and all $\leq \aleph_2$-closed posets.

Theorem (Hamkins & J. ’06)

The PFA lottery preparation of a strongly unfoldable cardinal $\kappa$ forces $PFA_{\aleph_2}$, with $c = \aleph_2 = \kappa$.

- If the given antichains have size at most $\aleph_2 = \kappa$, then they are small enough to be subsets of the elementary submodel $X \prec H_\lambda$ of size $\kappa$. The generic filter $G$ need not be $X$-generic, but it does meet all antichains inside of $X$. 
Question
Can we improve PFA(\(\mathfrak{N}_2\)-proper) to get PFA (\(\mathfrak{N}_3\)-preserving)?

(A poset is \(\delta\)-preserving if it does not collapse \(\delta\) as cardinal.)

Answer: Yes!

Main Theorem (Hamkins & J. ’07)
If \(\kappa\) is strongly unfoldable and \(0^\#\) does not exist, then the PFA lottery preparation of \(\kappa\) forces PFA (\(\mathfrak{N}_2\)-preserving) and PFA (\(\mathfrak{N}_3\)-preserving) and PFA\(_{\mathfrak{N}_2}\), with \(2^\omega = \kappa = \mathfrak{N}_2\).

Conclusion:
In order to extract significant strength from PFA, one must collapse \(\mathfrak{N}_3\) to \(\mathfrak{N}_1\)!
Combined with the equiconsistency result of Miyamoto ’98, we get:

**Corollary**

The following are *equiconsistent* over ZFC:

- There is a strongly unfoldable cardinal $\kappa$.
- $\text{PFA}(\aleph_2\text{-preserving}) + \text{PFA}(\aleph_3\text{-preserving}) + \text{PFA}_{\aleph_2} + 2^\omega = \aleph_2$
- $\text{PFA}_{\aleph_2}$

**Question**

Do any of the principles $\text{PFA}(\aleph_2\text{-preserving})$, $\text{PFA}(\aleph_3\text{-preserving})$, or $\text{PFA}_{\aleph_2}$ imply any of the others? Are the former principles equiconsistent with the latter?

- What happens if $0^\#$ does exist, to the PFA lottery preparation of a strongly unfoldable cardinal?
- Which fragment of PFA can we get from a weakly compact cardinal?
References


THANK YOU!