The Zariski topology on sets of semistar operations

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February 13th, 2015
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All rings $R$ will be commutative and unitary; we also assume that $R$ is an integral domain with quotient field $K$.

We will not assume that $R$ is Noetherian.

An overring is a ring comprised between $R$ and $K$.

$\text{Zar}(R)$ is the set of valuation overrings of $R$.

We denote by $\mathbf{F}(R)$ the set of $R$-submodules of $K$. 
Definition

A map $\star : \mathcal{F}(R) \longrightarrow \mathcal{F}(R)$, $I \mapsto I^\star$ is a semistar operation if, for every $I, J \in \mathcal{F}(R)$,

- it is extensive: $I \subseteq I^\star$;
- it is order-preserving: if $I \subseteq J$ then $I^\star \subseteq J^\star$;
- it is idempotent: $(I^\star)^\star = I^\star$;
- for every $x \in K$, $x \cdot I^\star = (xI)^\star$.

If $I = I^\star$, we say that $I$ is $\star$-closed.

The first three properties make sense in every partially ordered set, giving the general concept of closure operation.

Related kinds of closure operations: star and semiprime operations.
Definitions and examples

The order structure

Definition

Given two semistar operations $\star_1, \star_2$, we say that $\star_1 \leq \star_2$ if $I^{\star_1} \subseteq I^{\star_2}$ for every $R$-submodule $I$.

- The set $\text{SStar}(R)$ of all semistar operations, with this order, is a complete lattice.
- The infimum of $\{\star_\alpha\}_{\alpha \in A}$ is the operation $\star$ such that $I^\star = \bigcap_{\alpha \in A} I^{\star_\alpha}$.
- There is no general formula for the supremum $\star$ of $\{\star_\alpha\}_{\alpha \in A}$; however, $I = I^\star$ if and only if $I = I^{\star_\alpha}$ for every $\star_\alpha$. 
Finite type

Definition

Let $\star$ be a semistar operation. Then, define $\star_f$ as the map

$$I \mapsto I^{\star_f} := \bigcup \{ F^\star : F \subseteq I, \ F \text{ is finitely generated} \}.$$ 

- $\star_f$ is always a semistar operation.
- $\star_f \leq \star$.
- $I^\star = I^{\star_f}$ if $I$ is finitely generated.
- $\star$ is a semistar operation of finite type (or a finite-type operation) if $\star = \star_f$. 
Why operations of finite type?

- Finite-type closures depend only on finitely-generated fractional ideals (in particular, on ideals inside the ring).
  - For example, if $\star_1$ and $\star_2$ are of finite type, and we want to know whether $\star_1 = \star_2$, we only need to check it at ideals.

- More uniform behaviour (no “jumps”).
  - If $\star$ is of finite type and $L$ is $R$-flat, then $(IL)^* = I^*L$.

- Existence of $\star$-maximal ideals: say that $I$ is a quasi-$\star$-ideal if $I = I^* \cap R$. Then, $I$ is contained in a maximal quasi-$\star$-ideal, which moreover is prime.

- We can control the supremum:

$$I^{\text{sup}}(A) = \bigcup \{ I^*_{\star_1 \cdots \star_n} : \star_1, \ldots, \star_n \in A \}$$
Examples

- If $T$ is an overring, the extension $\wedge_T : I \mapsto IT$ is a semistar operation of finite type.
  - If $T = R$ we get the identity (denoted by $d$): $I^d = I$ for every submodule $I$.
  - If $T = K$ we get the trivial extension $\wedge_K: I^K = K$ for every $I \neq (0)$.
- Given a set $\Delta$ of overrings, $\star_\Delta := \inf\{\wedge_T : T \in \Delta\}$, i.e.,
  $$I^{\star_\Delta} = \bigcap_{T \in \Delta} IT.$$
- If $\Delta = \{R_P : P \in Y\}$ (with $Y \subseteq \text{Spec}(R)$) is a set of localizations, then we define $s_Y$ as $\star_\Delta$:
  $$I^{s_Y} = \bigcap_{P \in Y} IR_P,$$
Examples (2)

- The \( b \)-operation (or integral closure): if \( I \) is an ideal, \( I^b \) is the set of \( x \in K \) such that

\[
x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0
\]

for some \( a_i \in I^i \).

  ▶ Equivalently, \( b = \ast_{\text{Zar}}(R) \).
  ▶ Since we need only a finite amount of data (enough to generate the \( a_i \)) the \( b \)-operation is of finite type.

- The \( v \)-operation: \( I^v = (R : (R : I)) \), where \((F : G) := \{x \in K : xG \subseteq F\}\). More generally, it can be done by using any submodule \( L \) in place of \( R \).

  ▶ This is usually not of finite type: if \( V \) is a valuation domain with non-finitely generated maximal ideal \( M \), then \( M^v = V \) but \( I^v = I \) for every finitely generated ideal.
A Noetherian example

Let $R$ be a Noetherian domain such that $\dim(R) \geq 2$. Consider the set

$$\Delta := \{ V \in \text{Zar}(R) : V \text{ is a DVR} \}.$$

- If $I$ is an ideal of $R$, $I^\star_\Delta = I^b$; in particular, $\star_\Delta$ and $b$ coincide over finitely generated ideals: hence, $(\star_\Delta)_f = b_f = b$.

- If $W$ is a non-discrete valuation overring (for example, if $\dim(W) \geq 2$), then there is (at most) one $V \in \Delta$ above $W$. Hence, $W^\star_\Delta = V \neq W$, while $W^b = W$: hence, $\star_\Delta \neq b$.

- Therefore, $\star_\Delta \neq (\star_\Delta)_f$, that is, $\star_\Delta$ is not of finite type.
The Zariski topology

Definition

We define the Zariski topology on $S\text{Star}(R)$ to be the topology whose subbasic open set are those in the form

$$V_I := \{ \star \in S\text{Star}(R) : 1 \in I^* \}$$

as $I$ ranges among the $R$-submodule of $K$.

- We get the same topology if we use the sets of the form

$$V_{I,y} := \{ \star \in S\text{Star}(R) : y \in I^* \}$$

since $V_{I,y} = V_{y^{-1}I}$.

- However, if we want reduce to generalize the topology (for example, to semiprime operations or to rings with zerodivisors), then we have to use the $V_{I,y}$. 
Basic properties

- $\text{SStar}(R)$ is $T_0$.
- Link with the order structure: if $O$ is open, $\star \in O$ and $\star' \geq \star$, then $\star' \in O$.
  - In particular, the closure of $\{\star\}$ is $\{\star' : \star \leq \star'\}$.
- There is a unique closed point ($d$) and a unique generic point ($\wedge_K$).
- $\text{SStar}(R)$ is not $T_1$ nor $T_2$ (Haussdorff). [Unless $R = K$.]
- $\text{SStar}(R)$ is compact.
- Limited functoriality: if $A \subseteq B$ is an extension of integral domains, there is a continuous map $\text{SStar}(B) \rightarrow \text{SStar}(A)$, which is injective if $B$ is an overring of $A$. 
The Zariski topology on $\text{SStar}_f(R)$

- $\text{SStar}_f(R)$ is dense in $\text{SStar}(R)$.
- The map
  \[
  \psi_f : \text{SStar}(R) \to \text{SStar}_f(R)
  \]
  is a topological retraction.
- If $U_F := V_F \cap \text{SStar}_f(R)$, then
  \[
  \{ U_F : F \text{ is a finitely generated } R\text{-submodule of } K \}
  \]
  is a subbasis of the induced topology.
The Zariski topology on $\text{SStar}_f(R)$ (2)

- The map

$$\iota: \text{Over}(R) \longrightarrow \text{SStar}_f(R)$$

$$T \mapsto \Lambda_T$$

is a topological embedding.

- The topology on $\text{Over}(R)$ is generated by the sets $B_F := \text{Over}(R[F])$, as $F$ varies among the finite subsets of $K$.

- If $\Lambda \subseteq \text{SStar}_f(R)$ is compact, then $\inf \Lambda$ is of finite type.
  - If $\Delta \subseteq \text{Over}(R)$ is compact, then $\star_{\Delta}$ is of finite type.
  - If $R$ is Noetherian, $\dim(R) \geq 2$, then $\{ V \in \text{Zar}(R) : V$ is a DVR$\}$ is not compact.
  - The converse does not hold.

- $\text{SStar}_f(R)$ is a spectral space.
Spectral spaces

Definition

A spectral space is a topological space that is homemorphic to the prime spectrum of a commutative ring $R$, endowed with the Zariski topology.

- Spec$(R)$ (with the Zariski topology) is a spectral space.
- Zar$(R)$ and Over$(R)$ are spectral spaces.
- Every finite poset (endowed with the order topology) is a spectral space.
Spectral spaces can be characterized topologically [Hochster]: \( X \) is a spectral space if and only if the following properties hold:

- \( X \) is compact and \( T_0 \);
- every irreducible closed subset of \( X \) has a generic point (i.e., it is the closure of a single point);
- there is a basis of compact subsets that is closed by finite intersections.

If \( X = SStar(R) \) or \( X = SStar_f(R) \), then the first and the third point are easy (we use as a base the family of finite intersections of the sets \( U_F \)).

What about the second property?
Ultrafilters

Definition

Let $X$ be a set. A filter on $X$ is a subset $\mathcal{Y}$ of $\mathcal{P}(X)$ such that

- if $A \in \mathcal{Y}$ and $A \subseteq B$ then $B \in \mathcal{Y}$;
- if $A, B \in \mathcal{Y}$ then $A \cap B \in \mathcal{Y}$;
- $\emptyset \notin \mathcal{Y}$.

An ultrafilter is a maximal filter.

- Every filter is contained in an ultrafilter (consequence of Zorn’s lemma).
- Ultrafilters can be used to prove Tychonoff’s theorem.
Spectral spaces and ultrafilters

Through Hochster’s theorem, we can characterize spectral spaces in terms of ultrafilters:

Proposizione ([Finocchiaro, 2014])

Let $X$ be a $T_0$ space. Then, $X$ is a spectral space if and only if there is a subbasis $S$ of $X$ such that, for every ultrafilter $\mathcal{U}$ on $X$, the set

$$X_S(\mathcal{U}) := \{ x \in X : \forall B \in S, x \in B \iff B \in \mathcal{U} \}$$

is nonempty.

- The subbase $S$ matters.
- Very non-constructive criterion.
The Zariski topology

$\text{SStar}_f(R)$ as a spectral space

- $S := \{ U_F : F \text{ is finitely generated} \}$.
- The “natural” candidate is
  \[ \star := \sup \{ \inf(U_F) : U_F \in \mathcal{U} \} \]
- $\inf(U_F)$ is of finite type since $U_F$ is compact.
- Since all is of finite type, we can control the supremum.
- We can show that $\star \in X_S(\mathcal{U})$.

**Teorema**

$\text{SStar}_f(R)$ is a spectral space.
What kind of ring?

Suppose $\text{SSStar}_f(R) \simeq \text{Spec}(D)$.

- $\text{SSStar}_f(R)$ has a minimum and a maximum; hence $D$ is local and has a unique minimal prime (so we can take it as a domain).
- $\dim(D) \geq |\text{Spec}(R)|$.
  - $|\text{Spec}(R)| < \infty$: write $\text{Spec}(R) = \{P_1, \ldots, P_n\}$ in a way such that $P_i$ is a minimal element of $\{P_1, \ldots, P_n\}$. Then, if $\Delta_i := \{P_1, \ldots, P_k\}$, we have a descending chain $\land K = s_\emptyset > s_{\Delta_1} > s_{\Delta_2} > \cdots > s_{\Delta_n} = d$, and thus $\dim(D) \geq n = |\text{Spec}(R)|$.
  - $|\text{Spec}(R)| = \infty$: we can do the previous reasoning with arbitrary large finite subsets.
- In particular, if $\text{Spec}(R)$ is infinite, $\dim(D) = \infty$ and, being local, $D$ cannot not Noetherian.
More kinds of semistar operations

Definition

A semistar operation $\star$ is:

- **stable** if $(I \cap J)^\star = I^\star \cap J^\star$ for every $I, J \in F(R)$;
- **spectral** if $I^\star = \bigcap_{P \in \Delta} IR_P$ for some $\Delta \subseteq \text{Spec}(R)$.

- By flatness, a spectral operation is stable.
- The converse is not true: $\mathcal{V}$ a valuation domain with non-finitely generated maximal ideal, $\star = v$.
- However, a stable operation of finite type is spectral.
More kinds of semistar operations (2)

Definition

A semistar operation $\star$ is:

- **eab** if, whenever $F, G, H$ are finitely generated, $(FG)^\star \subseteq (FH)^\star$ implies $G^\star \subseteq H^\star$;
- **valutative** if $I^\star = \bigcap_{V \in \Delta} IV$ for some $\Delta \subseteq \text{Zar}(R)$.

- Valutative operations are eab.
- Not all eab operations are valutative: $V$ a valuation domain with non-finitely generated maximal ideal, $\star = v$.
- However, an eab operation of finite type is valutative.
Spectral and eab: differences

- The relations stable/spectral and eab/valuative are different: spectral is equivalent to stable and semifinite, but valuative is *not* equivalent to eab and semifinite [Fontana and Loper, 2009].
- While $\star$ valutative implies $\star_f$ valutative, $\star$ spectral *does not imply* $\star_f$ spectral [Anderson and Cook, 2000].
- The supremum of a family of finite-type spectral operations is spectral. Does the same holds for valutative operations?
Like $\text{Over}(R)$ is embedded in $\text{SStar}_f(R)$, the set of localizations of $R$ is embedded into $\text{SStar}_{f,sp}(R)$, while $\text{Zar}(R)$ is embedded into $\text{SStar}_{f,eab}(R)$.

Like for $\Psi_f : \text{SStar}(R) \longrightarrow \text{SStar}_f(R)$, we can define retractions $\Psi_{sp} : \text{SStar}(R) \longrightarrow \text{SStar}_{f,sp}(R)$ and $\Psi_a : \text{SStar}(R) \longrightarrow \text{SStar}_{f,eab}(R)$.

- However, while $\Psi_f(\star) \leq \star$ and $\Psi_{sp}(\star) \leq \star$, we have $\Psi_a(\star) \geq \star_f$, and we can’t in general compare $\Psi_a(\star)$ with $\star$.

$\text{SStar}_{f,sp}(R)$ is a spectral space.

- The proof follows the same path of the proof for $\text{SStar}_f(R)$. 

Finite type, spectral and eab: analogies (2)

- If $\star$ is of finite type and eab or spectral, then there is a ring $R_\star(X)$ such that $R[X] \subseteq R_\star(X) \subseteq K(X)$ and $I^\star = IR_\star(X) \cap K$ for every $I \in \mathcal{F}(R)$.

- If $\Delta \subseteq \text{Spec}(R)$ or $\Delta \subseteq \text{Zar}(R)$, then $s_\Delta$ (respectively, $\star_\Delta$) is of finite type if and only if $\Delta$ is compact.
  - New proof of the fact that $\text{Zar}(R)$ is compact.

- If $\Delta, \Lambda \subseteq \text{Spec}(R)$ are compact, then $\star_\Delta = \star_\Lambda$ if and only if $\Delta^\downarrow = \Lambda^\downarrow$
  - $Y^\downarrow := \{ P \in \text{Spec}(R) : P \subseteq Q \text{ for some } Q \in Y \}$ is the generization of $Y$.
  - An analogous criterion holds for valutative operations, but in the other way: $\star_\Delta = \star_\Lambda$ if and only if $\Delta^\uparrow = \Lambda^\uparrow$. 
The Kronecker function ring

Definition

The Kronecker function ring of $R$ is

$$\text{Kr}(R) := \left\{ \frac{f}{g} \in K(X) : f, g \in R[X], c(f) \subseteq c(g)^b \right\}$$

where $c(f)$ (the content of $f$) is the ideal of $R$ generated by the coefficients of $f$.

- $\text{Kr}(R)$ is always a Bézout domain.
- In particular, $\text{Spec}(\text{Kr}(R)) \simeq \text{Zar}(\text{Kr}(R))$.
- With the previous notation, $\text{Kr}(R) = R_b(X)$. 

The Kronecker function ring (2)

**Teorema**

The map $\Phi : \text{Zar}(R) \rightarrow \text{Zar}(\text{Kr}(R))$, $V \mapsto \text{Kr}(V)$ is an homeomorphism.
The Kronecker function ring (2)

Teorema

The map $\Phi : \text{Zar}(R) \rightarrow \text{Zar}(\text{Kr}(R))$, $V \mapsto \text{Kr}(V)$ is an homeomorphism.

Let $\star$ be a valutative operation of finite type.

- $\star = \star_\Delta$ for a unique $\Delta \subseteq \text{Zar}(R)$ such that $\Delta$ is compact and $\Delta = \Delta^\uparrow$;
- $\Phi(\Delta) \subseteq \text{Zar}(\text{Kr}(R))$ is compact and $\Phi(\Delta)^\uparrow = \Phi(\Delta)$;
- $\Phi(\Delta)$ corresponds to $Y \subseteq \text{Spec}(\text{Kr}(R))$, which is compact and such that $Y = Y^\downarrow$;
- $Y$ generates $s_Y$.

$$\text{SStar}_{f,eab}(R) \simeq \text{SStar}_{f,sp}(\text{Kr}(R))$$
Bibliography


