STRONGLY UNFOLDABLE CARDINALS
MADE INDESTRUCTIBLE

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Abstract. I provide indestructibility results for large cardinals consistent with \( V = L \), such as weakly compact, indescribable and strongly unfoldable cardinals. The Main Theorem shows that any strongly unfoldable cardinal \( \kappa \) can be made indestructible by \( <\kappa \)-closed, \( \kappa \)-proper forcing. This class of posets includes for instance all \( \kappa \)-closed posets that are either \( \kappa^+\)-c.c. or \( \leq \kappa \)-strategically closed as well as finite iterations of such posets. Since strongly unfoldable cardinals strengthen both indescribable and weakly compact cardinals, the Main Theorem therefore makes these two large cardinal notions similarly indestructible. Finally, I apply the Main Theorem to obtain a class forcing extension preserving all strongly unfoldable cardinals in which every strongly unfoldable cardinal \( \kappa \) is indestructible by \( <\kappa \)-closed, \( \kappa \)-proper forcing.

\( \S 1. \) Introduction. Determining which cardinals can be made indestructible by which classes of forcing has been a major interest in modern set theory. Laver [Lav78] made supercompact cardinals highly indestructible. Gitik and Shelah [GS89] treated strong cardinals and Hamkins [Ham00] obtained partial indestructibility for strongly compact cardinals. I aim to extend this analysis to some smaller large cardinals, such as weakly compact, indescribable or strongly unfoldable cardinals. Each of these cardinals is, if consistent with ZFC, consistent with \( V = L \). So is each of the large cardinal hypotheses used for the results of this paper.

The Main Theorem makes any given strongly unfoldable cardinal \( \kappa \) indestructible by \( <\kappa \)-closed, \( \kappa \)-proper forcing. This class of posets includes for instance all \( <\kappa \)-closed posets that are either \( \kappa^+\)-c.c. or \( \leq \kappa \)-strategically closed as well as finite iterations of such posets. Strongly unfoldable cardinals were introduced by Vialevees [Vil98] as a strengthening of both weakly compact cardinals and totally indescribable cardinals. The Main Theorem thus makes these two large cardinal notions similarly indestructible.

The only previously known method of producing a weakly compact cardinal \( \kappa \) indestructible by \( <\kappa \)-closed, \( \kappa^+\)-c.c. forcing was to start with a supercompact cardinal \( \kappa \) and apply the Laver preparation (or some alternative, such as the lottery

Received February 1, 2007.
2000 Mathematics Subject Classification. 03E55, 03E40.
Key words and phrases. strongly unfoldable cardinal, forcing, indestructibility.

The results in this article are based in part on my Ph.D. dissertation, written under the direction of Joel David Hamkins at the CUNY Graduate Center. I am deeply grateful to Professor Hamkins for his constant and invaluable support while writing this article.

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0022-4812/08/7104-0008/54.40

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preparation [Ham00]. Similarly, in order to obtain a totally indescribable cardinal $\kappa$ indestructible by all $\leq \kappa$-closed forcing, one had to at least start with a strong cardinal $\kappa$ and use the Gitik–Shelah method. It follows from the Main Theorem that it does, in fact, suffice to start with a strongly unfoldable cardinal $\kappa$, thereby reducing the large cardinal hypothesis significantly (see Corollary 20 and 36).

I am hoping that the theorems and and ideas of this paper will allow for similar reductions in other indestructibility results or relative consistency statements. Moreover, the described methods may help identify indestructibility for other large cardinals as well, such as for those cardinals that can be characterized by elementary embeddings which are sets. In Section 7, I obtain a global form of the Main Theorem: I prove that there is a class forcing extension which preserves every strongly unfoldable cardinal $\kappa$ and makes its strong unfoldability indestructible by $\leq \kappa$-closed, $\kappa$-proper forcing.

Given a strongly unfoldable cardinal $\kappa$, how indestructible can we make it? Of course, if $\kappa$ happens also to be supercompact, then the Laver preparation of $\kappa$ makes $\kappa$ indestructible by all $\leq \kappa$-directed closed forcing. In general we cannot hope to prove such wide indestructibility for $\kappa$ if we want to only rely on hypotheses consistent with $V = L$. Intuitively it seems that collapsing $\kappa^+$ to $\kappa$ poses a serious problem: A strongly unfoldable cardinal $\kappa$ gives for every transitive set of size $\kappa$ a certain elementary embedding. If $M \in V$ is a transitive set of size $\kappa$ in the forcing extension, yet $M$ has size $\kappa^+$ in $V$, then there seems little reason that the strong unfoldability of $\kappa$ in $V$ provides the necessary embedding for $M$. Results from inner model theory confirm that this intuition is correct. For instance, if $\kappa$ is weakly compact and indestructible by some $\leq \kappa$-closed forcing that collapses $\kappa^+$, then Jensen’s Square Principle $\Box_{\kappa}$ fails, as was pointed out to me by Grigor Sargsyan. But a failure of $\Box_{\kappa}$ for a weakly compact cardinal $\kappa$ implies AD in $L(\mathbb{R})$, which has the strength of infinitely many Woodin cardinals (for a detailed discussion, see Section 3 in [HJ]). If we want to rely on hypotheses consistent with $V = L$ only, we must therefore focus on indestructibility by posets which preserve $\kappa^+$. It is thus natural to ask for instance the following:

**Question 1.** Given a strongly unfoldable cardinal $\kappa$, can we make it indestructible by all $\leq \kappa$-directed closed forcing that is $\kappa^+$-c.c.? Or indestructible by all $\leq \kappa$-directed closed forcing?

Already suggested in [She80] and studied intensively more recently (e.g., [RS], [Eis03]), the $\kappa$-proper posets have been defined for cardinals $\kappa$ with $\kappa^{< \kappa} = \kappa$ as a higher cardinal analogue of proper posets. Similar to the proper posets, which include all forcing notions that are either c.c.c. or countably closed, the $\kappa$-proper posets include all forcing notions that are either $\kappa^+$-c.c. or $\leq \kappa$-closed. Every $\kappa$-proper poset preserves $\kappa^+$. Moreover, every finite iteration of $\leq \kappa$-closed, $\kappa$-proper posets is itself $\leq \kappa$-closed and $\kappa$-proper (Corollary 17). Recall that proper posets can be characterized by the way in which the posets interact with countable elementary submodels $X$ of $H_\lambda$ for sufficiently large cardinals $\lambda$. From this characterization one obtains the definition of a $\kappa$-proper poset by generalizing “countable” to higher cardinals $\kappa$ (see Section 4). This interaction with elementary submodels $X \prec H_\lambda$ of size $\kappa$ is exactly what allowed me to handle posets of arbitrary size in the proof of the Main Theorem.
**Main Theorem.** Let $\kappa$ be strongly unfoldable. Then there is a set forcing extension in which the strong unfoldability of $\kappa$ is indestructible by $\kappa$-closed, $\kappa$-proper forcing of any size. This includes all $\kappa$-closed posets that are either $\kappa^+\text{-c.c.}$ or $\leq\kappa$-strategically closed.

It follows that the existence of a strongly unfoldable cardinal $\kappa$ indestructible by $\kappa$-closed, $\kappa$-proper forcing is equiconsistent over ZFC with the existence of a strongly unfoldable cardinal. Moreover, since strongly unfoldable cardinals are totally indescribable and thus weakly compact, the theorem provides a method of making these two classic cardinal notions indestructible by $\kappa$-closed, $\kappa$-proper forcing.

The Main Theorem thus answers Question 1 affirmatively. At the beginning of Section 5, I will illustrate why the class of $\kappa$-proper posets is a natural collection of posets to consider when one tries to make strongly unfoldable cardinals indestructible. Observe that a strongly unfoldable cardinal $\kappa$ is not always indestructible by $\kappa$-closed, $\kappa$-proper forcing: If $\kappa \in V$ is strongly unfoldable, then $\kappa$ is strongly unfoldable in $L$ (see [Vi98]), but forcing over $L$ with for instance the poset to add a Cohen subset of $\kappa$, destroys the weak compactness of $\kappa$ and thus its strong unfoldability (see Fact 25). Moreover, Hamkins showed in [Ham98] that any nontrivial small forcing over any ground model makes a weakly compact cardinal $\kappa$ similarly destructible (see Theorem 26). Of course, the strong unfoldability of $\kappa$ is then destroyed as well.

Note that we do not insist on $\kappa$-directed closure in the statement of the Main Theorem. We insist merely on $\kappa$-closure. This is a significant improvement since the usual indestructibility results for measurable or larger cardinals (such as [La78], [GS89] and [Ham00]) can never obtain indestructibility by all $\kappa$-closed, $\kappa$-proper forcing. In fact, no ineffable cardinal $\kappa$ can ever exhibit this degree of indestructibility (see Fact 29).

The proof of the Main Theorem employs the lottery preparation, a general tool invented by Hamkins [Ham00] to force indestructibility. The lottery preparation of a cardinal $\kappa$ is defined relative to a function $f : \kappa \to \kappa$ and works best if $f$ has what Hamkins calls the Menas property for $\kappa$. Since Woodin’s fast function forcing adds such a function, the lottery preparation is often assumed to be performed after some preliminary fast function forcing. For a strongly unfoldable cardinal $\kappa$ though, it turns out that we do not need to do any prior forcing; a function with the Menas property for $\kappa$ already exists (see Section 3).

The Main Theorem uses the lottery preparation of a strongly unfoldable cardinal $\kappa$ to make it indestructible by all $\kappa$-closed, $\kappa$-proper forcing. The strategy is to take the embedding characterization of strongly unfoldable cardinals and borrow lifting techniques of strong cardinals as well as those of supercompact cardinals in order to lift the ground model embeddings. I thereby follow Hamkins’ strategy, who was first to use these kind of lifting arguments in the strongly unfoldable cardinal context [Ham01]. But can we obtain more indestructibility than the Main Theorem identifies? We saw the need to focus on posets which do not collapse $\kappa^+$, which therefore suggests the following question:

**Question 2.** Can any given strongly unfoldable cardinal $\kappa$ be made indestructible by all $\kappa$-closed, $\kappa^+$-preserving forcing?
In a forthcoming paper [HJ], Joel Hamkins and I were able to answer Question 2 affirmatively, thereby providing as much indestructibility for strongly unfoldable cardinals as one could hope for. In Section 7, I will apply the Main Theorem simultaneously to all strongly unfoldable cardinals and obtain the following result.

**Main Theorem (Global Form).** If \( V \) satisfies ZFC, then there is a class forcing extension \( V[G] \) satisfying ZFC such that

1. every strongly unfoldable cardinal of \( V \) remains strongly unfoldable in \( V[G] \).
2. in \( V[G] \), every strongly unfoldable cardinal \( \kappa \) is indestructible by \( \lt \kappa \)-closed, \( \kappa \)-proper forcing, and
3. no new strongly unfoldable cardinals are created.

I review strongly unfoldable cardinals in Section 2 and show in Section 3 that there exists a class function \( F : \text{Ord} \to \text{Ord} \) which exhibits the Menas property for every strongly unfoldable cardinal simultaneously. Section 4 reviews \( \kappa \)-proper posets and in Section 5, I prove the Main Theorem using lifting techniques similar to those of supercompact cardinals. I mention some consequences and limitations of the Main Theorem in Section 6 and also provide several destructibility results. The global result, which makes all strongly unfoldable cardinals simultaneously indestructible, is proved in Section 7. In Section 8, I apply the Main Theorem to totally indescribable cardinals and to partially indescribable cardinals. To do so, I first prove a local analogue of the Main Theorem for a \( \theta \)-strongly unfoldable cardinal with \( \theta \) a successor ordinal. Section 9 addresses and solves the issue one faces when trying to prove the corresponding analogue for a \( \theta \)-strongly unfoldable cardinal with \( \theta \) a limit ordinal. Interestingly, this result provides a second and quite different proof of the Main Theorem. The case when \( \theta \) is a limit ordinal seems to require lifting techniques similar to those of strong cardinals. The fact that strongly unfoldable cardinals mimic both supercompact cardinals and strong cardinals allows for these two different proofs. At the end of Section 9, I state the local version of the Main Theorem in its strongest form.

§2. Strongly unfoldable cardinals. Following [DH06], I review several characterizations of strongly unfoldable cardinals. In [Vil98] Villaveces introduced strongly unfoldable cardinals. It turns out that they are exactly what Miyamoto calls the \( (H,\kappa)\)-reflecting cardinals in [Miy98]. Strongly unfoldable cardinals strengthen weakly compact cardinals similarly to how strong cardinals strengthen measurable cardinals. Their consistency strength is well below measurable cardinals, and if they exist, then they exist in the universe of constructible sets \( L \). It was discovered independently that strongly unfoldable cardinals also exhibit some of the characteristics of supercompact cardinals (see [Miy98] and [DH06]).

While measurable cardinals are characterized by elementary embeddings whose domain is all of \( V \), strongly unfoldable cardinals carry embeddings whose transitive domain mimics the universe \( V \), yet is a set of size \( \kappa \). Let \( ZFC^- \) denote the theory ZFC without the Power Set Axiom. For an inaccessible cardinal \( \kappa \), we call a transitive structure of size \( \kappa \) a \( \kappa \)-model if \( M \models ZFC^- \), the cardinal \( \kappa \in M \) and \( M^{<\kappa} \subseteq M \).

Fix any \( \kappa \)-model \( M \). Induction shows that \( V_\kappa \subseteq M \) and the Replacement Axiom in \( M \) implies that \( V_\kappa \models M \). Note that \( M \) satisfies enough of the ZFC-Axioms to
allow forcing over $M$. Moreover, for inaccessible $\kappa$, there are plenty of $\kappa$-models. For instance, if $\lambda > \kappa$ is any regular cardinal, we may use the Skolem–Łoewenheim method to build an elementary submodel $Y$ of size $\kappa$ with $Y \prec H_\lambda$ and $\kappa \in Y$ such that $X^{<\kappa} \subseteq Y$. The Mostowski collapse of $X$ is then a $\kappa$-model. This argument also shows that any given set $A \in H_\kappa$ can be placed into a $\kappa$-model.

**Definition 3.** [Vil98] Fix any ordinal $\theta$. A cardinal $\kappa$ is $\theta$-strongly unfoldable if $\kappa$ is inaccessible and for any $\kappa$-model $M$ there is an elementary embedding $j : M \to N$ with critical point $\kappa$ such that $\theta < j(\kappa)$ and $V_\theta \subseteq N$. A cardinal $\kappa$ is strongly unfoldable if $\kappa$ is $\theta$-strongly unfoldable for every ordinal $\theta$.

One can show that $\kappa$ is weakly compact if and only if $\kappa$ is $\kappa$-strongly unfoldable [Vil98]. Unlike Villaveces, who requires $\theta \leq j(\kappa)$, I insist in Definition 3 on strict inequality between $\theta$ and $j(\kappa)$. The two definitions are equivalent, as one can see by an argument given in the context of unfoldable cardinals in [Ham].

From now on, when I write $j : M \to N$, then I mean implicitly that $j$ is an elementary embedding with critical point $\kappa$ and both $M$ and $N$ are transitive sets. I will refer to embeddings $j : M \to N$ where $M$ is a $\kappa$-model, $\theta < j(\kappa)$ and $V_\theta \subseteq N$ as $\theta$-strong unfoldability embeddings for $\kappa$. In this paper, we will use the following previously known characterizations of $\theta$-strong unfoldability:

**Fact 4.** Let $\kappa$ be inaccessible and $\theta \geq \kappa$ any ordinal. The following are equivalent.

1. $\kappa$ is $\theta$-strongly unfoldable.
2. (Extender embedding) For every $\kappa$-model $M$ there is a $\theta$-strongly unfoldability embedding $j : M \to N$ such that $N = \{ j(g)(n) \mid g : V_\kappa \to M \text{ with } g \in M \text{ and } s \in S^{<\theta} \}$ where $S = V_\theta \cup \{ \emptyset \}$.
3. (Hauser embedding) For every $\kappa$-model $M$ there is a $\theta$-strongly unfoldability embedding $j : M \to N$ such that $|N| = 2^\lambda$ and $j \in N$ has size $\kappa$ in $N$.
4. For every $A \subseteq \kappa$ there is a $\kappa$-model $M$ and a $\theta$-strong unfoldability embedding $j : M \to N$ such that $A \in M$.
5. For every $A \subseteq \kappa$ there is a transitive set $M$ satisfying ZFC$^-$ of size $\kappa$ containing both $A$ and $\kappa$ as elements with a corresponding elementary embedding $j : M \to N$ such that $V_\theta \subseteq N$ and $\theta < j(\kappa)$.

**Proof.** The implication (1)$\Rightarrow$(2) is proved the same way how one produces canonical extender embeddings for $\theta$-strong cardinals. The proof that (2) implies (3) essentially follows from Hauser’s trick of his treatment of indescribable cardinals [Hau91], for a proof see [DH06]. For the other assertions, since every subset of $\kappa$ can be placed into a $\kappa$-model, it suffices to prove that (5) implies (1). Thus, suppose that $M'$ is any $\kappa$-model. Code it by a relation $A$ on $\kappa$ via the Mostowski collapse, and fix $M$ and $j : M \to N$ with $A \in M$ as provided by (5). Since $M \models \text{ZFC}^-$, it can decode $A$, and thus we have $M' \in M$. As $M'$ is closed under $<\kappa$-sequences and $\theta < j(\kappa)$, it follows by elementarity that $N$ thinks that $V_\theta \subseteq j(M')$. $N$ is correct and we see that $j \upharpoonright M' : M' \to j(M')$ is the desired $\theta$-strong unfoldability embedding.

The next fact is crucial for the results in this paper. It illustrates the way in which strongly unfoldable cardinals mimic supercompact cardinals and allows us thereby to use lifting arguments similar to those of supercompact cardinals when proving the Main Theorem.
FACT 5. [DH06] If \( \kappa \) is \(( \theta + 1 \)-strongly unfoldable, then for every \( \kappa \)-model \( M \) there is a \(( \theta + 1 \)-strong unfoldability embedding \( j : M \to N \) such that \( N^{\Delta_\theta} \subseteq N \) and \( |N| = \Delta_{\theta+1} \). If \( \kappa \) is \( \theta \)-strongly unfoldable and \( \theta \) is a limit ordinal, then for every \( \kappa \)-model \( M \) there is a \( \theta \)-strong unfoldability embedding such that \( N^{<\text{cof}(\theta)} \subseteq N \) and \( |N| = \Delta_\theta \).

If the GCH holds at \( \delta = \Delta_\theta \), we obtain in Fact 5 a \(( \theta + 1 \)-strong unfoldability embedding \( j : M \to N \) such that \( N \) has size \( \delta^+ \) and \( N^\delta \subseteq N \). Note that \( \delta < j(\kappa) \), as \( j(\kappa) \) is inaccessible in \( N \). This special case allows for diagonalization arguments, as in Section 5. Moreover, by forcing if necessary, we can simply assume that the GCH holds at \( \Delta_\theta \) for any given \(( \theta + 1 \)-strongly unfoldable cardinal \( \kappa \).

LEMMA 6. If \( \kappa \) is \(( \theta + 1 \)-strongly unfoldable for some \( \theta \geq \kappa \) and \( \mathbb{P} \) is any \( \leq \Delta_\theta \)-distributive poset, then \( \kappa \) remains \(( \theta + 1 \)-strongly unfoldable after forcing with \( \mathbb{P} \). In particular, we can force the GCH to hold at \( \Delta_\theta \) while preserving any \(( \theta + 1 \)-strongly unfoldable cardinal \( \kappa \).

PROOF. Fix any \( \leq \Delta_\theta \)-distributive poset \( \mathbb{P} \). Let \( G \subseteq \mathbb{P} \) be \( V \)-generic. Fix any \( \kappa \)-model \( M \in V[G] \). As \( \mathbb{P} \) is \( \leq \kappa \)-distributive, we see that \( M \in V \). We may thus fix in \( V \) an embedding \( j : M \to N \) with \( N_{\theta+1} \subseteq N \) and \( \theta < j(\kappa) \). Because the forcing is \( \leq \Delta_\theta \)-distributive, it follows that \( (V_{\theta+1})^V = (V_{\theta+1})^{V[G]} \), and \( j \) is hence the desired \(( \theta + 1 \)-strong unfoldability embedding in \( V[G] \).

We will use the results from [Ham03] in Section 6 to show that after nontrivial forcing of size less than \( \kappa \), a strongly unfoldable cardinal \( \kappa \) becomes highly destructible. All applications of the Main Theorem from [Ham03] need a cofinal elementary embedding whose target is highly closed. Lemma 7 shows how this can be achieved for most \( \theta \)-strongly unfoldable cardinals \( \kappa \). Note first that a map \( j : M \to N \) with \( j \in N \) and \( N \models \text{ZFC}^- \) can never be cofinal: As \( j \in N \) and hence \( M \in N \), we have that \( j''M \) is a set in \( N \) and therefore certainly not an unbounded class in \( N \). It follows that \( \theta \)-strong unfoldability embeddings \( j : M \to N \) of \( \kappa \) with \( N^\kappa \subseteq N \) can never be cofinal. For the same reason, Hauser embeddings as in assertion (3) of Fact 4 are not cofinal.

LEMMA 7. Let \( \kappa \) be a \( \theta \)-strongly unfoldable cardinal for some \( \theta \geq \kappa \). Suppose that \( \theta \) is either a successor ordinal or \( \text{cof}(\theta) \geq \kappa \). Then for every \( \kappa \)-model \( M \) there is a \( \theta \)-strong unfoldability embedding \( j : M \to N \) such that \( j \) is cofinal and \( N^{<\kappa} \subseteq N \).

PROOF. Fix any \( \kappa \)-model \( M \). Suppose \( \theta \geq \kappa \) is either a successor ordinal or \( \text{cof}(\theta) \geq \kappa \). By Fact 5, there is a \( \theta \)-strong unfoldability embedding \( j : M \to N \) such that \( N^{<\kappa} \subseteq N \). As seen above, there is no reason to think that \( j \) is cofinal. Yet, by restricting the target of \( j \) to \( N_0 = \bigcup j''M \), I claim that \( j : M \to N_0 \) is the desired \( \theta \)-strong unfoldability embedding. It is crucial that \( j : M \to N_0 \) remains an elementary embedding. This is shown by induction on the complexity of formulas. It is then easy to see that \( j : M \to N_0 \) is a cofinal \( \theta \)-strong unfoldability embedding. To see that \( N_0 \) is closed under \( <\kappa \)-sequences, note first that \( \text{Ord}^M \) is an ordinal with cofinality \( \kappa \), since \( M^{<\kappa} \subseteq M \). It follows that \( \text{Ord}^{N_0} \) has cofinality \( \kappa \). If \( s \in (N_0)^{<\kappa} \) is any sequence of less than \( \kappa \) many elements from \( N_0 \), then \( s \in N \) by the closure of \( N \) and rank(\( s \)) is bounded in \( \text{Ord}^{N_0} \). This shows that \( s \in N_0 \) as desired. \( \square \)
§3. A function with the Menas property for all strongly unfoldable cardinals. I show in Theorem 10 that there is a function $F : \text{Ord} \rightarrow \text{Ord}$ such that for every strongly unfoldable cardinal $\kappa$, the restriction $F \upharpoonright \kappa$ is what Hamkins calls a Menas function for $\kappa$. This will allow us to use Hamkins’ lottery preparation directly, without any preliminary forcing to add such a function.

For a $\theta$-strongly unfoldable cardinal $\kappa$, I follow [Ham00] and say that a function $f : \kappa \rightarrow \kappa$ has the ($\theta$-strong unfoldability) Menas property for $\kappa$ if for every $\kappa$-model $M$ with $f \in M$, there is a $\theta$-strong unfoldability embedding $j : M \rightarrow N$ such that $j(f)(\kappa) \geq \beth_\theta^\kappa$. Note that $\beth_\theta^\kappa \geq \beth_0$ and we have equality if $\theta$ is a limit ordinal (see for instance the proof of Lemma 8). I insist that $j(f)(\kappa) \geq \beth_\theta^\kappa$ since I want $N$ to see that $|V_\theta| \leq j(f)(\kappa)$. This will be crucial for the lifting arguments of Theorem 42 in Section 9. Arguments in [Ham01] show that given a $\theta$-strongly unfoldable cardinal $\kappa$, a function with the Menas property for $\kappa$ can be added by Woodin’s fast function forcing. But, as assertion (2) of Theorem 10 shows below, we do not have to force to have such a function. A canonical function $f$ with the Menas property for $\kappa$ already exists.

Observe that we may assume without loss of generality that an embedding $j$ witnessing the Menas property of $f$ for $\kappa$ is an extender embedding. In order to see this, simply follow the proof of assertion (2) of Fact 4 and use the embedding $j$ to obtain an extender embedding $j_0 : M \rightarrow N_0$ with $j_0(f)(\kappa) \geq \beth_0^\kappa$. In fact, when given a function $f$ with the Menas property for $\kappa$, we may assume without loss of generality that an embedding $j$ witnessing the Menas property of $f$ satisfies any of the equivalent characterizations of Fact 4 or Fact 5. This follows again from the corresponding proofs of the two facts.

As expected, we say for a strongly unfoldable cardinal $\kappa$ that $f : \kappa \rightarrow \kappa$ has the (strong unfoldability) Menas property for $\kappa$, if for every ordinal $\theta$, the function $f$ has the $\theta$-strong unfoldability Menas property for $\kappa$. Again, fast function forcing adds such a function. But, as assertion (1) of Theorem 10 shows, we do not have to force to have such a function, because it already exists.

We first need two lemmas. Let us say that $\kappa$ is $<\theta$-strongly unfoldable if $\kappa$ is $\alpha$-strongly unfoldable for every $\alpha < \theta$. Note that for $\theta \leq \kappa$, every $<\theta$-strongly unfoldable cardinal is in fact $\kappa$-strongly unfoldable and thus weakly compact.

**Lemma 8.** Let $\kappa$ be a $\theta$-strongly unfoldable cardinal for some ordinal $\theta > \kappa$. If $M$ is a $\kappa$-model and $j : M \rightarrow N$ is a $\theta$-strongly unfoldability embedding for $\kappa$, then $\kappa$ is $<\theta$-strongly unfoldable in $N$.

**Proof.** Fix any $\theta$-strong unfoldability embedding $j : M \rightarrow N$ for $\kappa$. We know by assertion (2) of Fact 4 that for ordinals $\alpha \geq \kappa$ the $\alpha$-strong unfoldability of $\kappa$ is characterized by the existence of extender embeddings $j$ of transitive size $\beth_\alpha$. As $\theta > \kappa$, it thus suffices to show that for every $\alpha$ with $\kappa \leq \alpha < \theta$ the model $N$ contains all these extender embeddings as elements. Fix thus any such $\alpha$. I first claim that $\beth_\alpha^N = \beth_\alpha$ and $H_{\beth_\alpha} \subseteq N$ for every $\xi < \theta$. As $M$ is a $\kappa$-model, we see by elementarity that $\beth_\alpha^N$ exists for every $\xi \leq j(\kappa)$. As $V_\theta \subseteq N$, it follows by induction that $\beth_\zeta^N = \beth_\zeta$ for each $\xi < \theta$. Thus, for each $\xi < \theta$, $P(\beth_\xi) \subseteq N$ (since for ordinals $\zeta \geq \omega^2$ the power set $P(\beth_\xi)$ corresponds in $N$ to $P(V_{\xi})$ and $P(V_{\xi}) \subseteq V_\delta \subseteq N$). But elements of $H_{\beth_\zeta}$ are coded via the Mostowski collapse by elements of $P(\beth_\zeta)$ and the claim
follows. Since \( \alpha < \theta \), we see that \( H_{\aleph_\alpha} \subseteq N \). This shows that \( N \) contains all the necessary extender embeddings.

Assertion (4) of Fact 4 allows us to switch between \( \kappa \)-models and subsets of \( \kappa \) as we desire, while assertion (5) frees us from insisting that the domain \( M \) of the embeddings has to be closed under \( \kappa \)-sequences, a requirement that need not be upwards absolute. It follows that, if \( N \subseteq V \) is a transitive class with \( P(\kappa) \cup V_\theta \subseteq N \) and if \( N \) thinks that \( \kappa \) is \( \theta \)-strongly unfoldable, then \( \kappa \) is indeed \( \theta \)-strongly unfoldable.

**Lemma 9.** Suppose that \( \kappa \) is \( \theta \)-strongly unfoldable. For every \( \kappa \)-model \( M \) there is a \( \theta \)-strong unfoldability embedding \( j : M \to N \) such that \( \kappa \) is not \( \theta \)-strongly unfoldable in \( N \).

**Proof.** Fix any \( \kappa \)-model \( M' \). Let \( A \subseteq \kappa \) code \( M' \) via the Mostowski collapse. Fix an elementary embedding \( j : M \to N \) as in characterization (5) of Fact 4 with \( A \in M \) and \( V_\theta \subseteq N \), such that \( N \) has least Levy rank. The set \( A = j(A) \cap \kappa \) is an element of \( N \). But, in \( N \), there cannot exist a \( \theta \)-strong unfoldability embedding \( j_0 : M_0 \to N_0 \) with \( A \in M_0 \); such an embedding \( j_0 \in N \) would by absoluteness really be an embedding as in characterization (5) of Fact 4 which would thus contradict our choice of \( j \), as \( N_0 \in N \). It follows that \( \kappa \) is not \( \theta \)-strongly unfoldable in \( N \). The restriction \( j \upharpoonright M' : M' \to j(M') \) is then the desired embedding.

Lemma 8 and Lemma 9 have the following consequence.

**Theorem 10.** There is a function \( F : \text{Ord} \to \text{Ord} \) such that

1. If \( \kappa \) is strongly unfoldable, then \( F'' \kappa \subseteq \kappa \) and the restriction \( F \upharpoonright \kappa \) has the Menas property for \( \kappa \). Moreover, every \( \kappa \)-model contains \( F \upharpoonright \kappa \) as an element.
2. If \( \kappa \) is \( \theta \)-strongly unfoldable for some ordinal \( \theta \geq \kappa \), then the restriction \( F \cap (\kappa \times \kappa) \) has the \( \theta \)-strong unfoldability Menas property for \( \kappa \). Moreover, every \( \kappa \)-model contains \( F \cap (\kappa \times \kappa) \) as an element.
3. The domain of \( F \) does not contain any strongly unfoldable cardinals.

**Proof.** Let \( F : \text{Ord} \to \text{Ord} \) be defined as follows: If \( \xi \) is a strongly unfoldable cardinal, then let \( F(\xi) = \varnothing \) be undefined; otherwise let \( F(\xi) = \varnothing \) where \( \eta \) is the least ordinal \( \alpha \geq \xi \) such that \( \xi \) is not \( \alpha \)-strongly unfoldable. Note that \( F(\xi) \geq \xi \) for all \( \xi \in \text{dom}(F) \). This will be used to prove assertion (2) in the case when \( \theta = \kappa \).

For assertion (1), fix any strongly unfoldable cardinal \( \kappa \). Let us first see that \( F'' \kappa \subseteq \kappa \). Suppose that \( \xi \prec \kappa \) is \( \kappa \)-strongly unfoldable. I claim that \( \xi \) is in fact strongly unfoldable and thus \( \xi \notin \text{dom}(F) \). To verify the claim, fix any ordinal \( \theta \geq \kappa \), any \( \kappa \)-model \( M \) and a corresponding \( \theta \)-strong unfoldability embedding \( j : M \to N \) for \( \kappa \). In particular, \( \text{crit}(j) = \kappa \). Since \( M \) sees that \( \xi \) is \( \kappa \)-strongly unfoldable and \( \theta \neq j(\kappa) \), it follows by elementarity that \( N \) thinks that \( j(\kappa) \) is \( \theta \)-strongly unfoldable. As \( j(\xi) = \xi \) and \( V_\theta \subseteq N \), we see that \( N \) is correct. The cardinal \( \xi \) is thus \( \theta \)-strongly unfoldable in \( V \). Since \( \theta \) was arbitrary, we verified the claim and thus \( F'' \kappa \subseteq \kappa \).

To see that every \( \kappa \)-model contains \( F \upharpoonright \kappa \) as an element, suppose that \( \xi \prec \kappa \) is \( \alpha \)-strongly unfoldable for some \( \alpha \prec \kappa \). Since this is witnessed by extender embeddings which are elements of \( V_\kappa \), the definition of \( F \upharpoonright \kappa \) is absolute for any \( \kappa \)-model. Consequently, every \( \kappa \)-model contains \( F \upharpoonright \kappa \) as an element. As desired.
To verify the Menas property of $F \upharpoonright \kappa$ in assertion (1), fix any $\kappa$-model $M$. Let $\theta$ be any ordinal that is strictly bigger than $\kappa$. By Lemmas 8 and 9 there is a $\theta$-strong unfoldability embedding $j : M \rightarrow N$ such that $\kappa$ is not $\theta$-strongly unfoldable in $N$, yet $\kappa$ is $<\theta$-strongly unfoldable in $N$. Since the definition of $F \upharpoonright \kappa$ is absolute for $M$ and $F \upharpoonright \kappa \in M$, it follows that $j(F \upharpoonright \kappa)(\kappa) = \beth_\kappa$. This verifies the Menas property of $F \upharpoonright \kappa$ for $\kappa$ and completes the proof of assertion (1).

For assertion (2), fix any $<\kappa$-strongly unfoldable cardinal $\kappa$ for some ordinal $\theta \geq \kappa$. Restricting the domain of $F$ now to only those $\zeta < \kappa$ which are not $<\kappa$-strongly unfoldable makes the definition of $F \cap (\kappa \times \kappa)$ absolute for $\kappa$-models. Consequently, every $\kappa$-model contains $F \cap (\kappa \times \kappa)$ as an element. The Menas property of $F \cap (\kappa \times \kappa)$ follows thus exactly as in assertion (1) as long as $\theta$ is strictly bigger than $\kappa$. But if $\theta = \kappa$, we cannot use Lemma 8. In this case, since we defined $F$ in such a way that $F(\zeta) \geq \zeta$ for all $\zeta \in \text{dom}(F)$, it follows from Lemma 9 directly that $F \cap (\kappa \times \kappa)$ has the Menas property for $\kappa$. This completes the proof of assertion (2). Assertion (3) is clear.

Observe that in assertion (2) of Theorem 10 we cannot avoid restricting $F \upharpoonright \kappa$ to $F \cap (\kappa \times \kappa)$: If $\kappa$ is not $<\kappa$-strongly unfoldable for some $\theta \geq \kappa$, then any $\zeta < \kappa$ which is $<\theta$-strongly unfoldable, but not strongly unfoldable, will have $F(\zeta) > \theta \geq \kappa$. This shows that $F''\kappa \not\subseteq \kappa$. Consequently, $F \upharpoonright \kappa$ does not technically have the Menas property for $\kappa$ even though $F \cap (\kappa \times \kappa)$ does.

§4. $\kappa$-proper forcing. We review $\kappa$-proper posets as defined in [SR] and [Eis03] and provide a few facts and lemmas about them. Since several arguments in this section are direct analogues of well known arguments for proper forcing, the reader may also compare the following material with any standard source on proper forcing (e.g., [She98], [Jec03]).

Suppose $(N, \in)$ is a transitive model of $\text{ZFC}^-$. Let $(X, \in)$ be an elementary substructure of $(N, \in)$, not necessarily transitive. Assume $P \in X$ is a poset and $G \subseteq P$ any filter on $P$. Let $X[G] = \{\tau_G \mid \tau$ is a $P$-name with $\tau \in X\}$. If $G$ is an $N$-generic filter, it is a well known fact that $X[G] \prec N[G]$.

The filter $G$ is $X$-generic on $P$ if for every dense set $D \in X$, we have $G \cap D \cap X \neq \emptyset$. In other words, an $X$-generic filter meets every dense set $D \in X$ in $X$. For transitive sets $X$ this condition coincides with the usual requirement for a filter to be $X$-generic. If $\pi : (X, \in) \rightarrow (M, \in)$ is the Mostowski collapse of $X$, then $G$ is $X$-generic on $P$ if and only if $\pi''G$ is an $M$-generic filter on the poset $\pi(P)$. It is a standard result that a $V$-generic filter $G \subseteq P$ is $X$-generic if and only if $X[G] \cap V = X$. A condition $p \in P$ is said to be $X$-generic (or $(X,P)$-generic) if every $V$-generic filter $G \subseteq P$ with $p \in G$ is $X$-generic.

Proper posets were introduced by Shelah as a common generalization of c.c.c. posets and countably closed posets. Recall Shelah’s characterization of proper posets that looks at the way in which the posets interact with elementary submodels of $H_\lambda$:

**Definition 11.** A poset $P$ is proper if for all regular $\lambda > 2^{\text{card}(P)}$ and for all countable $X \prec H_\lambda$ with $P \in X$ and for all $p \in P \cap X$ there exists an $X$-generic condition below $p$. 
Already suggested in [She80], one obtains the definition of a \( \kappa \)-proper poset by essentially just generalizing “countable” to higher cardinalities \( \kappa \). There is a subtle difference though: It can be shown that properness can be defined equivalently by weakening the quantification “for all countable \( X \prec H_\lambda \)…” to “for a closed unbounded set of countable \( X \prec H_\lambda \)…”. This other characterization of \( \kappa \)-properness shows that properness is a reasonably robust property, one that is for instance preserved by isomorphisms. In the case of \( \kappa \)-properness, I will prove this preservation directly in Fact 13.

**Definition 12** (Shelah, [RS]). Assume that \( \kappa \) is a cardinal with \( \kappa^{< \kappa} = \kappa \). A poset \( P \) is \( \kappa \)-proper if for all sufficiently large regular \( \lambda \) there is an \( x \in H_\lambda \) such that for all \( X \prec H_\lambda \) of size \( \kappa \) with \( X^{< \kappa} \subseteq X \) and \( \{ \kappa, P, x \} \in X \), there exists for every \( p \in P \cap X \) an \( X \)-generic condition below \( p \).

Definition 12 is a bit subtle, as for every sufficiently large regular cardinal \( \lambda \) we have to consider possibly very different witnessing parameters \( x \in H_\lambda \), and restrict ourselves to only those elementary substructures \( X \prec H_\lambda \) which contain \( x \) as an element. In assertion (1) of Fact 13 we will make essential use of this technicality when proving that \( \kappa \)-properness is preserved by isomorphisms. We will call any such parameter \( x \in H_\lambda \) as in Definition 12 a \( \lambda \)-witness for (the \( \kappa \)-properness of) \( P \).

Note that proper posets are simply \( \aleph_0 \)-proper posets.\(^1\)

There are a few different definitions of \( \kappa \)-properness in the literature. Our definition is exactly the same as the one presented in [RS] and [Ros]. Moreover, the definition of a \( \kappa \)-proper poset as in [Eis03] is equivalent to our definition. This follows from the fact that for an uncountable cardinal \( \kappa \) with \( \kappa^{< \kappa} = \kappa \), every elementary submodel \( X \prec H_\lambda \) of size \( \kappa \) with \( X^{< \kappa} \subseteq X \) has what Eisworth calls a filtration of \( X \): If \( X = \{ x_\alpha \mid \alpha < \kappa \} \) is such an elementary submodel, then it is easy to construct a filtration \( (X_\alpha : \alpha < \kappa) \) of \( X \) inductively; simply take unions at limit steps and choose an elementary submodel \( X_{\alpha+1} \prec X \) of size less than \( \kappa \) such that \( \{ x_\alpha, (X_\beta : \beta \leq \alpha) \} \cup X_\alpha \subseteq X_{\alpha+1} \) at successor steps.

Definition 12 differs slightly from [She80], where the substructures \( X \) are not required to be \( \prec \)-closed and generic conditions are only required for a closed unbounded set of elementary substructures. Definition 12 also differs from the notion of a \( \kappa \)-proper poset as defined in [HR01]. There, the authors generalize Definition 11 directly and hence omit the use of \( \lambda \)-witnesses. They also insist that \( P \) is \( \prec \)-closed in order for \( P \) to be considered \( \kappa \)-proper.

Fact 13 generalizes corresponding statements about proper posets. It shows that for a cardinal \( \kappa \) with \( \kappa^{< \kappa} = \kappa \), we have many \( \kappa \)-proper posets. It also shows that \( \kappa \)-proper posets preserve \( \kappa^+ \). For the definition of \( \leq \)-strategic closure, see the remarks before Fact 22.

**Fact 13.** Suppose that \( \kappa \) is a cardinal with \( \kappa^{< \kappa} = \kappa \), and \( P \) and \( Q \) are any posets. Then:

1. If \( P \) is \( \kappa \)-proper and \( Q \) is isomorphic to \( P \), then \( Q \) is \( \kappa \)-proper.
2. If \( i : P \to Q \) is a complete embedding and \( Q \) is \( \kappa \)-proper, then \( P \) is \( \kappa \)-proper.
3. If \( i : P \to Q \) is a dense embedding, then \( P \) is \( \kappa \)-proper if and only if \( Q \) is \( \kappa \)-proper.

\(^1\)This is not to be confused with the very different definition of an \( \alpha \)-proper poset for a countable ordinal \( \alpha \) (see for instance in [She98]), which we will not be concerned with.
(4) If $\mathbb{P}$ is any $\kappa^+$-c.c. poset, then $\mathbb{P}$ is $\kappa$-proper.
(5) If $\mathbb{P}$ is any $\leq \kappa$-closed poset, then $\mathbb{P}$ is $\kappa$-proper.
(6) If $\mathbb{P}$ is any $\leq \kappa$-strategically closed poset, then $\mathbb{P}$ is $\kappa$-proper.
(7) If $\mathbb{P}$ is any $\kappa$-proper poset, then $\mathbb{P}$ preserves $\kappa^+$.

Proof. This is a straightforward generalization of the corresponding proofs for proper forcing. To illustrate, I will prove assertion (1) only. Suppose that $i: \mathbb{P} \to \mathbb{Q}$ is an isomorphism between the posets $\mathbb{P}$ and $\mathbb{Q}$. Suppose that $\mathbb{P}$ is a $\kappa$-proper poset.
Then there is a cardinal $\lambda_2$ such that all regular $\lambda \geq \lambda_2$ are sufficiently large to witness the $\kappa$-properness of $\mathbb{P}$ as in Definition 12. Fix now any $\lambda > \text{trcl}\{\mathbb{P}, \mathbb{Q}, i, \lambda_2\}$ and some corresponding $\lambda$-witness $x_\mathbb{P} \in H_\lambda$ for the $\kappa$-properness of $\mathbb{P}$. To see that $\mathbb{Q}$ is $\kappa$-proper, it suffices to show that for all $X \in H_\lambda$ of size $\kappa$ with $X^{<\kappa} \subseteq X$ and $\{\mathbb{P}, \mathbb{Q}, i, x_\mathbb{P}\} \in X$, there exists for every $q \in \mathbb{Q} \cap X$ an $(X, \mathbb{Q})$-generic condition below $q$. Fix thus any such elementary substructure $X \in H_\lambda$ and a condition $q \in \mathbb{Q} \cap X$. Since $\{\mathbb{P}, \mathbb{Q}, i\} \subseteq X$, it follows that $i^{-1}(q) \in \mathbb{P} \cap X$. As $\lambda$ is sufficiently large, we know that there exists an $(X, \mathbb{P})$-generic condition $p_\mathbb{P}$ below $i^{-1}(q)$. Since $i$ is an isomorphism, it follows that $p_\mathbb{P}$ is the desired $(X, \mathbb{Q})$-generic condition below $q$. This shows that $\{\mathbb{P}, i, x_\mathbb{P}\}$ is a $\lambda$-witness for the $\kappa$-properness of $\mathbb{Q}$. As $\lambda$ was chosen arbitrarily above $\text{trcl}\{\mathbb{P}, \mathbb{Q}, i, \lambda_2\}$, we see that $\mathbb{Q}$ is $\kappa$-proper as desired for assertion (1).

The search for generic filters in this paper involves the following closure fact (Fact 14) and diagonalization criterion (Fact 18).

Fact 14 (Closure Fact). Suppose $N$ is a transitive model of ZFC$^-$ and $X \in N$ is an elementary substructure of any size. Suppose $\mathbb{P} \in X$ is a poset and $\delta$ is an ordinal such that $X^{<\delta} \subseteq X$ in $V$. Let $G$ be a filter on $\mathbb{P}$. Then:

1. If $G \subseteq V$ is $N$-generic, then $X[G]^{<\delta} \subseteq X[G]$ in $V$.
2. If $\mathbb{P}$ is $\delta$-distributive in $V$ and $G$ is $V$-generic on $\mathbb{P}$, then $X^{<\delta} \subseteq X$ in $V[G]$ and $X[G]^{<\delta} \subseteq X[G]$ in $V[G]$.
3. Suppose $\mathbb{P} \subseteq X$. If $\mathbb{P}$ is $\delta$-c.c. in $V$ and $G$ is $V$-generic on $\mathbb{P}$, then $X[G]^{<\delta} \subseteq X[G]$ in $V[G]$.

Proof. For transitive sets $X$ (let $X = N$), this result is well known and frequently used. Using $X[G] \in N[G]$ it is easy to verify assertions (1) and (2). To see assertion (3), we follow the usual proof for the transitive case closely. Fix the ordinal $\delta$, the structure $X$ with $X^{<\delta} \subseteq X$ and the poset $\mathbb{P} \subseteq X$ which is $\delta$-c.c. in $V$.
Let $G \subseteq \mathbb{P}$ be $V$-generic. Observe that every antichain $\mathcal{A} \in V$ of $\mathbb{P}$ is in fact an element of $X$. As usual, I denote the canonical $\mathbb{P}$-name for the generic filter by $\check{G}$.
I first claim that if $\tau \in V$ is a name such that $\check{\mathbb{P}} \Vdash "\tau \in X[G]"$, then we can find a name $\sigma \in X$ such that $\check{\mathbb{P}} \Vdash \tau = \sigma$. To see this, fix a name $\tau \in V$ as above. Working in $V$, we see that the set $D = \{p \in \mathbb{P} \mid \exists \sigma \in X \text{ such that } p \Vdash \sigma = \tau\}$ is dense in $\mathbb{P}$. Let $A \subseteq D$ be a maximal antichain in $V$ and choose for each $a \in A$ a witness $\sigma_a \in X$ such that $a \Vdash \sigma_a = \tau$. By our earlier observation, we know that $A \in X$ and thus $\langle \sigma_a : a \in A \rangle \in X$ also. By mixing these names in $X$, we obtain a single name $\sigma \in X$ such that $\check{\mathbb{P}} \Vdash \sigma = \tau$ hereby proving the claim. To verify that $X[G]$ is closed under $\delta$-sequences in $V[G]$, fix now any $s \in X[G]^\delta \cap V[G]$ for some $\beta < \delta$. We may assume that $s$ has a name $\check{s}$ in $V$ such that $\check{\mathbb{P}} \Vdash "\check{s} is a $\beta$-sequence of elements of $X[G]"$. For each $\alpha < \beta$, we
may fix by the claim a name \( \sigma_\alpha \in X \) such that \( \mathbb{I}_P \models \langle s(\alpha) = \sigma_\alpha \rangle \). In particular, 
\[ s(\alpha) = s_G(\alpha) = (\sigma_\alpha)_G. \]
The closure of \( X \) shows that \( \langle \sigma_\alpha : \alpha \prec \beta \rangle \in X \). As \( G \in X[G] \), it follows that \( s = \langle (\sigma_\alpha)_G : \alpha \prec \beta \rangle \in X \), as desired.

Note that assertion (3) of Fact 14 is false, if we omit the hypothesis \( P \subseteq X \). As a counterexample, suppose that \( \delta \) is an uncountable cardinal with \( \delta \prec \delta \). Let \( X \prec H_{\delta^+} \) have size \( \delta \) such that \( \delta \in X \) and \( \delta \prec \delta \subseteq X \in V \). Let \( \mathbb{P} = \text{Add}(\omega, \delta^+) \) be the poset which adds \( \delta^+ \) many Cohen reals. The poset \( \mathbb{P} \) is an element of \( H_{\delta^+} \), and since \( \mathbb{P} \) is definable there, it follows also that \( \mathbb{P} \in X \). Moreover, \( \mathbb{P} \) is certainly \( \delta \)-c.c. and preserves \( \delta^+ \). If \( G \subseteq \mathbb{P} \) is \( V \)-generic, it follows that we have at least \( \delta^+ \) many reals in \( V[G] \), yet \( X[G] \) has size \( \delta \) only. This shows that \( X[G] \not\subseteq X[G] \).

Fact 14 helps to establish some sufficient conditions for a finite iteration of \( \kappa \)-proper posets to be \( \kappa \)-proper.

**Lemma 15.** Suppose \( \mathbb{P} \) is a \( \kappa \)-distributive, \( \kappa \)-proper poset and \( \mathbb{Q} \) is a \( \mathbb{P} \)-name which necessarily yields a \( \kappa \)-proper poset. Then \( \mathbb{P} \star \mathbb{Q} \) is \( \kappa \)-proper.

**Proof.** Fix \( \mathbb{P} \) and \( \mathbb{Q} \) as in the statement of the lemma. There is a cardinal \( \lambda_\mathbb{P} \) such that all regular \( \lambda \geq \lambda_\mathbb{P} \) are sufficiently large to witness the \( \kappa \)-properness of \( \mathbb{P} \). Moreover, since \( \mathbb{P} \) is a set, we can find in \( V \) a cardinal \( \lambda_\mathbb{Q} \) such that \( \mathbb{I}_P \) forces that all regular \( \lambda \geq \lambda_\mathbb{Q} \) are sufficiently large to witness the \( \kappa \)-properness of \( \mathbb{Q} \). Without loss of generality, assume trcl(\( \mathbb{P} \)) \( < \lambda_\mathbb{P} \) and trcl(\( \mathbb{Q} \)) \( < \lambda_\mathbb{Q} \). To see that \( \mathbb{P} \star \mathbb{Q} \) is \( \kappa \)-proper, fix now any regular cardinal \( \lambda \geq \max(\lambda_\mathbb{P}, \lambda_\mathbb{Q}) \). As \( \lambda \geq \lambda_\mathbb{P} \), we may fix a \( \lambda \)-witness \( x_\mathbb{Q} \) for \( \mathbb{P} \). Since \( \mathbb{I}_P \) forces that there exists a \( \lambda \)-witness for \( \mathbb{Q} \) also, we may by mixing find a \( \mathbb{P} \)-name \( \check{x}_\mathbb{Q} \in V \) that is forced by \( \mathbb{I}_P \) to be a \( \lambda \)-witness for \( \mathbb{Q} \). In fact, we can find such a \( \mathbb{P} \)-name \( \check{x}_\mathbb{Q} \) with trcl(\( \check{x}_\mathbb{Q} \)) \( < \lambda \). We will show that \( \{x_\mathbb{P}, \check{x}_\mathbb{Q}\} \) serves as a \( \lambda \)-witness for the \( \kappa \)-properness of \( \mathbb{P} \star \mathbb{Q} \).

Fix thus any elementary submodel \( X \prec H_\kappa \) of size \( \kappa \) with \( X \prec \mathcal{K} \subseteq X \) such that \( \{\kappa, \mathbb{P} \star \mathbb{Q}, x_\mathbb{P}, \check{x}_\mathbb{Q}\} \in X \). Fix also any condition \( r_1 \in (\mathbb{P} \star \mathbb{Q}) \cap X \). We will find an \( (X, \mathbb{P} \star \mathbb{Q}) \)-generic condition \( r \in \mathbb{P} \star \mathbb{Q} \) below \( r_1 \). Let \( r_1 = \langle p_1, q_1 \rangle \) with \( p_1 \in \mathbb{P} \) and \( q_1 \in \text{dom}(\mathbb{Q}) \) and \( p_1 \models q_1 \in \mathbb{Q} \). Since \( \lambda \geq \lambda_\mathbb{P} \) and \( x_\mathbb{P} \in X \), there exists an \( (X, \mathbb{P}) \)-generic condition \( p_0 \in \mathbb{P} \) below \( p_1 \). Let \( \mathbb{G} \) be the canonical \( \mathbb{P} \)-name for the \( V \)-generic filter on \( \mathbb{P} \). Note that \( \mathbb{I}_P \) forces that \( \lambda \) is a sufficiently large regular cardinal, that \( X[G] \) is an elementary submodel of \( H_{\mathbb{I}_P[G]} \), and that \( \check{x}_\mathbb{Q} \in X[G] \) is a \( \lambda \)-witness for \( \mathbb{Q} \). Moreover, \( \mathbb{I}_P \) also forces that \( X[G] \) is closed under \( \kappa \)-sequences. This follows from assertion (2) of Fact 14 and the \( \kappa \)-distributivity of \( \mathbb{P} \). We thus see that \( p_1 \models \exists x \in \mathbb{Q} \) below \( q_1 \) which is \( (X[G], \mathbb{Q}) \)-generic”. Let \( p \preceq p_1 \) and \( \check{q} \in \text{dom}(\mathbb{Q}) \) such that \( p \models \check{q} \prec q_1 \) and \( \check{q} \in \mathbb{Q} \) is \( (X[G], \mathbb{Q}) \)-generic”. Then \( r = \langle p, \check{q} \rangle \) is an element of \( \mathbb{P} \star \mathbb{Q} \) below \( r_1 \).

I claim that \( r \in \mathbb{P} \star \mathbb{Q} \) is the desired \( (X, \mathbb{P} \star \mathbb{Q}) \)-generic condition below \( r_1 \). Clearly \( r \leq r_1 \). Thus, fix any \( V \)-generic filter \( G \ast H \subseteq \mathbb{P} \star \mathbb{Q} \) where \( G \in \mathbb{P} \) is \( V \)-generic and \( H \) is \( V[G] \)-generic on \( \mathbb{Q} = \check{Q}_G \) such that \( r \in G \ast H \). It follows that \( G \subseteq \mathbb{P} \) is \( X \)-generic since \( p \in G \) and thus \( X \cap \text{Ord} = X[G] \cap \text{Ord} \). Moreover, since \( \check{q}_G \in H \) it follows that \( H \subseteq \mathbb{Q} \) is \( X[G] \)-generic and thus \( X[G] \cap \text{Ord} = X[G][H] \cap \text{Ord} \). Thus \( X[G \ast H] \) has the same ordinals as \( X \), that is \( G \ast H \subseteq \mathbb{P} \star \mathbb{Q} \) is an \( X \)-generic filter. This proves the claim and hence that \( \{x_\mathbb{P}, \check{x}_\mathbb{Q}\} \) is a \( \lambda \)-witness for the \( \kappa \)-properness of \( \mathbb{P} \star \mathbb{Q} \). Since \( \lambda \geq \max(\lambda_\mathbb{P}, \lambda_\mathbb{Q}) \) was arbitrary, this concludes the proof of the fact. \( \dagger \)
The next lemma is crucial for the proof of the Main Theorem, where I precede a \( \kappa \)-proper forcing \( \mathbb{Q} \) with the lottery preparation \( \mathbb{P} \) of \( \kappa \).

**Lemma 16.** Assume that \( \kappa \) is a cardinal with \( \kappa^{<\kappa} = \kappa \). If \( \mathbb{P} \) is a \( \kappa \)-c.c. poset of size \( \kappa \) and \( \mathbb{Q} \) is a \( \mathbb{P} \)-name which necessarily yields a \( \kappa \)-proper poset, then \( \mathbb{P} \upharpoonright \mathbb{Q} \) is \( \kappa \)-proper.

**Proof.** Fix \( \mathbb{P} \) and \( \mathbb{Q} \) as in the statement of the lemma. Since \( \mathbb{P} \) has size \( \kappa \), and \( \kappa \)-properness is preserved by isomorphisms (Fact 13), we may assume without loss of generality that \( \mathbb{P} \subseteq \kappa \). The rest of the argument is identical to the proof of Lemma 15, except that we use now assertion (3) of Fact 14 instead of assertion (2). The hypotheses of assertion (3) hold since \( X^{<\kappa} \subseteq X \) implies that \( \kappa \subseteq X \).

**Corollary 17.** A finite iteration of \( \kappa \)-closed, \( \kappa \)-proper posets is itself \( \kappa \)-closed and \( \kappa \)-proper. A finite iteration of \( \kappa \)-distributive, \( \kappa \)-proper posets is itself \( \kappa \)-distributive and \( \kappa \)-proper.

**Proof.** Finite forcing iterations of \( \kappa \)-distributive posets are \( \kappa \)-distributive. Similarly, \( \kappa \)-closure is preserved by finite iterations. Apply Lemma 15 finitely often.

**Fact 18** (Diagonalization Criterion). Let \( \delta \) be an ordinal. Suppose \( (N, \in) \) is a transitive model of \( \text{ZFC}^- \). Let \( (X, \in) \) be an elementary substructure of \( (N, \in) \), not necessarily transitive. Assume \( \mathbb{P} \in X \) is a poset. If the following criteria are satisfied:

1. \( X \) has at most \( \delta \) many dense sets for \( \mathbb{P} \).
2. \( \mathbb{P} \) is \( \delta \)-closed in \( X \) and
3. \( X^{<\delta} \subseteq X \),

then for any \( p \in \mathbb{P} \cap X \) there is an \( X \)-generic filter \( G \subseteq \mathbb{P} \) with \( p \in G \).

**Proof.** The proof is similar to the method of building generic filters for countable transitive models of set theory. Indeed, using conditions (2) and (3) we can meet \( \delta \) many dense sets of \( X \) inside of \( X \). This descending chain of \( \delta \) many elements of \( X \) generates in \( V \) a filter \( G \subseteq \mathbb{P} \) that is \( X \)-generic.

**§5. The Main Theorem.** I will now prove the Main Theorem that makes a strongly unfoldable cardinal \( \kappa \) indestructible by \( \kappa \)-closed, \( \kappa \)-proper forcing. First, I will describe the basic strategy that one would like to use, illustrate some immediate problems and show how to overcome them. I will also review Hamkins’ lottery preparation [Ham00] briefly. Suppose \( \kappa \) is strongly unfoldable and we want to make \( \kappa \) indestructible by some nontrivial forcing \( \mathbb{Q} \). Let \( \theta \) be an ordinal with rank(\( \mathbb{Q} \)) < \( \theta \), and \( G \subseteq \mathbb{Q} \) a \( V \)-generic filter. To show that \( \kappa \) is \( \theta \)-strongly unfoldable in \( V[G] \), it is our goal (by assertion (5) of Fact 4) to place in \( V[G] \) any given \( A \in V[G] \) with \( A \subseteq \kappa \) into a transitive set \( M^* \) satisfying \( \text{ZFC}^- \) of size \( \kappa \) containing \( \kappa \) as an element with a corresponding embedding \( j^*: M^* \rightarrow N^* \) for which \((V_\theta \subseteq N^*)^{V[G]} \) and \( \theta < j^*(\kappa) \).

To illustrate the basic method, suppose first that \( \mathbb{Q} \) has size at most \( \kappa \), say \( \mathbb{Q} \in H_{\kappa^+} \). If \( A \in V[G] \) with \( A \subseteq \kappa \), then \( A \) has a nice \( \mathbb{Q} \)-name \( \dot{A} \in V \). Since \( \mathbb{Q} \in H_{\kappa^+} \), it follows that \( A \in H_{\kappa^+} \) also. In \( V \), we can thus place both \( A \) and \( \dot{A} \) into a \( \kappa \)-model \( M \). As \( \kappa \) is \( \theta \)-strongly unfoldable in \( V \), there exists in \( V \) a \( \theta \)-strong unfoldability embedding \( j: M \rightarrow N \). Note that \( “V_\theta \subseteq N^*” \) holds in \( V \), but fails in \( V[G] \). As \( \dot{A} \in M \), we can force with \( \dot{A} \) over \( M \) using the \( M \)-generic filter \( G \subseteq \mathbb{Q} \). If we manage to lift the embedding \( j \) to \( j^*: M[G] \rightarrow N[H] \) such that \( G \in N[H] \), then I claim that we have fulfilled our goal and \( j^* \) is the desired embedding. Clearly
$A = A_G \in M[G]$. To verify that \( V_\theta \subseteq N[H] \) holds in \( V[G] \), let us denote the rank initial segment \((V_\theta)^{V[G]}\) by \( V[G]_\theta \). It is a standard fact about forcing that for ordinals \( \alpha > \text{rank}(\mathbb{Q}) \) every \( x \in V[G]_\theta \) has a \( \mathbb{Q} \)-name \( \dot{x} \in V_\alpha \times V_\alpha \). By means of a suitable pairing function, a flat pairing function, which does not increase rank, we may assume that \( V_\alpha \times V_\alpha \subseteq V_\gamma \) for all infinite ordinals \( \alpha \) (see for instance [Ham]). It follows that \( V[G]_\lambda \subseteq V[G] \) for all \( \alpha > \text{rank}(\mathbb{Q}) \). Since \( \theta > \text{rank}(\mathbb{Q}) \), the filter \( G \in N[H] \), and \( V_\theta \subseteq N \), we see that \( V[G]_\theta \subseteq N[H] \). This verifies the claim. A necessary and sufficient condition for the embedding \( j \) to lift to \( j^* \), the lifting criterion, is that \( H \) is an \( N \)-generic filter on \( j(\mathbb{Q}) \) such that \( j^*G \subseteq H \). We will use Silver’s master condition argument to verify the lifting criterion when proving the Main Theorem.

Suppose now that the poset \( \mathbb{Q} \) has size bigger than \( \kappa \). The above strategy fails completely: We cannot place \( \mathbb{Q} \) into a \( \kappa \)-model \( M \) and thus we cannot force with \( \mathbb{Q} \) over \( M \). Of course, the \( \mathbb{Q} \)-name \( A \) for the subset of \( \kappa \) may also be too big to fit into \( M \). Yet, the next lemma provides a solution to the problem: If we succeed in putting \( \mathbb{Q} \) into an elementary substructure \( X \prec H_\lambda \) of size \( \kappa \) (where \( \lambda \) is some sufficiently large regular cardinal) such that the filter \( G \subseteq \mathbb{Q} \) is both \( X \)-generic and \( V \)-generic. Then we can follow the above strategy with the collapsed version of \( \mathbb{Q} \). An easy density argument given in the proof of the Main Theorem shows that \( \kappa \)-properness of \( \mathbb{Q} \) suffices to achieve this goal.

**Lemma 19.** Suppose \( N \) is a transitive model of ZFC\(^-\) and \( \mathbb{P} \) any poset. Suppose \( X \prec N \) is an elementary substructure of any size. \( \mathbb{P} \in X \) a poset and \( G \subseteq \mathbb{P} \) is a filter on \( \mathbb{P} \) that is both \( X \)- and \( N \)-generic. Let \( \pi : (X, \varepsilon) \to (M, \varepsilon) \) be the Mostowski collapse of \( X \) and let \( G_0 = \pi^*G \). Then:

1. \( G_0 \) is \( M \)-generic on \( \pi(\mathbb{P}) \) and \( \pi \) lifts to \( \pi_1 : X[G] \to M[G_0] \), which is the Mostowski collapse of \( X[G] \) in \( V[G] \).
2. Suppose \( \kappa \) is a cardinal with \( \kappa + 1 \subseteq X \). If \( A \in X \) is a \( \mathbb{P} \)-name which necessarily yields a subset of \( \kappa \), then \( A_G = \pi(\dot{A})_{G_0} \).

**Proof.** To verify assertion (1), recall that we mentioned earlier that \( X \)-genericity of \( G \) is equivalent to \( G_0 \) being \( M \)-generic on \( \pi(\mathbb{P}) \). Since every object in \( X[G] \) is the interpretation of a \( \mathbb{P} \)-name \( \tau \in X \) by the generic filter \( G \), we must let \( \pi_1(\tau_G) = \pi(\tau)_{G_0} \). I claim that \( \pi_1 \) is a well defined map. For, if \( \sigma, \tau \in X \) are \( \mathbb{P} \)-names with \( \sigma_G = \tau_G \), then consider the Boolean value \( b = [\sigma = \tau]^\mathbb{P} \). Since \( G \) is \( N \)-generic on \( \mathbb{P} \), it follows that \( b \in G \). Moreover \( b \in X \), as \( b \) is definable from \( \sigma, \tau \) and \( \mathbb{P} \). Since \( b \models_{\mathbb{P}} \sigma = \tau \) holds in \( N \) and hence in \( X \), it follows that \( M \) thinks that \( \pi(b) \models_{\pi(\mathbb{P})} \pi(\sigma) = \pi(\tau) \). Since \( G_0 = \pi^*(G \cap X) \) is \( M \)-generic and \( \pi(b) \in G_0 \), we see that \( \pi(\sigma)_{G_0} = \pi(\tau)_{G_0} \), which shows that \( \pi_1 \) is well defined. One checks similarly that \( \pi_1 \) preserves the membership relation, extends \( \pi \) and is a bijection. Since \( M[G_0] \) is transitive, \( \pi_1 \) must be the Mostowski collapse of \( X[G] \) in \( V[G] \).

To see assertion (2), fix any \( \alpha \in A_G \). Since \( A \) is a name which necessarily yields a subset of \( \kappa \), we have \( \alpha \in X \). Consider the Boolean value \( b = [\alpha \in A]^\mathbb{P} \). It follows as in (1) that \( b \in G \cap X \). Elementarity of \( \pi \) yields \( \alpha = \pi(\alpha) \in \pi(\dot{A})_{G_0} \). This establishes \( A_G \subseteq \pi(A)_{G_0} \). The converse inclusion is similar.

The Main Theorem uses the lottery preparation, a general tool developed by Hamkins [Ham00] to force indestructibility for various large cardinal notations. The lottery preparation of \( \kappa \) is defined relative to a function \( f : \kappa \to \kappa \). Usually, one
assumes that \( f \) has the Menas property for the particular large cardinal \( \kappa \) (see Section 3). The basic building block of the lottery preparation is the lottery sum \( \bigoplus A \) of a collection \( A \) of posets. Also commonly called side-by-side forcing, \( \bigoplus A \) is the poset \( \{ \langle Q, p \rangle : Q \in A \text{ and } p \in Q \} \cup \{ 1 \} \), ordered with \( 1 \) above everything and \( \langle Q, p \rangle \leq \langle Q', p' \rangle \) when \( Q = Q' \) and \( p \leq p' \). Because compatible conditions must have the same \( Q \), the forcing effectively holds a lottery among all the posets in \( A \). a lottery in which the generic filter selects a ‘winning’ poset \( Q \) and then forces with it. The lottery preparation \( \mathbb{P} \) of \( \kappa \) relative to \( f \) is an Easton support \( \kappa \)-iteration which at stage \( \gamma < \kappa \), if \( \gamma \in \text{dom}(f) \) and \( f^\gamma \gamma \subseteq \gamma \), forces with the lottery sum of all \( \langle \gamma, \delta \rangle \)-closed posets \( Q \in H_{f^\gamma(\delta)}^\gamma \) in \( V^\mathbb{P} \). (Note: Insisting on \( \langle \gamma, \delta \rangle \)-closure is slightly less general than developed in [Ham00], but sufficient for the purpose of this paper.) Generically, if \( f(\gamma) \) is large, then the stage \( \gamma \) forcing of \( \mathbb{P} \) selects from a wide variety of posets. so that if \( j : M \to N \) is an embedding such that both \( \mathbb{P} \) and \( f \) are elements of \( M \), then \( j(\mathbb{P}) \) selects from a wide variety of posets. It follows that the stage \( \kappa \) lottery of \( j(\mathbb{P}) \) typically includes a sufficiently rich collection of posets so that we can work below a condition \( z \in j(\mathbb{P}) \) that opts at stage \( \kappa \) for a particular desired forcing notion. For instance, suppose \( Q \) is any \( \langle \kappa, \alpha \rangle \)-closed poset in the forcing extension of \( V \) after forcing with \( \mathbb{P} \). Using the strong unfoldability of \( \kappa \) and the Menas property of \( f \), we can fix an ordinal \( \theta \), a \( \mathbb{P} \)-name \( \dot{Q} \in V_\theta \) for \( Q \) and a \( \theta \)-strong unfoldability embedding \( j : M \to N \) with \( f \in M \) and thus \( \mathbb{P} \in M \) such that \( j(f)(\kappa) \geq \| \mathcal{N} \|_\theta \). Recall that \( \| \mathcal{N} \|_\kappa \geq \| \mathcal{N} \|_{\| \mathcal{N} \|_\kappa} \). It follows that the stage \( \kappa \) lottery of \( j(\mathbb{P}) \) includes the poset \( Q \). By simply working below a condition \( z \in j(\mathbb{P}) \) that opts at stage \( \kappa \) for \( Q \), we see that \( j(\mathbb{P}) \upharpoonright z \) forces at stage \( \kappa \) with \( \mathbb{P} \) and thus factors as \( j(\mathbb{P}) \upharpoonright z = \mathbb{P} * \dot{Q} * \mathbb{P}_{\text{tail}} \). Moreover, since \( j(f)(\kappa) \geq \| \mathcal{N} \|_\theta \), the next nontrivial forcing after stage \( \kappa \) occurs after stage \( \| \mathcal{N} \|_\theta \). It follows that \( \mathbb{P}_{\text{tail}} \) is \( \| \mathcal{N} \|_\theta \)-closed in \( N \). This flexibility to make \( \mathbb{P}_{\text{tail}} \) highly closed is crucial for the tail forcing arguments.

**Main Theorem.** Let \( \kappa \) be strongly unfoldable. Then after the lottery preparation of \( \kappa \) relative to a function with the Menas property for \( \kappa \), the strong unfoldability of \( \kappa \) becomes indestructible by \( \kappa \)-closed, \( \kappa \)-proper forcing.

**Proof.** Let \( \kappa \) be strongly unfoldable. By assertion (1) of Theorem 10 we know that there is a function \( f : \kappa \to \kappa \) with the Menas property for the strongly unfoldable cardinal \( \kappa \). Let \( \mathbb{P} \) be the lottery preparation of \( \kappa \) relative to \( f \). We will show that after forcing with \( \mathbb{P} \) the strong unfoldability of \( \kappa \) becomes indestructible by \( \kappa \)-closed, \( \kappa \)-proper forcing. The poset \( \mathbb{Q} \) certainly preserves the inaccessibility of \( \kappa \) (see [Ham00]). Fix any \( \mathbb{P} \)-name \( \dot{Q} \) which necessarily yields a \( \kappa \)-closed, \( \kappa \)-proper poset. Since \( \dot{Q} \) is the name of a \( \kappa \)-distributive poset, it follows that \( \kappa \) is inaccessible after forcing with \( \mathbb{P} * \dot{Q} \). It remains to show that for every ordinal \( \theta \), the poset \( \mathbb{P} * \dot{Q} \) preserves the embedding property of the \( \theta \)-strongly unfoldable cardinal \( \kappa \). Note that \( \mathbb{P} \) is \( \kappa \)-proper, as it has size \( \kappa \) and \( \kappa^{< \kappa} = \kappa \) (Fact 13). Moreover, since \( \mathbb{P} \) is \( \kappa \)-c.c. and \( \mathbb{P} \) has size \( \kappa \), Lemma 16 shows that \( \mathbb{P} * \dot{Q} \) is \( \kappa \)-proper. In view of characterization (4) of Fact 4, fix any ordinal \( \theta \geq \kappa \) and any \( \mathbb{P} * \dot{Q} \)-name \( \dot{A} \) which necessarily yields a subset of \( \kappa \). We may assume that \( \theta \) is large enough so that \( \dot{Q} \) and \( \dot{A} \) are elements of \( V_{\theta+1} \). Consider the following subset \( D \) of \( \mathbb{P} * \dot{Q} \).

\[
D = \{ r \in \mathbb{P} * \dot{Q} : r \models " \dot{A} \text{ can be placed into a } \kappa \text{-model } M \text{ with an embedding } j : M \to N \text{ with } \theta < j(\kappa) \text{ and } V_\theta \subseteq N " \}.
\]
To prove the theorem, it suffices to show that $D$ is dense in $P \ast \dot{Q}$. To do so, consider any $r' \in P \ast \dot{Q}$. Note that $\text{tr}(\{\kappa, P, f, \dot{Q}, \dot{A}, \theta\}) \leq \Delta_\theta$. Let $\lambda > \Delta_\theta$ be a sufficiently large regular cardinal to witness the $\kappa$-properness of $P \ast \dot{Q}$ as in Definition 12, and let $x \in H_\lambda$ be a corresponding $\lambda$-witness for $P \ast \dot{Q}$. Since $\kappa < \lambda$, we may use the Skolem–Löwenheim method in $V$ to build an elementary submodel $X \prec H_\lambda$ of size $\kappa$ with $X^{<\kappa} \subseteq X$ such that $\{\kappa, r', P, f, \dot{Q}, \dot{A}, \theta, x\} \subseteq X$. Note that $V_{\kappa} \subseteq X$ by induction. As $\lambda$ is sufficiently large and $x \in X$, we may thus fix an $(X, P \ast \dot{Q})$-generic condition $r \in P \ast \dot{Q}$ such that $r \leq r'$. The rest of the proof will show that $r \in D$, and hence that $D$ is dense.

Let $G \ast g \subseteq P \ast \dot{Q}$ be any $V$-generic filter containing $r$ so that $G \subseteq P$ is a $V$-generic filter and $g \subseteq \dot{Q} = \dot{Q}_G$ is a $V[G]$-generic filter. Since $r \in G \ast g$ is an $(X, P \ast \dot{Q})$-generic condition, we see that $G \ast g$ is an $X$-generic filter on $P \ast \dot{Q}$. Let $\pi: \langle X, \in \rangle \rightarrow \langle M, \in \rangle$ be the Mostowski collapse of $X$. The construction of $X$ shows that $M$ is a $\kappa$-model. Since $\pi \restriction V_{\kappa} = \text{id}$, we see that $\pi$ also fixes $\kappa$, the poset $P$, and the Menas function $f$. Let $\pi(\dot{Q}) = \dot{Q}_0$ and $\pi(\dot{A}) = \dot{A}_0$. It follows in $M$ that $\dot{Q}_0$ is a $P$-name for a $\kappa$-closed poset and that $\dot{A}_0$ is a $P \ast \dot{Q}_0$-name for a subset of $\kappa$. Moreover, the image $G \ast g_0 = \pi''(G \ast g)$ is an $M$-generic filter on $P \ast \dot{Q}_0$ by Lemma 19. Note that the poset $P \ast \dot{Q}_0$ is isomorphic to $(P \ast \dot{Q}) \cap X$. The next diagram illustrates the situation.

Let $A = \dot{A}_{G \ast g}$ be the subset of $\kappa$ which we need to put into the domain of an elementary embedding $j^* \in V[G \ast g]$. We saw that $G \ast g \subseteq P \ast \dot{Q}$ is an $X$-generic filter, which implies that $X[G \ast g] \cap V = X$. It follows that $G$ is $X$-generic on $P$ and $g$ is $X[G]$-generic on $M$. Since $\kappa$ is a $\text{tr}(\{\kappa, P, f, \dot{Q}, \dot{A}, \theta\})$ holds, unfoldable in $V$. By Fact 5, a $(\theta + 1)$-strong unfoldability embedding $j: M \rightarrow N$ with $N^{\Delta_\theta} \subseteq N$ and $|N| = \Delta_{\theta + 1}$. Since $f$ has the Menas property for $\kappa$, we may assume that $f(\kappa) \geq \Delta_\theta$ and $\Delta_\theta < f(\kappa)$. Let $\dot{\kappa} = \Delta_\theta$. Since $V_{\theta + 1} \subseteq N$, we see that $\Delta_\theta = \Delta_\theta = \delta$. Elementarity
of $j$ and $\delta < j(\kappa)$ shows that $N \models "H_{\delta^+} \text{ exists}"$. It follows that $H^{j(\kappa)}_{\delta^+} = H_{\delta^+}$. Summarizing, we know that the $(\theta+1)$-strong unfoldability embedding $j: M \rightarrow N$ with $\delta < j(\kappa)$ and $|N| = 2^\theta$ satisfies $N^\delta \subseteq N$ and $j(f)(\kappa) \geq \delta$. As indicated in the remark after Fact 5, we would like $N$ to have size $\delta^+$ in order to allow for diagonalization arguments over $N$. For simplicity, let us for the moment assume that $2^\theta = \delta^+$ in $V$. I shall show at the end of this proof how to modify the arguments in the case if $2^\theta \neq \delta^+$.

The next diagram illustrates our strategy of lifting the elementary embedding $j: M \rightarrow N$ in $V[G * g]$ in two steps. While lifting $j$, we will also lift the isomorphism $\pi: X \rightarrow M$ twice.

![Diagram]

**STEP 1.** In $V[G * g]$, lift the embedding $j: M \rightarrow N$ to $j: M[G] \rightarrow N[j(G)]$.

Since $P \in M$, we can certainly embed the $V$-generic filter $G \subseteq P$ over $M$. By elementarity, $j(P)$ is the lottery preparation of $j(\kappa)$ relative to $j(f)$ computed in $N$. Since $Q \in V_{\theta+1} \subseteq H_{\delta^+}$ and $H_{\delta^+} \in N$, we see that $N[G] \models (Q \in H_{\delta^+}$ and $Q$ is $\kappa$-closed). As $j(f)(\kappa) \geq \delta$, we may hence opt for $Q$ at the stage $\kappa$ lottery of $j(P)$. Consequently, below the condition $p$ which opts for $Q$ at stage $\kappa$, the forcing $j(P)$ factors as $P \ast Q \ast P_{\text{tail}}$. To satisfy the lifting criterion we first need an $N$-generic filter $j(G)$ for the poset $P \ast Q \ast P_{\text{tail}}$. Since $G$ is $N$-generic on $P$ and $g$ is $N[G]$-generic on $Q$, it suffices to find in $V[G * g]$ a filter $G_{\text{tail}} \subseteq P_{\text{tail}}$ that is $N[G * g]$-generic. To do so, we verify the diagonalization criterion in $V[G * g]$ for $N[G * g]$ (see Fact 18): $N$ has size $\delta^+$ in $V$, and $N[G * g]$ has thus size $\delta^+$ in $V[G * g]$. Since $j(f)(\kappa) \geq \delta$, the poset $P_{\text{tail}}$ is $\leq \delta$-closed in $N[G * g]$ by the definition of $j(P)$. Lastly, since $N^\delta \subseteq N$ in $V$, and both $P$ and $Q$ are $\delta^+$-c.c., the closure fact shows that $N[G * g]^\delta \subseteq N[G * g]$ in $V[G * g]$. So, by diagonalization in $V[G * g]$, we may construct a filter $G_{\text{tail}} \subseteq P_{\text{tail}}$ which is $N[G * g]$-generic. If we let $j(G) = G * g * G_{\text{tail}}$, then $j''G \subseteq j(G)$, and we satisfy the lifting criterion, and $j$ lifts. Note that $N[j(G)]^\delta \subseteq N[j(G)]$. This concludes Step 1.

Since $P$ is $\kappa$-c.c. and $Q$ is $\kappa$-distributive, the closure fact (Fact 14) shows that $M[G]$ is still a $\kappa$-model in $V[G * g]$. Since our goal is to put $A = A_{G, \kappa}$ into a $\kappa$-model and the domain of the embedding so far is only $M[G]$, we need to lift the embedding again. But in general $Q$ will not be an element of $M[G]$ and thus we cannot force with $Q$ over $M[G]$. Instead, we shall force with the collapsed version of $Q$—namely $(Q_0)_G$—over $M[G]$. That this is at all possible is a crucial step of the argument.
We had $X \prec H_\kappa$ and the Mostowski collapse $\pi : (X, \in) \to (M, \in)$. We also saw that $G$ is both $X$-generic and $V'$-generic for $\mathbb{P}$. Recall that $\pi \upharpoonright V_\kappa = \text{id}$ and hence $\pi''G = G$. We can thus apply Lemma 19 and see that $\pi$ lifts to $\pi_1 : X[G] \to M[G]$ by $\pi_1(\sigma_G) = \pi(\sigma)|_G$, where $\pi_1$ is the Mostowski collapse of $X[G]$ in $V[G]$. Since $\dot{Q} \in X$, we have $\dot{Q} = \dot{Q}_G \in X[G]$. Let $\pi_1(\dot{Q}) = \dot{Q}_0$ be the collapsed version of $\dot{Q}$, which is an element of $M[G]$. We have $X[G] \prec H_\kappa[G]$ as usual. Recall that $\kappa$-properness of $\dot{Q}$ (and hence of $\mathbb{P} * \dot{Q} \dot{Q}$) enabled us to pick an $X$-generic condition $r \in \mathbb{P} * \dot{Q}$ below $r'$. This showed that $g$ is $X[G]$-generic for $\dot{Q}$, which now allows for the crucial application of Lemma 19 in $V[G]$ to $g \subseteq \dot{Q}$. It follows that $g_0 = \pi_1''g$ is $M[G]$-generic for $\dot{Q}_0$ and forcing with $g_0 \subseteq \dot{Q}_0$ over $M[G]$ thus makes sense. Moreover, $A = A_{G * g} = \pi(A)_{G * g_0} = (A_0)_{G * g_0}$ is an element of $M[G * g_0]$. Since $\dot{Q}_0$ is $\kappa$-closed in $M[G]$, Fact 14 shows that $M[G * g_0]$ is a $\kappa$-model in $V[G * g]$. Thus, to proceed showing that $r \in D$, let us lift the embedding $j$ once more:

**Step 2.** In $V[G * g]$, lift $j : M[G] \to N[j(G)]$ to $j : M[G * g_0] \to N[j(G) * j(g_0)]$.

Since $\dot{Q}_0$ is a $\kappa$-distributive poset in $M[G]$, I claim that $M[G * g_0]$ thinks that $g_0$ is a $\kappa$-closed subset of $\dot{Q}_0$. To see this, fix any $\beta < \kappa$ and any descending sequence $s = (s_\xi : \xi < \beta)$ in $M[G * g_0]$ of elements in $g_0$. Then $s \subseteq M[G]$ by the distributivity of $\dot{Q}_0$. Consider therefore in $M[G]$ for each $\xi < \beta$ the dense open set $D_\xi = \{q \in \dot{Q}_0 \mid q \leq s_\xi \text{ or } q \perp s_\xi\}$. The distributivity of $\dot{Q}_0$ shows that $D = \bigcap_{\xi < \beta} D_\xi$ is a dense open subset of $\dot{Q}_0$ in $M[G]$. As $g_0$ is an $M[G]$-generic filter on $\dot{Q}_0$, it meets the set $D$. Let $t \in D \cap g_0$. Since $t$ is compatible with $s_\xi$ and $t \in D_\xi$ for each $\xi < \beta$, it follows that $t$ lies below all $s_\xi$'s, which proves the claim.

As $M[G * g_0]$ is closed under $\kappa$-sequences in $V[G * g]$, the model $M[G * g_0]$ is correct, and we have that $V[G * g] \models (g_0$ is a $\kappa$-closed subset of $\dot{Q}_0)$. Moreover, $g_0$ is a directed set of size $\kappa$ in $V[G * g]$. Consequently, we see in $V[G * g]$ that there is a descending chain $(q_\xi : \xi < \kappa)$ of elements of $g_0$ such that every element of $g_0$ lies above $q_\xi$ for some $\xi < \kappa$. It follows that every element of $j''g_0$ lies above $j(q_\xi)$ for some $\xi < \kappa$. Consider therefore in $V[G * g]$ the descending chain $\bar{c} = (j(q_\xi) : \xi < \kappa)$ of elements of $j''g_0 \subseteq j(\dot{Q}_0)$. Since $N[j(G)]$ is closed under $\kappa$-sequences in $V[G * g]$ and $\bar{c} \in N[j(G)]^\mathbb{P}$, we have $\bar{c} \in N[j(G)]$. Moreover, $N[j(G)]$ thinks that $j(\dot{Q}_0)$ is $\kappa$-closed, and we can hence find a condition $q \in j(\dot{Q}_0)$ below all the $j(q_\xi)$'s. Since $q \leq x$ for all $x \in j''g_0$, we see that $q$ is the desired master condition. Finally, we verify the diagonalization criterion in $V[G * g]$ easily and build an $N[j(G)]$-generic filter $j(g_0) \subseteq j(\dot{Q}_0)$ containing $q$ as an element. Since $j''g_0 \subseteq j(g_0)$, we satisfy the lifting criterion, and $j$ lifts. This concludes Step 2.

To see that $r \in D$, recall that we used a flat pairing function which now implies that $V[G * g]|_{r + 1} \subseteq V_{r + 1}[G * g] \subseteq N[j(G) * j(g_0)]$. Since $A \in M[G * g_0]$ and $j$ is elementary in $V[G * g]$ with a $\kappa$-model as its domain, we have verified that $r \in D$. Since $r \leq r'$ and $r'$ was arbitrary in $\mathbb{P} * \dot{Q}$, we established that $D$ is dense in $\mathbb{P} * \dot{Q}$. This completes the proof in the easy case when $2^\omega = \delta^+$ holds in $V$.

Let me point out that we did not only show that $r \in D$, but we also established that $V[G * g]|_{r + 1} \subseteq N[j(G) * j(g_0)]$ in the easy case when $2^\omega = \delta^+$. This subtle point—$V_\theta$ versus $V_{\theta + 1}$—will be useful for the proof of Theorem 37.

But what to do if GCH fails at $\delta$? As before, we will show that the set $D \subseteq \mathbb{P} * \dot{Q}$ is dense. The arguments are essentially the same as before, until we diagonalize over
\\(N[G \ast g]\) in Step 1. It was crucial that \(|N[G \ast g]| = \delta^+\) in \(V[G \ast g]\). Instead, we now force \(2^\kappa = \delta^+\). In \(V[G \ast g]\), let \(R = \text{Add}(\delta^+, 1)\), the poset that adds a Cohen subset of \(\delta^+\) using conditions of size at most \(\delta\). If \(H \subseteq R\) is \(V[G \ast g]\)-generic, then \(V[G \ast g \ast H] = (2\kappa = \delta^+)\) and thus \(N[G \ast g]\) has size \(\delta^+\) in \(V[G \ast g \ast h]\) as needed. Rather than lifting in \(V[G \ast g]\), we now lift the embedding \(j : M \rightarrow N\) in \(V[G \ast g \ast H]\) to \(j : M[G \ast g_0] \rightarrow N[j(G) \ast j(g_0)]\). Since \(R\) is \(<\delta\)-distributive and \(|V[G \ast g]| = \delta\) in \(V[G \ast g]\), it then follows that \(j\) is a \((\theta + 1)\)-strong unfoldability embedding in \(V[G \ast g \ast H]\). Of course, \(j\) need not be an element of \(V[G \ast g]\).

Yet, the embedding \(j\) naturally induces in \(V[G \ast g \ast H]\) a \(\theta\)-strong unfoldability embedding \(j_0 : M[G \ast g_0] \rightarrow N_0\) such that \(N_0\) has size \(\delta\) only, by using seeds in \(V_0[G \ast g \ast H] \cup \{\theta\}\). The embedding \(j_0\) has thus hereditary size \(\delta\), which shows by the \(<\delta\)-distributivity of \(\mathbb{R}\) that \(j_0\) already exists in \(V[G \ast g]\).

Since \(A \in \text{dom}(j_0)\), we see that \(r \in D\) and thus \(D\) is dense in \(P \ast \dot{Q}\). This completes the proof of the Main Theorem.

\[\square\]

§6. Consequences, limitations, and destructibility. In this section, I discuss some corollaries and limitations of the Main Theorem. Moreover, in contrast to the indestructibility that I obtained in the Main Theorem, I also show that a strongly unfoldable cardinal \(\kappa\) becomes highly destructible after forcing with posets of size less than \(\kappa\). In particular, the cardinal \(\kappa\) may then get destroyed by some \(<\kappa\)-closed, \(\kappa\)-proper forcing.

Since strongly unfoldable cardinals strengthen weakly compact cardinals, we have the following corollary of the Main Theorem:

**Corollary 20.** If \(\kappa\) is strongly unfoldable, then there is a set forcing extension in which the weak compactness of \(\kappa\) is indestructible by \(<\kappa\)-closed, \(\kappa\)-proper forcing.

**Proof.** This is an immediate consequence of the Main Theorem.

**Corollary 20** suggests the following question, to which I would very much like to know the answer:

**Question 21.** Can every weakly compact cardinal \(\kappa\) be made indestructible by all \(<\kappa\)-closed, \(\kappa\)-proper forcing? Or indestructible at least by all \(<\kappa\)-closed, \(\kappa^+\text{-c.c.}\) forcing?

In the statement of the Main Theorem, the reader may wonder whether the \(<\kappa\)-closure assumption can be relaxed. Can we relax it to \(<\kappa\)-strategical closure? The answer is no.

Recall that a poset \(P\) is \(<\kappa\)-strategically closed if there is a strategy for the second player in the game of length \(\kappa\) allowing her to continue play, where the players alternate play to build a descending sequence \(\langle p_\xi \mid \xi < \kappa\rangle\) of conditions in \(P\), with the second player playing at limit stages.\(^2\) Every \(<\kappa\)-closed poset is of course \(<\kappa\)-strategically closed. A poset is \(P\) is \(<\kappa\)-strategically closed if the second player has a strategy for the game of length \(\kappa + 1\).

**Fact 22.** A weakly compact cardinal \(\kappa\) cannot be indestructible by all \(<\kappa\)-strategically closed forcing of size \(\kappa\). Specifically, the Jech–Prikry–Silver poset to

\(^2\)Some authors call this property \(\kappa\)-strategic closure, while defining \(<\kappa\)-strategic closure to mean the weaker property where the second player only needs a strategy for the games of length less than \(\kappa\).
add a \( \kappa \)-Suslin tree has size \( \kappa \) and is \(<\kappa\)-strategically closed, but it destroys the weak compactness of \( \kappa \).

**Proof.** Suppose that \( \kappa \) is weakly compact and \( \mathbb{P} \) is the Jech–Prikry–Silver poset for adding a \( \kappa \)-Suslin tree (see for instance p. 248 in [Kun99]). Since \( \mathbb{P} \) adds a \( \kappa \)-Suslin tree, it is clear that forcing with \( \mathbb{P} \) destroys the tree property of \( \kappa \) and thus its weak compactness. The poset \( \mathbb{P} \) has size \( \kappa \). Moreover, \( \mathbb{P} \) is \(<\kappa\)-strategically closed. This can be seen essentially by the same argument which is given in [Kun99] and which shows that \( \mathbb{P} \) is \(<\kappa\)-distributive. Note that \( \mathbb{P} \) is not \( \omega_1 \)-closed. \( \dashv \)

Since posets of size \( \kappa \) with \(<\kappa \mathopen{<} \kappa = \kappa \) are \( \kappa \)-proper, we know that any attempt to generalize the Main Theorem to \(<\kappa\)-strategically closed, \( \kappa \)-proper posets has to fail necessarily. In fact, such an attempt fails in the second lifting argument (Step 2 of the proof of the Main Theorem), where we built a master condition for \( \mathbb{Q} \). One way to avoid the second lift is by assuming that \( \mathbb{Q} \) does not add any new subsets to \( \kappa \).

**Theorem 23.** Let \( \kappa \) be strongly unfoldable. Then after the lottery preparation of \( \kappa \) relative to a function with the Menas property for \( \kappa \), the strong unfoldability of \( \kappa \) becomes indestructible by all set forcing which does not add subsets to \( \kappa \) and which is also \(<\delta\)-strategically closed for every \( \delta < \kappa \).

**Proof.** Let \( \kappa \) be strongly unfoldable. Recall that I defined the lottery preparation \( \mathbb{P} \) of \( \kappa \) relative to a function \( f : \kappa \to \kappa \) in such a way that the stage \( \gamma \) lottery in \( \mathbb{P} \), included all posets \( \mathbb{Q} \in H_{\gamma \mathopen{<} \gamma} \) that were \(<\gamma\)-closed. But the slightly more general form as presented in [Ham00] considers at stage \( \gamma \) the lottery sum of all those posets \( \mathbb{Q} \in H_{\gamma \mathopen{<} \gamma} \) which are \(<\delta\)-strategically closed for every \( \delta < \gamma \). Let \( \mathbb{P} \) be the lottery preparation of \( \kappa \) relative to a function \( f \) with the Menas property for \( \kappa \) in this more general form. Let \( G \subseteq \mathbb{P} \) be a \( V \)-generic filter. As shown in [Ham00], \( \mathbb{P} \) preserves the inaccessibility of \( \kappa \). Let \( \mathbb{Q} \) be any poset in \( V[G] \) which does not add subsets to \( \kappa \) and which is also \(<\delta\)-strategically closed for every \( \delta < \kappa \). Let \( g \subseteq \mathbb{Q} \) be a \( V[G] \)-generic filter on \( \mathbb{Q} \). If \( A \subseteq \kappa \) is a set in \( V[G \ast g] \), then \( A \in V[G] \) by the choice of \( \mathbb{Q} \). The set \( A \) has thus a \( \mathbb{P} \)-name \( A \in V \) whose transitive closure has size at most \( \kappa \). We can hence put the function \( f \), the poset \( \mathbb{P} \) and the \( \mathbb{P} \)-name \( A \) into a \( \kappa \)-model \( M \) in \( V \). Let \( \theta \) be sufficiently large so that \( \mathbb{Q} \in V[G]_\theta \) and let \( j : M \to N \) be a \((\theta + 1)\)-strong unfoldability embedding for \( \kappa \) as in Fact 5. We may assume without loss of generality that \( j(f)(\kappa) \geq \beth_\theta^N \). Since the poset \( \mathbb{Q} \) enters the stage \( \kappa \) lottery of \( j(\mathbb{P}) \), we may factor \( j(\mathbb{P}) \) below a condition which opts for \( \mathbb{Q} \) as \( j(\mathbb{P}) = \mathbb{P} \ast \mathbb{Q} \ast \mathbb{P}_{\mathrm{fail}} \). By following the arguments in Step 1 of the proof of the Main Theorem (and the corresponding modification if the GCH fails at \( \beth_\theta \)), we can show that \( j \) lifts in \( V[G \ast g] \) to a \( \theta \)-strong unfoldability embedding \( j : M[G] \to N[j(G)] \) where \( j(G) = G \ast g \ast G_{\mathrm{fail}} \). The point is that we avoid the need for a second lift, since \( A \in M[G] \) and \( V_\theta \subseteq N \) implies \( V[G \ast g][N] \subseteq N[G \ast g] \subseteq N[j(G)] \). Since \( A \subseteq \kappa \) was arbitrary in \( V[G \ast g] \), we showed that \( \kappa \) is \( \theta \)-strongly unfoldable in \( V[G \ast g] \). It follows that \( \kappa \) is strongly unfoldable in \( V[G \ast g] \). \( \dashv \)

The method of proving Theorem 23 provides a local analogue even if \( \kappa \) is only partially strongly unfoldable:

**Corollary 24.** Let \( \kappa \) be \( \theta \)-strongly unfoldable for some ordinal \( \theta \geq \kappa \). Then after the lottery preparation of \( \kappa \) relative to a function with the Menas property for \( \kappa \), the
\(\theta\)-strong unfoldability of \(\kappa\) becomes indestructible by forcing of rank less than \(\theta\) which does not add subsets to \(\kappa\) and which is also \(\delta\)-strategically closed for every \(\delta < \kappa\).

**Proof.** Sketch. If \(\theta\) is a successor ordinal such that the GCH holds at \(\aleph_\theta\), then it is easy to modify the proof of Theorem 23 to prove this corollary. Unfortunately, if the GCH fails at \(\aleph_\theta\) or if \(\theta\) is a limit ordinal, one needs a more refined method of proof. I will give the necessary arguments in Section 9 when proving Theorem 43. It is then straightforward to modify the proof of Theorem 23 to prove this corollary, even if the GCH fails at \(\aleph_\theta\) or if \(\theta\) is a limit ordinal.

While the Main Theorem establishes that indestructibility is possible, let me in contrast now show various possibilities for destructibility.

**Fact 25.** Suppose that \(\kappa\) is weakly compact in \(V = L\). Then any \(\kappa\)-distributive forcing which adds a subset to \(\kappa\) will destroy the weak compactness of \(\kappa\).

**Proof.** Let \(\kappa\) be weakly compact in \(L\). Let \(\mathbb{Q}\) be a \(\kappa\)-distributive poset and let \(\mathcal{G} \subseteq \mathbb{Q}\) be an \(L\)-generic filter. Let \(A \subseteq \kappa\) be a set that is in \(L[G]\) but not in \(L\). Assume towards contradiction that \(\kappa\) is weakly compact in \(L[G]\). Recall that this is equivalent to \(\kappa\) being \(\kappa\)-strongly unfoldable in \(L[G]\). The distributivity of \(\mathbb{Q}\) shows that \(\mathbb{Q}\) does not add elements of rank less than \(\kappa\) and so \(L[G]_\kappa = L_\kappa\). In \(L[G]\), fix a \(\kappa\)-model \(M\) with \(A \in M\) and a corresponding \(\kappa\)-strong unfoldability embedding \(j : M \to N\). Note that \(A = j(A) \cap \kappa\) is an element of \(N\). Since \(M\) sees that \(L[G]_\kappa = L_\kappa\), and \(A \in N\) has rank less than \(j(\kappa)\), it follows by elementarity that \(N\) thinks that \(A\) is constructible, a contradiction.

Since weakly compact cardinals are downwards absolute to \(L\) and there are \(\kappa\)-closed, \(\kappa\)-proper posets which add subsets to \(\kappa\) (the poset \(\text{Add}(\kappa, 1)\) to add a Cohen subset to \(\kappa\) for instance), we see that forcing over \(L\) with a \(\kappa\)-closed, \(\kappa\)-proper poset can possibly destroy the weak compactness, and hence the strong unfoldability, of a cardinal \(\kappa\).

Results from [Ham98] free us from forcing over \(L\) and show that any nontrivial small forcing over any ground model makes a weakly compact cardinal \(\kappa\) similarly destructible as in Fact 25. Recall that a poset \(\mathbb{P}\) is **small** relative to \(\kappa\), if \(\mathbb{P}\) has size less than \(\kappa\).

**Theorem 26.** [Ham98] After nontrivial small forcing, any \(\kappa\)-closed forcing which adds a subset to \(\kappa\) will destroy the weak compactness of \(\kappa\).

This result is the second Main Theorem of [Ham98]. The essential idea is that every elementary embedding witnessing the weak compactness in the forcing extension lifts a ground model embedding. This is then, similarly as in Fact 25, easily seen to be impossible. I will give the proof after Theorem 28 using methods from [Ham03].

Theorem 26 thus shows that the indestructibility obtained in the Main Theorem is necessarily destroyed by any nontrivial small forcing. But how about Theorem 23 and its Corollary 24? These results concern indestructibility of a strongly unfoldable cardinal \(\kappa\) by certain posets that do not add subsets to \(\kappa\). Such posets preserve of course the weak compactness of a cardinal \(\kappa\), yet there are various possibilities that they destroy the strong unfoldability of \(\kappa\):

**Fact 27.** Suppose that \(\kappa\) is strongly unfoldable in \(V = L\). Then any \(\kappa\)-distributive poset which adds a subset to \(\theta\) will destroy the \((\theta + 1)\)-strong unfoldability of \(\kappa\).
Proof. The proof is similar to the proof of Fact 25. If $A \subseteq \theta$ is a set that is in $L[g]$ but not in $L$, then $A \in V_{\theta+1}$. In general, $A$ will not fit into a given $\kappa$-model $M$, but if $j$: $M \to N$ is any $(\theta+1)$-strong unfoldability embedding, then $A \in N$. This suffices to obtain the same contradiction as in the proof of Fact 25.

In particular, Fact 27 shows that forcing over $L$ with any nontrivial $<\kappa$-distributive poset $P$ necessarily destroys the strong unfoldability of $\kappa$. Moreover, as before, we have again that any nontrivial small forcing over any ground model makes a strongly unfoldable cardinal $\kappa$ similarly destructible:

**Theorem 28.** After nontrivial small forcing, any $<\kappa$-closed forcing which adds a subset to $\theta$ will destroy the $(\theta+1)$ strong unfoldability of $\kappa$.

Proof. I use the general results on the approximation and cover properties from [Ham03]. The essential idea is that every elementary embedding witnessing the $(\theta+1)$-strong unfoldability in the forcing extension lifts a ground model embedding. This is then, similar as in Fact 27, seen to be impossible.

Suppose that $P$ is any nontrivial poset of size less than $\kappa$. We may assume that $\mathbb{P} \in V_{\kappa}$. Suppose that $\hat{Q}$ is a name for a poset which necessarily adds a subset to $\theta$ while being $<\kappa$-closed. Let $g \ast G \subseteq \mathbb{P} \ast \hat{Q}$ be a $V$-generic filter. Let $A \subseteq \theta$ be a set that is in $V[g \ast G]$ but not in $V[g]$. The poset $\hat{Q} = \hat{Q}_E$ is $<\kappa$-closed in $V[g]$, which implies $\theta \geq \kappa$. Assume towards a contradiction that $\kappa$ is $(\theta+1)$-strongly unfoldable in $V[g \ast G]$.

The cardinal $\kappa$ is inaccessible in $V[g \ast G]$. Let $\lambda > \kappa$ be any regular cardinal above $2^{|\mathbb{P} \ast \hat{Q}|}$. In $V[g \ast G]$, use the Skolem–Löwenheim method to build an elementary substructure $\hat{X} \prec H_\lambda[g \ast G]$ of size $\kappa$ in the language with a predicate for $V$, so that $(\hat{X}, \mathcal{E}) \prec (H_\lambda[g \ast G], \mathcal{E})$ where $X = \hat{X} \cap V$. We may assume that $\mathcal{E}^{<\kappa} \subseteq \mathcal{E}$ in $V[g \ast G]$ and that $\{\kappa, \mathbb{P}, \hat{Q}, g, G\} \subseteq \hat{X}$. Let $\pi$: $(\hat{X}, \mathcal{E}) \to (\hat{M}, \mathcal{E})$ be the Mostowski collapse of $\hat{X}$. It follows that $\hat{M}$ is a $\kappa$-model in $V[g \ast G]$. The isomorphism $\pi$ fixes both $\mathbb{P}$ and the filter $g \subseteq \mathbb{P}$. Let $Q_0 = \pi(\hat{Q})$ be the collapsed poset and let $G_0 = \pi(G)$. As $\pi$ is an isomorphism, it follows that $g$ is an $M$-generic filter on $\mathbb{P}$ and $G_0$ is an $M[g \ast G]$-generic filter on $Q_0$. Consequently, the model $\hat{M}$ decomposes as $\hat{M} = M[g \ast G_0]$. Note also that $Q_0$ is $<\kappa$-closed in $M[g \ast G_0]$.

Let $j$: $\hat{M} \to \hat{N}$ be a $(\theta+1)$-strong unfoldability embedding for $\kappa$ in $V[g \ast G]$. By Lemma 7, we may assume that $j$ is cofinal and $\mathcal{N}^{<\kappa} \subseteq \mathcal{N}$ in $V[g \ast G]$. Since $\hat{M} = M[g \ast G_0]$ and $j$ is cofinal, the model $\hat{N}$ decomposes by elementarity into $\hat{N} = N[g \ast j(G_0)]$, where $N = \bigcup j^{-1}M$. Since $A \subseteq \theta$ is an element of $V[g \ast G]_0 \subseteq \hat{N}$, it follows that $A \in N[g]$ by the $<j(\kappa)$-closure of the poset $j(Q_0)$ in $N[g]$. I now claim that this is impossible:

To see this claim, I use some results from [Ham03]. First observe that $P \ast \hat{Q}$ has the $\delta$ approximation and cover properties for $\delta = |P|^+$ by Lemma 13 in [Ham03]. Moreover, by the proof of Lemma 15 in [Ham03], it follows from our construction of $\hat{X}$ that $M = M \cap V$. Thus, the embedding $j$: $\hat{M} \to \hat{N}$ satisfies all hypotheses of the Main Theorem in [Ham03]. The elementary embedding $j | M$: $M \to N$ exists therefore in $V$. In particular, $N \subseteq V$. Combined with $A \in N[g]$ this implies $A \in V[g]$, contradicting the choice of $A$. This verifies the claim and hence completes the proof.

$\diamondsuit$
In particular, Theorem 28 shows that any nontrivial small forcing followed by any nontrivial $\prec\kappa$-closed forcing necessarily destroys the strong unfoldability of $\kappa$. The proof of Theorem 28 can be modified to also show Theorem 26:

**Proof of Theorem 26.** We follow the proof of Theorem 28 closely. Fix therefore the cardinal $\kappa$, the poset $\mathbb{P} \ast \dot{Q}$ and the filter $g * G$ as before. Assume towards contradiction that $\kappa$ is weakly compact in $V[g * G]$. Given a subset $A \subseteq \kappa$ which is in $V[g * G]$ but not in $V[g]$, we simply make sure that the elementary substructure $\check{X} \prec H_\kappa[g * G]$ contains $A$ as an element. It then follows that $\check{M}$, the Mostowski collapse of $\check{X}$, contains the set $A$ also. Let $j: \check{M} \rightarrow \check{N}$ be a $\kappa$-strong unfoldability embedding in $V[g * G]$ witnessing the weak compactness of $\kappa$. We can assume by Lemma 7 that $j$ is cofinal and $\check{N}^{<\kappa} \subseteq \check{N}$ in $V[g * G]$. Since $A = j(A) \cap \kappa$ is an element of $\check{N}$, we may assume as in the proof of Theorem 28 to reach a contradiction. It follows that $\kappa$ is not weakly compact in $V[g * G]$, which completes the proof of Theorem 26.

Lastly, I want to mention that the indestructibility which we obtained in the Main Theorem for a strongly unfoldable cardinal $\kappa$ can never be achieved for an ineffable cardinal. In particular, as measurable cardinals are ineffable, no measurable or strong or supercompact cardinal can ever exhibit such indestructibility. Fact 29 states and proves this well known result (see also the discussions in [KY] or [Ham]).

Recall that an uncountable regular cardinal $\kappa$ is **ineffable** if for every sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of sets with $A_\alpha \subseteq \alpha$ for each $\alpha < \kappa$, there exists a set $A \subseteq \kappa$ such that $\{\alpha \in \kappa | A \cap \alpha = A_\alpha\}$ is stationary. Every measurable cardinal is ineffable, and every ineffable cardinal is weakly compact. If $\kappa$ is ineffable, then $\kappa$ is ineffable in $L$. Furthermore, a tree $T$ of height $\kappa$ is a $\kappa$-Kurepa tree if $T$ has at least $\kappa^+$ many paths and every level of $T$ has size less than $\kappa$. A tree $T$ is **slim** if for each infinite ordinal $\alpha$, the $\alpha^+$ level of $T$ has size at most $|\alpha|$. It is clear that the complete binary tree $2^{<\kappa}$ for an inaccessible cardinal $\kappa$ is always a $\kappa$-Kurepa tree. In contrast, Jensen and Kunen showed that for an ineffable cardinal $\kappa$ there can never exist a slim $\kappa$-Kurepa tree (see [Dev84]). As usual, I denote the set of all paths trough a tree $T$ by $[T]$.

**Fact 29.** An ineffable cardinal $\kappa$ can never be indestructible by all $\prec\kappa$-closed, $\kappa$-proper forcing. Specifically, if $\kappa^{<\kappa} = \kappa$, then there is a $\prec\kappa$-closed, $\kappa^+$-c.c. poset of size $\kappa^+$ which adds a slim $\kappa$-Kurepa tree, thereby destroying the ineffability of $\kappa$.

**Proof.** I will follow the proof given in [Ham] closely. Suppose that $\kappa^{<\kappa} = \kappa$. Let $\mathbb{P}$ be the partial order with the top element $\mathbb{1}_\mathbb{P} = (\emptyset, 0)$ and conditions $(\langle t, f \rangle)$ below $\mathbb{1}_\mathbb{P}$, where $t \subseteq 2^{<\beta}$ is a slim tree of height $\beta$ for some $\beta < \kappa$, and $f : t : \kappa^+ \rightarrow [t]$ is a function with $1 \leq |\text{dom}(f')| \leq |\beta|$ (in particular, $[t] \neq \emptyset$). The conditions of $\mathbb{P}$ are ordered as follows: $(\langle t, f \rangle) \leq (\langle t', f' \rangle)$ iff either $t = t'$ and $f = f'$ or the tree $t$ is a proper end-extension of $t'$ and $f(\xi)$ extends $f'(\xi)$ for every $\xi \in \text{dom}(f')$. Note that the poset $\mathbb{P}$ is atomless, since we made sure that $[t] \neq \emptyset$ for every nontrivial condition $\langle t, f \rangle \in \mathbb{P}$. While $\mathbb{P}$ is not $\omega_1$-directed closed, it is easy to verify that $\mathbb{P}$ is $\prec\kappa$-closed and that $|\mathbb{P}| = \kappa^+$. A standard application of the $\Delta$-system lemma shows that $\mathbb{P}$ is $\kappa^+$-c.c. It follows that $\mathbb{P}$ preserves all cardinals. If $G \subseteq \mathbb{P}$ is a $V$-generic filter, then the union of the first coordinates of the elements of $G$ is seen to be a slim $\kappa$-tree $T$. Furthermore, it is clear that the pointwise union of the paths given by the second coordinates of elements in $G$ naturally produces a partial function $F : \kappa^+ \rightarrow [T]$. Density arguments show that $F$ is one-to-one and that $\text{dom}(F) = \kappa^+$. 
The tree $T$ is thus a slim $\kappa$-Kurepa tree, which shows via the Jensen–Kunen result that $\kappa$ is not ineffable in $V[G]$.

§7. Global indestructibility. We will obtain in Theorem 33 a class forcing extension $V[G]$ such that every strongly unfoldable cardinal of $V$ is preserved and every strongly unfoldable cardinal $\kappa$ in $V[G]$ is indestructible by $\lt\kappa$-closed, $\kappa$-proper forcing. There is a subtle issue in our goal of obtaining the model $V[G]$. We need to make sure that the process of making the strongly unfoldable cardinals of $V$ indestructible in $V[G]$ does not create any new strongly unfoldable cardinals. Such new large cardinals would have little reason to exhibit the desired indestructibility in $V[G]$. The question when forcing does not create any new large cardinals is the main focus of [Ham03]. The following fact suffices for our purposes in this section.

FACT 30. [Ham03] Suppose $\lambda$ is any cardinal. Suppose that $\mathbb{P}$ is nontrivial forcing, $|\mathbb{P}| \leq \lambda$, and $\dot{Q}$ is a $\mathbb{P}$-name for a necessarily $\leq \lambda$-closed poset. If $G \subseteq \mathbb{P} \ast \dot{Q}$ is a $V$-generic filter, then every strongly unfoldable cardinal above $\lambda$ in $V[G]$ is strongly unfoldable in $V$.

PROOF. The result follows directly from Lemma 13 and Corollary 20 in [Ham03].

Fact 30 suggests to precede the lottery preparation $\mathbb{P}$ by some small forcing. I will follow this idea in Theorem 33 when we add a Cohen real at stage $\omega$ of the lottery preparation.

Forcing with classes generalizes the usual set forcing, and the main ideas to do so are sketched for instance in [Kun99] or [Jec03]. A rigorous exposition of class forcing that includes class versions of the usual set forcing results can be found in [Rei06]. The main problem when forcing with a proper class $\mathbb{P}$ of forcing conditions is that there is little reason for the forcing extension to satisfy the axioms of set theory. Nevertheless, many commonly used class iterations are unproblematic. For instance, every progressively closed class forcing iteration [Rei06] preserves the ZFC axioms. In essence, a progressively closed class iteration $\mathbb{P}$ is an $\text{Ord}$-length iteration of complete subposets $\mathbb{P}_\alpha$, where $\mathbb{P} = \bigcup_{\alpha \in \text{Ord}} \mathbb{P}_\alpha$ with the additional requirement that the tail forcing becomes more and more closed as we progress through the iteration.

Theorem 33 relies on Lemma 32 below, which shows that the indestructibility of $\kappa$ that I obtained in the Main Theorem for a strongly unfoldable cardinal $\kappa$ is itself preserved by a wide variety of forcing notions, including partially ordered classes. The situation is easy for partially ordered sets:

**Lemma 31.** Suppose that $\kappa$ is any large cardinal which is indestructible by $\lt\kappa$-closed, $\kappa$-proper set forcing. Then any $\lt\kappa$-closed, $\kappa$-proper set forcing preserves $\kappa$ and its indestructibility.

**Proof.** This is clear by Corollary 17.
indestructibility of a strongly unfoldable cardinal $\kappa$ by different types of class forcing notions also:

**Lemma 32.** Suppose $\kappa$ is a strongly unfoldable cardinal which is indestructible by $<\kappa$-closed, $\kappa$-proper set forcing. Let $\mathbb{P}$ be a class forcing notion that preserves ZFC such that for unboundedly many cardinals $\delta$, the class $\mathbb{P}$ factors as $\mathbb{P} = \mathbb{P}_1 \ast \mathbb{P}_2$ where $\mathbb{P}_1$ is a $<\kappa$-closed, $\kappa$-proper poset and $\mathbb{P}_2$ is the name for a necessarily $<\delta$-closed class. Then:

1. Forcing with $\mathbb{P}$ preserves the strong unfoldability of $\kappa$.
2. Forcing with $\mathbb{P}$ preserves the indestructibility of $\kappa$.

**Proof.** Fix the cardinal $\kappa$ and the class $\mathbb{P}$ as in the theorem. Suppose that $G \subseteq \mathbb{P}$ is a $V$-generic class filter on the partially ordered class $\mathbb{P}$.

For assertion (1), fix any ordinal $\theta \geq \kappa$. Let $\delta > \beth^\mathbb{P}[G]$ be a cardinal such that $\mathbb{P}$ factors as $\mathbb{P} = \mathbb{P}_1 \ast \mathbb{P}_2$ where $\mathbb{P}_1$ is a $<\kappa$-closed, $\kappa$-proper poset and $\mathbb{P}_2$ is a $<\delta$-closed class forcing notion in $V^{\mathbb{P}_1}$. Both $\mathbb{P}_1$ and $\mathbb{P}_2$ are $<\kappa$-distributive, which shows that $\mathbb{P} = \mathbb{P}_1 \ast \mathbb{P}_2$ preserves the inaccessibility of $\kappa$. Moreover, $\mathbb{P}_1$ preserves the strong unfoldability of $\kappa$ by hypothesis. As $\mathbb{P}_2$ is $<\delta$-closed, it does not add new elements of rank less than $\theta$, which shows that every $\theta$-strong unfoldability embedding in $V^{\mathbb{P}_1}$ is in fact a $\theta$-strong unfoldability embedding in $V[G]$. The cardinal $\kappa$ is thus $\theta$-strongly unfoldable in $V[G]$. Since $\theta$ was arbitrary, we verified assertion (1).

For assertion (2), let $\mathbb{Q} \in V[G]$ be any $<\kappa$-closed, $\kappa$-proper poset in $V[G]$. Let $H \subseteq \mathbb{Q}$ be a $V[G]$-generic filter. As $\mathbb{Q}$ is $<\kappa$-distributive, it preserves the inaccessibility of $\kappa$. To verify that $\mathbb{Q}$ preserves the strong unfoldability of $\kappa$, fix any sufficiently large ordinal $\theta \geq \kappa$ such that $\mathbb{Q} \in V[G]_{\theta}$. Let $\delta > \beth^\mathbb{Q}[G]$ be a cardinal such that $\mathbb{P}$ factors as $\mathbb{P} = \mathbb{P}_1 \ast \mathbb{P}_2$ where $\mathbb{P}_1$ is a $<\kappa$-closed, $\kappa$-proper poset and $\mathbb{P}_2$ is a $<\delta$-closed class forcing notion in $V^{\mathbb{P}_1}$. If we factor the filter $G$ correspondingly as $G = G_1 \ast G_2$, it then follows that $\mathbb{Q}$ is an element of $V[G_1]$. We thus see that the two-step iteration $\mathbb{P}_2 \ast \mathbb{Q}$ is isomorphic to the product $\mathbb{P}_2 \times \mathbb{Q}$ in $V[G_1]$. As $\mathbb{P}_2$ is $\leq |\mathbb{Q}|$-closed, it follows that we may reverse the order of the factors of the product $\mathbb{P}_2 \times \mathbb{Q}$ and see that $G_2$ is in fact $V[G_1][H]$-generic on $\mathbb{P}_2$. Moreover, while $\mathbb{P}_2$ may not be $<\delta$-closed in $V[G_1][H]$, forcing with $\mathbb{Q}$ does preserve the $<\delta$-distributivity of $\mathbb{P}_2$. The class versions of these standard properties of products of posets are given in [Rei06]. We thus have that $V[G][H] = V[G_1][H][G_2]$. Lemma 31 shows that the poset $\mathbb{P}_1$ preserves the indestructibility of $\kappa$ by $<\kappa$-closed, $\kappa$-proper posets. Moreover, we may assume without loss of generality that $\theta$ was chosen large enough, so that $V[G_1][H] \subsetneq V[G_1]$ in $V[G]$ implies that $\mathbb{Q}$ is $<\kappa$-proper in $V[G_1]$. As $\mathbb{Q}$ is certainly $<\kappa$-closed in $V[G_1]$, it follows that $\kappa$ is strongly unfoldable in $V[G_1][H]$. But as in the proof of assertion (1), this means that $\kappa$ is $\theta$-strongly unfoldable in $V[G][H]$. Since $\theta$ was arbitrary, we verified the strong unfoldability of $\kappa$ in $V[G][H]$ and thus assertion (2).

In particular, the Main Theorem from Section 5 shows that any strongly unfoldable cardinal $\kappa$ becomes also indestructible by class forcing notions as described in Lemma 32. We can now combine the Main Theorem, Theorem 10 and Lemma 32 to obtain the following global indestructibility result.

**Theorem 33.** If $V$ satisfies ZFC, then there is a class forcing extension $V[G]$ satisfying ZFC such that
(1) every strongly unfoldable cardinal of \( V \) remains strongly unfoldable in \( V[G] \).
(2) in \( V[G] \), all strongly unfoldable cardinals \( \kappa \) are indestructible by \( \kappa \)-proper set forcing, and
(3) no new strongly unfoldable cardinals are created.

**Proof.** We will force with the class lottery preparation relative to a suitable Menas function in order to prove the theorem. Let \( F : \text{Ord} \to \text{Ord} \) be the class function as defined in Theorem 10. We saw that \( F \) has the Menas property for every strongly unfoldable cardinal \( \kappa \in \text{Ord} \) and that \( \text{dom}(F) \) does not contain any strongly unfoldable cardinals. Let \( \mathbb{P} \) be the class lottery preparation \( \mathbb{P} \) relative to the function \( F \). This is the direct limit of an \( \text{Ord} \)-stage forcing iteration with Easton support which at stage \( \gamma \), if \( \gamma \in \text{dom}(F) \) and \( F''\gamma \subseteq \gamma \), forces with the lottery sum of all \( \text{cf} \)-closed posets \( Q \in H_{F(\gamma)} \) in \( V^{\mathbb{P}} \). Since \( F(\omega) = \omega \), we know that the forcing to add a Cohen real, \( \text{Add}(\omega,1) \), enters the stage \( \omega \) lottery. Let \( p \in \mathbb{P} \) be a condition opting for \( \text{Add}(\omega,1) \). Let \( G \subseteq \mathbb{P} \) be a \( V \)-generic filter containing \( p \).

I first sketch that \( V[G] \models \text{ZFC} \). Note that for any \( \delta \) which is closed under \( F \), the lottery preparation \( \mathbb{P} \) factors as \( \mathbb{P}_\delta \times \mathbb{P}_{\text{tail}} \), where \( \mathbb{P}_\delta \) is the set lottery preparation using \( F \upharpoonright \delta \) and \( \mathbb{P}_{\text{tail}} \) is the class lottery preparation defined in \( V^{\mathbb{P}_\delta} \) using the restriction of \( F \) to ordinals greater than or equal to \( \delta \). It follows that the tail forcing \( \mathbb{P}_{\text{tail}} \) is necessarily \( \delta \)-closed. Since the class of closure points of \( F \) is unbounded in \( \text{Ord} \) and \( \mathbb{P} \) is the direct limit at \( \text{Ord} \) of all the previous stages, one can show that \( \mathbb{P} \) is progressively closed and consequently that \( V[G] \models \text{ZFC} \) (for details, see for instance [Rei06]).

We verify assertion (3) next. Observe that the class forcing iteration \( \mathbb{P} \) factors below \( p \) as \( \text{Add}(\omega,1) \ast \mathbb{P}_{(\omega,\infty)} \) where \( \mathbb{P}_{(\omega,\infty)} \) is a name for a necessarily \( \leq \omega \)-closed class iteration. Note that Fact 30 generalizes to class forcing iterations: The definition of the approximation and cover properties also applies to class forcing extensions and the proof of Lemma 13 in [Ham03] works well no matter whether the tail forcing is a set or a proper class. It follows that forcing with \( \mathbb{P} \) does not create any new strongly unfoldable cardinals, which proves assertion (3).

We can now verify assertions (1) and (2) simultaneously. For assertion (2), note first that it suffices to make all cardinals \( \kappa \) which are strongly unfoldable in \( V \) indestructible in \( V[G] \): By assertion (3), we do not have to worry about any other possibly strongly unfoldable cardinals in \( V[G] \). Fix thus any strongly unfoldable cardinal \( \kappa \in V \). Since \( F''\kappa \subseteq \kappa \), we know that \( \mathbb{P} \) factors at stage \( \kappa \). Moreover, as \( \kappa \notin \text{dom}(F) \), we have that the stage \( \kappa \) forcing of \( \mathbb{P} \) is trivial. This means that \( \mathbb{P} \) factors as \( \mathbb{P}_\kappa \ast \mathbb{P}_{\text{tail}} \) where \( \mathbb{P}_\kappa \) is the set lottery preparation of \( \kappa \) relative to \( F \upharpoonright \kappa \) and \( \mathbb{P}_{\text{tail}} \) is the \( \leq \kappa \)-closed class lottery preparation in \( V^{\mathbb{P}_\kappa} \) relative to the restriction of \( F \) to ordinals above \( \kappa \). Since \( F \upharpoonright \kappa \) has the Menas property for \( \kappa \), it follows from the Main Theorem that \( \mathbb{P}_\kappa \) preserves the strong unfoldability of \( \kappa \) and makes \( \kappa \) indestructible by forcing with \( \kappa \)-closed, \( \kappa \)-proper posets. Note that there are unboundedly many \( \delta \) such that \( \mathbb{P}_{\text{tail}} \) factors as \( \mathbb{P}_{\text{tail}} = \mathbb{P}_{(\kappa,\delta)} \ast \mathbb{P}_{[(\delta,\infty)} \) where \( \mathbb{P}_{(\kappa,\delta)} \) is a \( \leq \kappa \)-closed poset and \( \mathbb{P}_{[(\delta,\infty)} \) is a \( \delta \)-closed class forcing iteration. Lemma 32 thus shows that \( \mathbb{P}_{\text{tail}} \) preserves the strong unfoldability of \( \kappa \) and its indestructibility by \( \k \)-closed, \( \k \)-proper set forcing. This proves assertion (1) and (2). Planish 33 makes every strongly unfoldable cardinal \( \kappa \) indestructible by \( \k \)-closed, \( \k \)-proper set forcing. Combined with assertion (1) of Lemma 31 this also
implies indestructibility of every strongly unfoldable cardinal $\kappa$ by a wide variety of class forcing notions.

§8. An application to indescribable Cardinals. When Villaveces introduced strongly unfoldable cardinals in [Vil98], he observed that they also strengthen indescribable cardinals. Referring to the embedding characterization due to Hauser [Hau91], classically, for $m, n \in \mathbb{N}$, a $\Pi^m_n$-indescribable cardinal $\kappa$ is characterized by a certain reflection property of $V_\kappa$ for $\Pi^m_n$-formulas. A totally indescribable cardinal is then a cardinal $\kappa$ that is $\Pi^m_n$-indescribable for every $m, n \in \mathbb{N}$. Hauser’s embedding characterization introduced the idea of $\Sigma^m_n$-correctness at $\kappa$. Following [Ham], we say for $m \geq 1$ and $n \geq 0$ that a transitive set $N$ is $\Sigma^m_n$-correct at $\kappa$ if $(V_{\kappa+m})^N \prec \Sigma^m_n V_{\kappa+m}$ and $V_{\kappa+m-1} \subseteq N$. (Since $V_{\alpha}$ is $\Sigma^1_1$-definable in $V_{\alpha+1}$, it follows that the latter condition is redundant if $n > 0$.) Note that unlike [Hau91], we do not insist that $N$ is closed under $(\Sigma^m_{\kappa+m-2})$-sequences. As noted in [DH06], this closure requirement can easily be dropped, as the following fact shows.

**Fact 34.** [Hau91, DH06] Let $\kappa$ be an inaccessible cardinal. Let $m \geq 1$ and $n \geq 1$ be natural numbers. The following are equivalent:

1. $\kappa$ is a $\Pi^m_n$-indescribable cardinal.
2. For every $\kappa$-model $M$ there is an embedding $j : M \rightarrow N$ with critical point $\kappa$ such that $N$ is $\Sigma^m_{n-1}$ correct at $\kappa$.
3. For every $\kappa$-model $M$ there is an embedding $j : M \rightarrow N$ with critical point $\kappa$ such that $N$ is $\Sigma^m_{n-1}$ correct at $\kappa$, the model $N$ has size $\exists^\kappa_{\kappa+m-1}$ and $\exists^\kappa_{\kappa+m-2} \subseteq N$ (meaning $N^{\kappa} \subseteq N$ when $m = 1$).

**Proof.** Hauser [Hau91] provided the characterization of $\Pi^m_n$-indescribable cardinals as in assertion (3). Dzamonja and Hamkins observed in [DH06] that assertions (2) and (3) are equivalent. Their proof of Fact 5 of this paper can be modified in a straightforward manner to establish the equivalence between assertion (2) and (3).

Assertion (2) of Fact 34 implies for instance the classic result due to Hanf and Scott that a cardinal $\kappa$ is weakly compact if and only if $\kappa$ is $\Pi^1_1$-indescribable. More generally, we have the following.

**Corollary 35.** [DH06] Let $m \geq 0$ be a natural number. A cardinal $\kappa$ is $\Pi^m_{n+1}$-indescribable if and only if $\kappa$ is $(\kappa + m)$-strongly unfoldable. A cardinal $\kappa$ is totally indescribable if and only if $\kappa$ is $(\kappa + m)$-strongly unfoldable for every $m \in \mathbb{N}$.

**Proof.** This is immediate by characterization (2) of Fact 34.

The Main Theorem has thus the following corollary.

**Corollary 36.** If $\kappa$ is strongly unfoldable, then there is a forcing extension in which the total indescribability of $\kappa$ is indestructible by $\lt \kappa$-closed, $\kappa$-proper forcing.

**Proof.** This is immediate by the Main Theorem and Corollary 35.

If $\kappa$ is a $\Pi^m_n$-indescribable cardinal which is not strongly unfoldable, we need a local version of the Main Theorem for $\theta$-strong unfoldability in order to make $\kappa$ indestructible. This is fairly straightforward if $\theta$ is a successor ordinal. It is more difficult to obtain the local version if $\theta$ is a limit ordinal (see Section 9).

**Theorem 37.** Let $\kappa$ be a $(\theta + 1)$-strongly unfoldable cardinal for some ordinal $\theta \geq \kappa$. Assume that the GCH holds at $\exists^\kappa_{\theta}$. Then after the lottery preparation of $\kappa$
relative to a function with the Menas property for $\kappa$, the $(\theta + 1)$-strong unfoldability of $\kappa$ becomes indestructible by $\langle \kappa \rangle$-closed, $\kappa$-proper forcing of size at most $\beth_\theta$.

**Proof.** This is what we essentially argued when proving the Main Theorem. Let $\kappa$ be $(\theta + 1)$-strongly unfoldable for some ordinal $\theta \geq \kappa$. By assertion (4) of Theorem 10, we know that there is a function $f : \kappa \rightarrow \kappa$ with the Menas property for $\kappa$. Let $P$ be the lottery preparation of $\kappa$ relative to $f$ and let $G \subseteq P$ be $V$-generic. If $\bar{Q}$ is $\langle \kappa \rangle$-closed and $\kappa$-proper of size at most $\beth_\theta$ in $V[G]$, we may assume by assertion (1) of Fact 13 that $\bar{Q} \in V_{\theta+1}[G]$. Following the proof of the Main Theorem, we thus know that $\theta$ is large enough so that in $V$ we have names $\bar{Q}$ and $A$ that are elements of $V_{\theta+1}$. The GCH assumption brings us to the easy case when $2^\theta = \delta^+$ in $V$. We remind the reader that in this case we were able to prove not only the set $D$ to be dense in $P \ast \bar{Q}$, but actually the set

$$D^* = \{ r \in P \ast \bar{Q} : r \Vdash "A" \text{ can be placed into a } \kappa \text{-model } M \text{ with an embedding } j : M \rightarrow N \text{ with } \theta < j(\kappa) \text{ and } V_{\theta+1} \subseteq N" \}.$$  

to be dense in $P \ast \bar{Q}$. But density of $D^*$ proves that $\kappa$ remains $(\theta + 1)$-strongly unfoldable after forcing with $P \ast \bar{Q}$, as desired.

Note that the GCH assumption at $\beth_\theta$ in Theorem 37 is not too restrictive: If the assumption fails, we can simply force the GCH at $\beth_\theta$ first, which by Lemma 6 preserves the $(\theta + 1)$-strong unfoldability of $\kappa$. Moreover, we will see in Section 9 that the GCH assumption at $\beth_\theta$ is in fact an unnecessary hypothesis for the conclusion of Theorem 37 (see Theorem 42). Using this slightly stronger result from Section 9, we have the following indescribability result for indescribable cardinals:

**Corollary 38.** Let $\kappa$ be $\Pi_1^{\omega+1}$-indescribable for some natural number $m \geq 1$. Then, after the lottery preparation of $\kappa$ relative to a function with the Menas property for $\kappa$, the $\Pi_1^{\omega+1}$-indescribability of $\kappa$ becomes indestructible by $\langle \kappa \rangle$-closed, $\kappa$-proper forcing of size at most $\beth_{\kappa+m-1}$.

**Proof.** This is immediate from Theorem 42 and Corollary 35.

**Corollary 39.** Let $\kappa$ be totally indestructible. Then after the lottery preparation of $\kappa$ relative to a function with the Menas property for $\kappa$, the total indescribability of $\kappa$ is indestructible by $\langle \kappa \rangle$-closed, $\kappa$-proper forcing of size less than $\beth_{\kappa+\omega}$.

**Proof.** This is immediate from Corollaries 35 and 38.

**§9. The limit case.** In Section 8, I needed and found a local analogue of the Main Theorem for $(\theta + 1)$-strong unfoldability. But can we make a $\theta$-strongly unfoldable cardinal $\kappa$ for a limit ordinal $\theta$ also indestructible? The answer is yes (see Theorem 41). But it seems that the method of proof as in the Main Theorem does not quite work. If $j : M \rightarrow N$ with $V_\theta \subseteq N$ and $|N| = \beth_\theta$, then we can make $N$ closed under sequences of length less than $\text{cof}(\theta)$ (see Fact 5), but not under $\text{cof}(\theta)$-sequences (since $\beth_0^{\text{cof}(\theta)} > \beth_\theta$). In general, this closure is not sufficient in order to apply diagonalization for $N$ (unless $\theta = \beth_\theta = \text{cof}(\theta)$, that is unless $\theta$ is inaccessible). Instead, I will use a $\theta$-strong unfoldability embedding $j : M \rightarrow N$ where $N$ is generated by less than $j(\kappa)$ many seeds. Woodin [CW] was first to show how to use factor methods to lift such embeddings, and I will follow the
modification of Woodin’s technique by Gitik and Shelah [GS89] that they used to make strong cardinals indestructible by $\leq \kappa$-closed forcing.

There is a slight problem though that prevents us from directly using extender embeddings as in assertion (2) of Fact 4: When we built a master condition in Step 2 of the proof of the Main Theorem, we relied on $j: M \rightarrow N$ being an embedding with $N^\kappa \subseteq N$. The closure of $N$ under sequences of length $\kappa$ implied that $N[j(G)]^\kappa \subseteq N[j(G)]$ in $V[G \ast g]$, which in turn enabled us to find the desired master condition. But, if $j: M \rightarrow N$ is an extender $\theta$-strong unfoldability embedding such that $N = \{ j(g)(s) \mid g: V_\alpha \rightarrow M \text{ with } g \in M \text{ and } s \in S^{\kappa \text{cf} \theta} \}$ for $S = V_\theta \cup \{ \emptyset \}$, then $j''M$ is cofinal in $N$ and thus $j \notin N$. It follows that extender embeddings as in assertion (2) of Fact 4 can never satisfy $N^\kappa \subseteq N$. The solution to this problem is quite easy. We did not really need $N[j(G)]^\kappa \subseteq N[j(G)]$ in $V[G \ast g]$ in Step 2 of the Main Theorem in order to find the desired master condition. It would have been sufficient to have $j \upharpoonright g_0 \in N[j(G)]$. The next lemma shows how this can be achieved while still keeping the necessary properties of an extender embedding.

**Lemma 40.** Assume that $\kappa$ is a $\theta$-strongly unfoldable cardinal for an ordinal $\theta \geq \kappa$. Suppose that $M$ is a $\kappa$-model, $\alpha \in M$ an ordinal and $B \in M$ a set such that $M \models (V_\alpha \text{ exists and } B \in V_\alpha)$. Then there is a $\theta$-strong unfoldability embedding $j: M \rightarrow N$ such that $N = \{ j(h)(s) \mid h: D^{\kappa \text{cf} \theta} \rightarrow M \text{ with } h \in M \text{ and } s \in S^{\kappa \text{cf} \theta} \}$ where $D = V_\alpha^M$ and $S = V_\emptyset \cup \text{trcl}(B) \cup \{ \emptyset, j \upharpoonright B \}$. In particular, $j \upharpoonright B \in N$.

**Proof.** The proof uses arguments from seed theory. Assume ordinals $\kappa, \theta, \alpha$ and sets $M, B, D$ are given as above. Without loss of generality assume $\alpha \geq \kappa$. We may fix a Hauser embedding $j: M \rightarrow N$ with $\theta < j(\kappa)$ and $V_\emptyset \subseteq N$ such that $j \notin N$. Let $b = j \upharpoonright B$. Since $M \in N$ and $j \in N$, we have $b \in N$. As $B \in D = V_\alpha^M$, we see that both $B$ and $j(B)$ are elements of $j(D)$. Since $B \subseteq B \times j(B)$, it follows that $b \in j(D)$. This is easy to see for a limit ordinal $\alpha$. If $\alpha$ is a successor ordinal, one again needs to use a flat pairing function (see Section 5) instead of the usual von Neumann code of ordered pairs. Let $S = V_\emptyset \cup \text{trcl}(B) \cup \{ \emptyset, b \}$. Since $\alpha \geq \kappa$, it follows that $S \subseteq j(D)$. It hence makes sense to define the seed hull of $S$ via $j$ in $N$, namely the set $X_S = \{ j(h)(s) \mid h: D^{\kappa \text{cf} \theta} \rightarrow M \text{ with } h \in M \text{ and } s \in S^{\kappa \text{cf} \theta} \}$. As usual, $X_S \prec N$ is an elementary substructure (by the Tarski–Vaught test) such that $\text{ran}(j) \subseteq X_S$ and $S \subseteq X_S$. Let $\pi: X_S \rightarrow N_0$ be the Mostowski collapse of $X_S$. The composition map $j_0 = \pi \circ j$ is elementary with critical point $\kappa$ and $\theta < j_0(\kappa)$. With $S_0 = \pi''S$ it follows by elementarity of $\pi$ that $j_0: M \rightarrow N_0$ is an embedding with $N_0 = \{ j_0(f)(t) \mid f: D^{\kappa \text{cf} \theta} \rightarrow M \text{ with } f \in M \text{ and } t \in S_0^{\kappa \text{cf} \theta} \}$. Since $V_\emptyset \cup \text{trcl}(B)$ is a transitive subset of $X_S$, we see that $\pi \upharpoonright V_\emptyset = \text{id}$ and $\pi \upharpoonright \text{trcl}(B) = \text{id}$. This means that $\pi$ fixes each element of $S$ except possibly $b$. Let $b_0 = \pi(b)$. Moreover, since $B \subseteq X_S$ and $\pi \upharpoonright B = \text{id}$, we have $b_0 = j_0 \upharpoonright B$ as desired. It follows that $j_0: M \rightarrow N_0$ is the desired embedding.

As usual, one checks easily that the embedding characterization of Lemma 40 preserves the Menas property of any function $f: \kappa \rightarrow \kappa$ that has the Menas property for $\kappa$.

**Theorem 41.** Let $\kappa$ be $\theta$-strongly unfoldable for a limit ordinal $\theta \geq \kappa$. Then after the lottery preparation of $\kappa$ relative to a function with the Menas property for $\kappa$, the
θ-strong unfoldability of κ becomes indestructible by <κ-closed, κ-proper forcing of size less than Δ_θ.

**Proof.** We will follow the proof of the Main Theorem closely. Let κ be θ-strongly unfoldable for some limit ordinal θ ≥ κ. By assertion (2) of Theorem 10, we know that there is a function f : κ → κ with the Menas property for κ. Let P be the lottery preparation of κ relative to f. The poset P certainly preserves the inaccessibility of κ. Fix any P-name Q which necessarily yields a <κ-closed, κ-proper poset of size less than Δ_θ. We may assume without loss of generality that Q is the name of a poset of rank less than θ, and consequently we may assume that Q ∈ V_θ. In view of characterization (4) of Fact 4, fix any P∗Q-name A which necessarily yields a subset of κ. As both P and Q are elements of V_θ and θ ≥ κ, we may assume that A ∈ V_θ also. As before we shall prove that the set

D = {r ∈ P∗Q: r ⪰ “A can be placed into a κ-model M with
an embedding j : M → N with θ < j(κ) and V_θ ⊆ N”}.

is dense in P∗Q. We fix any r′ ∈ P∗Q and let λ > Δ_θ be a sufficiently large regular cardinal witnessing the κ-properness of P. Again, we let x ∈ H_λ be a λ-witness for the κ-properness of P∗Q. We use the Skolem–Löwenheim method in V to build X < H_λ of size κ with X^κ ⊆ X such that {κ, r′, P, f, Q, A, θ, x} ⊆ X. As λ is sufficiently large and x ∈ X, we can thus fix an (P∗Q)-generic condition r ∈ P∗Q such that r ≤ r'. The rest of the proof will again show that r ∈ D, and hence that D is dense.

Let G ∩ g ⊆ P∗Q be any V-generic filter containing r as an element so that G ⊆ P is a V-generic filter and g ⊆ Q = Q_0 is a V[G]-generic filter. Let A = A_{G,λ} be the subset that has to be put into the domain of a θ-strong unfoldability embedding j ∈ V[G ∩ g]. As before, G is X-generic on P and g is X[G]-generic for Q. Again, let π : (X, ∈) → (M, ∈) be the Mostowski collapse. M is a κ-model in V. Let π(Q) = Q_0 and π(A) = A_0. Let θ_0 denote the ordinal π(θ). We want to use Lemma 40 for the κ-model M and the set Q_0 ⊆ M. Since H_λ sees that Q ∈ V_θ, it follows that M ⊨ (V_{θ_0} exists and Q_0 ∈ V_{θ_0}). Moreover, since θ_0 ≤ θ and Q_0 ∈ V_θ, we have trcl(Q_0) ⊆ V_θ. Since κ is θ-strongly unfoldable in V, Lemma 40 now provides a θ-strong unfoldability embedding j : M → N such that N = {j(h)(s) | h : D^{<κ} → M with h ∈ M and s ∈ S^{<κ}} where S = V_θ ∪ {θ, j | Q_0} and D = (V_{θ_0})^M. The purpose of putting j | Q_0 into S and hence N is clear: it will allow us to build a master condition later on. Note that the target model N is generated by |S| = Δ_θ many elements, and Δ_θ is less than j(κ). Let δ = Δ_θ^N. Since f has the Menas property for κ, we may assume that j(f)(κ) ≥ δ and δ < j(κ). Note that S ⊆ N and thus S ∈ N has size δ in N. As θ is a limit ordinal and V_θ ⊆ N, we have that Δ_θ^N = Δ_θ. Let b = j | Q_0.

As in the proof of the Main Theorem, our strategy is to lift the embedding j in V[G ∩ g] in two steps.

**Step 1.** In V[G ∩ g], lift j : M → N to j : M[G] → N[j(G)].

We force with G ⊆ P over M. As before we may opt for Q at the stage κ lottery of j(P). Thus j(P) factors as P ∩ Q ∩ P_{tail}. Since G ∩ g is V-generic and hence N-generic on P ∗ Q, it suffices to find in V[G ∩ g] a filter G_{tail} ⊆ P_{tail} which is
$N[G*g]$-generic. The key to solving this problem is the fact that $j: M \to N$ is an embedding where $N$ is generated by less than $j(\kappa)$ many seeds $s \in S^{<\omega}$. We shall use the diagonalization criterion (Fact 18) in $V[G*g]$ not for $N[G*g]$, but for a suitable elementary substructure $Y[G*g] \prec N[G*g]$. The structure $Y[G*g]$ will have size $\kappa$ and will be closed under $\kappa$-sequences, which in turn will allow for diagonalization over $Y[G*g]$. Since $\delta \leq j(f)(\kappa)$, the next nontrivial stage of forcing in $P_{\text{tail}}$ is beyond $\delta$, and so $P_{\text{tail}}$ is $\leq \delta$-closed in $N[G*g]$. Observe that $a = \pi \upharpoonright \mathbb{Q}$ is an element of $V_\theta$ (since $\theta$ is a limit ordinal) and thus $a \in N$. Recall that $b = j \upharpoonright \mathbb{Q}_0 \subseteq N$ by construction of $j$.

Let $Y = \{j(h)(\kappa, \theta, a, b) \mid h: D^4 \to M$ with $h \in M\}$. As usual, $\langle \kappa, \theta, a, b \rangle \cup \text{ran}(j) \subseteq Y$ and $Y \prec N$. Since $P_{\text{tail}}$ is definable from $j(P)$ and $\kappa$, it follows that $P_{\text{tail}} \in Y$. Clearly $\text{ran}(Y) = |M| = \kappa$. Moreover, $M^{<\kappa} \subseteq M$ in $V$ implies $Y^{<\kappa} \subseteq Y$ in $V$. Consider the forcing extension $Y[G*g] = \{\tau_{G*g}: \tau \in Y$ is a $\mathbb{P} \ast \mathbb{Q}$-name. Since $\mathbb{P}$ is $\kappa$-c.c. in $V$ and $\mathbb{P} \subseteq V_\kappa \subseteq Y$, it follows by assertion (3) of Fact 14 that $Y[G*g]^{<\kappa} \subseteq Y[G*g]$ in $V[G]$. Assertion (2) of the same fact shows that $Y[G*g]^{<\kappa} \subseteq Y[G*g]$ in $V[G*g]$, since $\mathbb{Q}$ is $\kappa$-distributive in $V[G]$. We have $Y[G*g] \prec N[G*g]$ as usual. It follows that $P_{\text{tail}}$ is (much more than) $\kappa$-closed in $Y[G*g]$. By the diagonalization criterion for $Y[G*g]$, we may construct in $V[G*g]$ a filter $G_{\text{tail}} \subseteq P_{\text{tail}}$ which is $Y[G*g]$-generic on $P_{\text{tail}}$.

The crucial claim now is that $G_{\text{tail}}$ is actually $N[G*g]$-generic. For, if $E \in N[G*g]$ is any dense open set in $P_{\text{tail}}$, then $E = \bar{E}$ for some $\mathbb{P} \ast \mathbb{Q}$-name $\bar{E} \in N$. Thus, $\bar{E} = j(h_0)(s_0)$ for some function $h_0 \in M$ and some $s_0 \in S^{<\omega}$. Consider in $N[G*g]$ the set

$$\bar{E} = \bigcap \{\tau_{G*g}: \tau = j(h_0)(s) \text{ for some } s \in S^{<\omega} \text{ where } \tau_{G*g} \subseteq P_{\text{tail}} \text{ is open dense} \}.$$

Recall that $S \subseteq N$ and $S^{<\omega}$ has size $\delta$ in $N$. As $P_{\text{tail}}$ is $\leq \delta$-distributive in $N[G*g]$, it follows that $\bar{E}$ is dense in $P_{\text{tail}}$. Note that $S \in Y$ since it is definable in $N$ from parameters $\kappa$ and $b$ which are both elements of $Y$. Moreover, $\bar{E}$ is definable in $N[G*g]$ from parameters $j(h_0)$, the seed set $S$, the tail forcing $P_{\text{tail}}$ and the filter $G*g$. As all these parameters are elements of $Y[G*g]$, it follows that $\bar{E} \subseteq Y[G*g]$. Since $G_{\text{tail}}$ is a $Y[G*g]$-generic filter on $P \ast \mathbb{Q}$, we see that $G_{\text{tail}} \cap \bar{E} \neq \emptyset$. As $\bar{E} \subseteq E$, we established that $G_{\text{tail}}$ is indeed $N[G*g]$-generic and thus proved the claim. We thus let $j(G) = G*g \ast G_{\text{tail}}$, then $G \cong j[G] \subseteq j(G)$. This satisfies the lifting criterion and $j$ hence lifts to $j: M[G] \to N[j(G)]$. This concludes Step 1.

As in the proof of the Main Theorem, we will force with the collapsed version of $\mathbb{Q}$—we called it $\mathbb{Q}_0$—over $M[G]$. Since $g$ was $X[G]$-generic on $\mathbb{Q}$, Lemma 19 applied and yielded the $M[G]$-generic filter $g_0 \subseteq \mathbb{Q}_0$. Recall that $g_0 = \pi_1^*g$ where $\pi_1$ is the Mostowski collapse of $X[G]$ in $V[G]$, defined by $\pi_1(\tau_G) = \pi(\tau)_G$. The crucial application of Lemma 19 in $V[G*g]$ showed that $g_0$ is $M[G]$-generic on $\mathbb{Q}_0$ and that $A = A_{G*g} = \pi(A)_{G_{\text{tail}}} = (A_0)_{G_{\text{tail}}}$ is an element of $M[G*g]$. As before, $M[G*g_0]$ is a $\kappa$-model in $V[G*g]$ and $V[G*g_0] \subseteq V[G*g] \subseteq N[G*g]$. Thus, to finish showing that $r \in D$, it suffices to lift the embedding $j$ once more:

Step 2. In $V[G*g]$, lift $j: M[G] \to N[j(G)]$ to $j: M[G*g_0] \to N[j(G)*j(g_0)]$. 


Again, we will verify the lifting criterion. Similar to Step 1, we will build a $Y[j(G)]$-generic filter $j(g_0) \subseteq j(Q_0)$ and then argue that $j(g_0)$ is actually $N[j(G)]$-generic. In order to satisfy the necessary condition $j''g_0 \subseteq j(g_0)$, let us first find a master condition $q \in j(Q_0)$ below $j''g_0$ so that $q \in Y[j(G)]$.

Observe that $Q_0 \in V_\theta$ and $\pi \upharpoonright Q : V_\theta \to V_\theta$ both have size $\kappa$ in $V_\theta$ (since $\theta$ is a limit ordinal). Since $V_\theta \subseteq N$ and $\pi_1$ is definable from $\pi$ and $G$, it follows that $g_0 = \pi_1''g \in N[G \ast g]$ has size $\kappa$ in $N[G \ast g]$. Moreover, since $Q_0$ is $\kappa$-closed in $M[G]$, the same density argument as before shows that $g_0$ is a $\kappa$-closed subset of $Q_0$ in $M[G \ast g]$, and also in $V[G \ast g]$. Absoluteness shows that $N[G \ast g] \models \langle g_0 \rangle$ is a $\kappa$-closed subset of $Q_0 \cap |g_0| = \kappa \cap g_0$ is directed). Thus, in $N[G \ast g]$, the filter $g_0$ is generated by a descending chain $q_\xi : \xi < \kappa$. By applying $j$ in $V[G \ast g]$, we see that $j''g_0$ is generated by the descending chain $\tilde{c} = \langle j(q_\xi) : \xi < \kappa \rangle$ in $V[G \ast g]$. Since we constructed $\tilde{c}$ in such a way that $j \upharpoonright Q_0 \in N$, it follows that $j \upharpoonright Q_0 \in N[j(G)]$ and thus $\tilde{c} \in N[j(G)]$. Moreover, $N[j(G)]$ thinks that $j(Q_0)$ is $\kappa$-closed, and we can hence find a master condition $q \in j(Q_0)$ below $\tilde{c}$. But in order to apply the diagonalization criterion to build a $Y[G \ast g]$-generic filter on $j(Q_0)$ containing $q$, we need a master condition $q$ that exists in $Y[G \ast g]$.

I claim that we may assume without loss of generality that $q \in Y[j(G)]$. For, since $Y[j(G)] \prec N[j(G)]$ and $N[j(G)] \models \langle \exists q \in j(Q_0) \text{ below } j''g_0 \rangle$, it suffices to show that the parameters $j(Q_0)$ and $j''g_0$ are elements of $Y[j(G)]$. Since we put $a = \pi \upharpoonright Q$ and $b = j \upharpoonright Q_0$ into $Y$ when defining $Y$, it follows that $g_0 = \pi''g \in Y[G \ast g]$ and thus $j''g_0 \in Y[j(G)]$. Since $j(Q_0) \in Y$, we have that the poset $j(Q_0)$ exists in $Y[j(G)]$. This proves my claim and we may therefore fix a master condition $q \in j(Q_0) \cap Y[G \ast g]$ below all of $j''g_0$.

Lastly, we will use diagonalisation to build a $Y[j(G)]$-generic filter $j(g_0) \subseteq j(Q_0)$ containing the element $q$. Observe that in $V[G \ast g]$, the structure $Y[j(G)]$ has size $\kappa$ and $j(Q_0) \in Y[j(G)]$. Moreover, since $G_{\text{tail}} \in V[G \ast g]$ is $N[G \ast g]$-generic on $\mathbb{P}_{\text{tail}}$, assertion (1) of Fact 14 shows that $Y[j(G)]^{\leq \kappa} \subseteq Y[j(G)]$ in $V[G \ast g]$. Since $Y[j(G)] \prec N[j(G)]$ we see that $j(Q_0)$ is much more than $\kappa$-closed in $Y[j(G)]$. By the diagonalization criterion in $V[G \ast g]$, we can thus construct a filter $j(g_0) \subseteq j(Q_0)$ which is $Y[j(G)]$-generic on $j(Q_0)$ such that $q \in j(g_0)$. Similar to Step 1, one verifies that this filter on $j(Q_0)$ is actually $N[j(G)]$-generic. Since $q \in j(g_0)$ we have $j''g_0 \subseteq j(g_0)$. The lifting criterion is satisfied and $j$ lifts to $j : M[G \ast g_0] \to N[j(G) \ast j(g_0)]$ in $V[G \ast g]$ as desired. This concludes Step 2.

We thus established that $D$ is dense in $\mathbb{P} \ast Q$ and the proof is complete.

Since the limit ordinals are unbounded in all the ordinals, it follows that Theorem 41 certainly implies the Main Theorem. We thus found a proof of the Main Theorem that is quite different from the one presented in Section 5. The original proof used the fact that strongly unfoldable cardinals have embeddings similar to those of supercompact cardinals, while this second proof uses their embedding characterization that mimics strong cardinals.

Interestingly, the proof of Theorem 41 can be modified in a straightforward manner to also prove the successor case (i.e., the strengthening of Theorem 37 where the GCH assumption is omitted), since we never used $\theta$ being a limit ordinal in an essential way: For instance, when we fixed $j : M \to N$ with $V_\theta \subseteq N$ and let $\delta = 2^\omega_\theta$, we concluded that $2^\omega_\theta = \mathfrak{d}_\theta$. If $\theta$ is a successor ordinal, one can merely
infer that $\beth_\delta^\beta \geq \beth_\delta$. But, in fact we never used the equality between $\delta$ and $\beth_\delta$; all we worked with was that $j(f)(\kappa) \geq \delta$ and that $V_\delta$ had size $\delta$ in $N$. This is exactly why we insisted that $j(f)(\kappa) \geq \beth_\delta^\beta$ when defining the Menas property for a $\theta$-strongly unfoldable cardinal $\kappa$ in Section 3. Another use of $\theta$ being a limit ordinal occurred when we observed that $\alpha = \pi \upharpoonright \bar{Q}$ was an element of size $\kappa$ in $V_\delta$. If $\theta$ is a successor ordinal and one uses the usual von Neumann code for ordered pairs, then $\alpha$ need not be an element of $V_\theta$. But, as discussed before, if we use a flat pairing function instead, it follows that $V_\theta \times V_\theta \subseteq V_\theta$ for all infinite ordinals $\theta$ and $\alpha$ has thus size $\kappa$ in $V_\theta$. Lastly, note that the poset $Q$ must have size at most $\beth_\delta$, in order for it to have an isomorphic copy in $V_{\theta+1}$. With these modifications, the proof of Theorem 41 yields the following:

**Theorem 42.** Let $\kappa$ be $(\theta+1)$-strongly unfoldable for some ordinal $\theta \geq \kappa$. Then after the lottery preparation of $\kappa$ relative to a function with the Menas property for $\kappa$, the $(\theta+1)$-strong unfoldability of $\kappa$ is indestructible by $<\kappa$-closed, $\kappa$-proper forcing of size at most $\beth_\theta$.

Theorem 42 improves Theorem 37 by freeing us from any GCH assumption for $\beth_\delta$. We thus obtain the following strongest local version of the Main Theorem:

**Theorem 43.** Let $\kappa$ be $\theta$-strongly unfoldable for some ordinal $\theta \geq \kappa$. Then after the lottery preparation of $\kappa$ relative to a function with the Menas property for $\kappa$, the $\theta$-strong unfoldability of $\kappa$ is indestructible by $<\kappa$-closed, $\kappa$-proper forcing of rank less than $\theta$.

**Proof.** This is immediate by Theorems 41 and 42.

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