Master’s project

A GEOMETRIC CONSTRUCTION OF MINIMAL GENERATING SEQUENCES

Samar El Hitti
University of Missouri, Department of Mathematics, Columbia, MO 65203, USA samar@math.missouri.edu

Advisor S.Dale Cutkosky
University of Missouri, Department of Mathematics, Columbia, MO 65203, USA dale@math.missouri.edu

July 2006
1. Definitions

Let $k$ be an algebraically closed field of characteristic $\geq 0$ and let $K$ be a 2 dimensional function field over $k$.

Let $\nu$ be a valuation of $K/k$ and let $\Gamma$ be the value group of $\nu$. Assume that $\Gamma \subseteq \mathbb{Q}$ is a non-discrete subgroup. Let $V$ be the valuation ring of $\nu$, and $m_\nu$ the maximal ideal of $V$.

We have $V/m_\nu = k$.

Assume that $R$ is a regular local ring which is essentially of finite type over $k$ with quotient field $K$ such that $\nu$ dominates $R$.

**Definition 1.1.** A set of elements $\{Q_i\} \in R$ (possibly infinite) is called a generating sequence for $\nu$; if and only if $\forall \gamma \in \Gamma$ the ideal $I_\gamma = \{x \in R \text{ s.t. } \nu(x) \geq \gamma\}$ is generated by the set $\{\prod_j Q_j^{r_j} \text{ s.t. } r_j \in \mathbb{N} \text{ and } \sum_j r_j \nu(Q_j) \geq \gamma\}$.

We say that $\{Q_i\}$ described above is a minimal generating sequence of $\nu$ if no proper subset of $\{Q_i\}$ is a generating sequence.

We have the following: if $\nu(Q_i) \notin \sum_{j=0}^{i-1} \mathbb{Z} \nu(Q_j)$, $\forall i \geq 1$, $\Rightarrow \{Q_i\}$ is a minimal generating sequence of $\nu$.

**Definition 1.2.** Consider the following sequence $R = R_0 \to R_1 \to \cdots \to R_n \to \cdots$ of regular local rings with maximum ideals $m_i$ dominated by $V$ defined as follows:

Let $f \in m_i$ be such that $\nu(f) = \min\{\nu(g) \text{ s.t. } g \in m_i\}$.

Define $R_{i+1} = R_i \left[\frac{m_i}{f}\right]_{m_i \cap R}$ where $f$ is part of a regular system of parameters of $m_i$.

**Definition 1.3.** We say that $R_i$ is free if the reduced exceptional divisor of $R \to R_i$ is a single irreducible divisor $E_i$.

**Definition 1.4.** Let $r'_1 = \tilde{r}_0 = 0$.

For all $i \geq 1$ let $(r'_{i+1}, \tilde{r}_i)$ be the pair of integers with the following properties:

1. $\tilde{r}_i$ is the largest integer $r \geq r'_i$ such that $R_r$ is free $\forall r'$ with $r'_i \leq r' \leq r$.
2. $r'_{i+1}$ is the smallest integer $r > \tilde{r}_i$ such that $R_r$ is free.

**Remark 1.5.** Suppose $(x, y)$ are regular parameters in $R_{r'_i}$ such that $x = 0$ is the local equation of $E_{r'_i}$.

Then $\exists l \in \mathbb{N}$ and a change of variables $\tilde{y} = y - \sum_{i=1}^{l} a_i x^{r_i}$ with $a_i \in k$ such that $\nu(\tilde{y}) = n_0 \nu(x) + \lambda$ with $0 < \lambda < \nu(x)$ and $\lambda \in \mathbb{Q}^+$ thus $(x, \tilde{y})$ are regular parameters of $R_{r'_i}$ with $\nu(\tilde{y}) \notin \mathbb{Z} \nu(x)$.

**Proof.** $\Gamma \subset \mathbb{Q}$ is non-discrete.

Assume that $\nu(y) \in \mathbb{Z} \nu(x) \Rightarrow \nu(y) = n_0 \nu(x)$ for some positive integer $n_0$. 
Let \( y_0 = y - a_0 x^{n_0} \) where \( a_0 \) is the residue of \( \frac{y}{x^{n_0}} \) in \( V/m_\nu = k \), and let \( \xi_0 = \nu(y_0) \) and notice that \( \nu\left(\frac{y - a_0 x^{n_0}}{x^{n_0}}\right) = \nu(y - a_0 x^{n_0}) - n_0 \nu(x) = \nu\left(\frac{y}{x^{n_0}} - a_0\right) > 0 \)
\[ \Rightarrow \xi_0 > n_0 \nu(x) \geq \nu(x) \Rightarrow \xi_0 > \nu(x). \]
and let \( y_s = y - \sum^{s}_{i=1} a_i x^{n_i} \) with \( n_i \in \mathbb{N} \) and \( \xi_s = \nu(y_s) \) as long as \( \xi_{s-1} \in \mathbb{Z} \nu(x) \).

Notice that \( \xi_s > \xi_{s-1} > \cdots > \xi_0 > \nu(x) \).

Assume that \( \forall j \geq 0; \ \nu(y_j) \in \mathbb{N} \nu(x) \) then \( \{y_s\} \) is a Cauchy sequence in \( R_{\nu} \) and \( \nu(y_s) \to \infty. \)

This implies that there exists \( a \in R_{\nu} \) such that \( \nu(a) = \infty \Rightarrow \Gamma \) is discrete; contradiction.

Thus \( \exists l \in \mathbb{N} \) such that \( \tilde{y} = y - \sum^{l}_{i=1} a_i x^{n_i} \) and \( \nu(\tilde{y}) = \xi_l \notin \mathbb{N} \nu(x) \), moreover take \( a_l = 0 \) if necessary \( \Rightarrow \exists \lambda \in \mathbb{Q}^+ \) such that \( \xi_l = \nu(\tilde{y}) = n_l \nu(x) + \lambda \) with \( 0 < \lambda < \nu(x), \lambda \in \mathbb{Q}^+. \)

**Definition 1.6.** We say that a sequence of elements \( \{P_j\}_{j \geq 0} \) of \( R \) are in Weierstrass form if:

\[
x = P_0, \ y = P_1 \text{ are regular parameters of } R \text{ and for } j \geq 2, \text{ there is an expansion:}
\]

\[
P_{j+1} = P_j^{\nu x_j} - \theta_j \prod^{j-1}_{i=0} P_{i}^{m_{j,i}} - \sum^{j}_{i=0} \lambda_{j} I_j \prod^{j}_{i=0} P_{i}^{m'_{j,i}}
\]

with \( I_j = (m'_{j,0}, m'_{j,1}, \ldots, m'_{j,j}) \) satisfying the following conditions:

1. \( 0 \leq m_{j,i} < a_i \) for \( 1 \leq i \leq j \\
   m_{j,0}, m'_{j,0} \geq 0 \)

2. \( \theta_j, \lambda_{j} I_j \) are non-zero elements of \( k, \forall j, I_j \).

3. If \( \Gamma_j = \{\nu(P_j)\}_{0 \leq i \leq j} \) then for \( j \geq 1, \nu(\prod^{j-1}_{i=0} P_{i}^{m_{j,i}}) \) has order \( a_j \) in \( \Gamma_{j-1}/a_j \Gamma_{j-1} \cong \mathbb{Z}/a_j \mathbb{Z}. \)

4. \( \nu(P_{j+1}) > a_j \nu(P_j) = \sum^{j-1}_{i=0} m_{j,i} \nu(P_i). \)

5. \( \sum^{j}_{i=0} m'_{j,i} \nu(P_i) > \nu(P_j^{\nu x_j}), \forall j \geq 1 \text{ and } \forall I_j. \)

Note: we say that the sequence of elements \( \{P_j\}_{j \geq 0} \) described above are in Generalized Weierstrass form if condition 2 is replaced by: \( \theta_j, \lambda_{j} I_j \) are units in \( \tilde{R} \).

**Lemma 1.7.** Suppose that \( \{P_j\}_{j \geq 0} \) are in Weierstrass form.

Let \( d_j = \deg y P_j \) for \( j \geq 0. \) Then \( d_{k+1} = a_k d_k \) for \( k \geq 1. \)

**Proof.** Proof is by induction on \( k. \)

For \( k = 1 \) \( P_2 = Y^{a_1} - \theta_1 X^{m_{1,0}} - \sum_{l_j} \lambda_{1 \bar{l}_j} X^{m'_{1,0}} Y^{m'_{1,j}} \)
with \( m'_{1,1} < a_1 \)
\[ \Rightarrow \deg y P_2 = d_2 = a_1 = a_1,1 = a_1 d_1. \]
Assume $d_{j+1} = a_j d_j$ for $1 \leq j \leq k-1$.

\[ P_{k+1} = P_k^a - \theta_k \prod_{i=0}^{k-1} P_{i+1}^a - \sum_{i} \lambda_{k,i} \prod_{i=0}^k P_{m_i}^a \]

\[ \Rightarrow d_{k+1} = \max \left\{ a_k d_k, \sum_{i=0}^{k-1} d_i m_{k,i}, \sum_{i=0}^{k} d_i m'_{k,i} \right\} \]

Notice that $m_{k,0} = 0$ since $d_0 = 0$ and $m_{k,h+1} = m_{k,h+1} d_{h+1} = (1 + m_{k,h+1}) d_{h+1} \leq a_k d_h + m_{k,h+1} d_{h+1}$ for $1 \leq h \leq k-2$ and $m'_{k,h+1} d_{h+1} < a_k d_h + m'_{k,h+1} d_{h+1} = d_{h+1} + m'_{k,h+1} d_{h+1} = (1 + m'_{k,h+1}) d_{h+1} \leq a_k d_h + m_{k,h+1} d_{h+1}$ for $1 \leq h \leq k-1$, by induction and by (1) of definition 1.6, thus $\sum_{i=0}^{k-1} d_i m_{k,i} < a_k d_{k-1} < d_k < a_k d_k$; and

\[ \sum_{i=0}^{k} d_i m'_{k,i} < a_k d_k \Rightarrow d_{k+1} = a_k d_k \text{ for } k \geq 1. \]

\[ \square \]

2. Arithmetics

Suppose that $\frac{\nu(p)}{\nu(q)} = \frac{p}{q}$ with $p$ and $q$ positive integers and $(p, q) = 1$.

Consider the Euclidean algorithm for finding the greatest common divisor of $p$ and $q$:

\[ r_0 = f_1 r_1 + r_2 \]
\[ r_1 = f_2 r_2 + r_3 \]
\[ \ldots \]
\[ r_{N-2} = f_{N-1} r_{N-1} + 1 \]
\[ r_{N-1} = f_N \cdot 1, \]

where $r_0 = p$, $r_1 = q$ and $r_1 > r_2 > \cdots > r_{N-1} > r_N = 1$. Denote by $N = N(p, q)$ the number of divisions in the Euclidean algorithm for $p$ and $q$ and by $f_1, f_2, \ldots, f_N$ the coefficients in the Euclidean algorithm for $p$ and $q$. Define $F_i = f_1 + \cdots + f_i$ and $F_N = f_1 + \cdots + f_N$, Let $a$ and $b$ be integers such that $0 < a \leq p$, $0 \leq b < q$, and $aq - bp = \pm 1$.

Remark 2.1. With notations as above, $\frac{p}{q} = \frac{1}{f_2 + \cdots + \frac{1}{f_N}}$.

Let $\{ P_k(z_1, \ldots, z_k) \}_{k \in \mathbb{N}}$ be a sequence of polynomials as in [5]. So $P_k(z_1, \ldots, z_k) \in \mathbb{N}[z_1, \ldots, z_k]$ is a polynomial in $k$ variables with nonnegative integer coefficients such that for any set of numbers $c_1, \ldots, c_K$ we have

\[ \frac{c_1 + \frac{1}{c_2 + \cdots + \frac{1}{c_K}}}{P_K(c_1, \ldots, c_K)} = \frac{P_K(c_1, \ldots, c_K)}{P_{K-1}(c_2, \ldots, c_K)}. \]

We also assume that $P_0 = 1$ and set $P_{-1} = 0$. 
Then it follows from properties (1.2)-(1.6) in [5] that

\[ p = P_N(f_1, \ldots, f_N), \]
\[ q = P_{N-1}(f_2, \ldots, f_N), \]
\[ a = P_{N-1}(f_1, \ldots, f_{N-1}), \quad b = P_{N-2}(f_2, \ldots, f_{N-1}), \quad \text{if } N \text{ is odd}, \]
\[ a = p - P_{N-1}(f_1, \ldots, f_{N-1}), \quad b = q - P_{N-2}(f_2, \ldots, f_{N-1}), \quad \text{if } N \text{ is even}. \]

We also recall property (1.5) from [5] here since it will be used in the sequel

\[ P_k(f_1, \ldots, f_k) = f_kP_{k-1}(f_1, \ldots, f_{k-1}) + P_{k-2}(f_1, \ldots, f_{k-2}), \]
\[ P_{k-1}(f_2, \ldots, f_k) = f_kP_{k-2}(f_2, \ldots, f_{k-1}) + P_{k-3}(f_2, \ldots, f_{k-2}). \]

3. Quadratic Transforms

We will now consider a sequence

\[ R = R_{r_1} \rightarrow R_{r_1+1} \rightarrow R_{r_1+2} \ldots R_{r_1} \rightarrow R_{r_1+1} \ldots \rightarrow R_{r_1} \rightarrow \ldots \]

of quadratic transforms along \( \nu \) as defined in Definition 1.2.

Suppose that \( R := R_{r_1} \) has regular parameters \((X, \tilde{Y})\), with \( \tilde{Y} \) constructed as in remark 1.5. Then, since \( \nu(\tilde{Y}) \neq \mathbb{Z}\nu(X) \) then we can choose regular parameters \((X_1', Y_1')\) for \( R_{r_1+1} \) as follows:

a) If \( \nu(X) < \nu(\tilde{Y}) \) then \( X_1' = X \) and \( Y_1' = \frac{\tilde{Y}}{X} \).

b) If \( \nu(X) > \nu(\tilde{Y}) \) then \( X_1' = \frac{X}{Y} \) and \( Y_1' = \tilde{Y} \).

Our goal is to describe explicitly the sequence of quadratic transforms of \( R \) along \( \nu \).

Assume that \((X, \tilde{Y})\) is a system of regular parameters in \( R \). Let \( p \) and \( q \) be positive coprime integers such that \( \frac{\nu(\tilde{Y})}{\nu(X)} = \frac{p}{q} \). We let \( \nu(X) = \mu \). Let \( N = N(p, q) \), \( f_1, \ldots, f_n \) and \( F_1, \ldots, F_N \) be defined by the Euclidean algorithm for \( p \) and \( q \) as in Section 2.

Let \( a \) and \( b \) be integers such that \( 0 < a \leq p \), \( 0 \leq b < q \) and \( aq - bp = \pm 1 \). We will investigate the following sequence of quadratic transforms along \( \nu \):

\[ R \rightarrow R_1 \rightarrow \ldots R_{F_1} \rightarrow \ldots R_{F_3} \rightarrow \ldots R_{F_N}. \]
If $N > 1$ then for all $0 \leq j \leq F_1$, the ring $R_j$ is free and has a permissible system of parameters $(X, \frac{X}{X^t})$. In particular,

$$(X_1, Y_1) = \left( X, \frac{Y}{X^{f_1}} \right)$$

$$= \left( \frac{X^{P_0}}{Y^{P_{-1}}}, \frac{Y^{P_0}}{X^{P_{f_1}}} \right)$$

is a system of regular parameters in $R_{F_1}$ with $\nu(X_1) = \mu$ and $\nu(Y_1) = \frac{r_2}{q} \mu$.

If $N = 1$ then $R = R_0 \rightarrow R_1 \rightarrow \cdots \rightarrow R_{F_N}$ is a sequence of free rings and $R_{F_N}$ has a permissible system of parameters

$$(X_N, Y_N) = (X_1, Y_1) = \left( X, \frac{Y}{X^{f_1}} - c \right) = \left( \frac{X^u}{X^x}, \frac{Y^u}{X^x} - c \right),$$

where $c \in k$ is the residue of $\frac{Y^u}{X^x}$.

If $N > 2$ then for all $0 < j \leq F_2$, the ring $R_{F_1+j}$ is not free and has a system of regular parameters $(\frac{X}{Y^t}, Y_1)$.

In particular, $(X_2, Y_2) = \left( \frac{X}{Y^t}, Y_1 \right) = \left( \frac{X^r+X^e}{Y^t} \right)$ and $\nu(X_2) = \frac{r_2}{q} \mu, \nu(Y_2) = \frac{r_2}{q} \mu$.

In general, for all $1 < k < N$ and $0 < j \leq f_k$, the ring $R_{F_{k+1}+j}$ is not free and has a system of regular parameters $(\frac{X}{Y^t}, Y_{k-1})$ if $k$ is even or $(X_{k-1}, \frac{Y}{X^t})$ if $k$ is odd. In particular, if $k$ is even then $R_{F_k}$ has a system of regular parameters

$$(X_k, Y_k) = \left( \frac{X}{Y^t}, Y_{k-1} \right)$$

where $\nu(X_k) = \frac{r_{k+1}}{q} \mu$ and $\nu(Y_k) = \frac{r_2}{q} \mu$. We also notice that since $\frac{X_{k-1}}{Y^t} = \frac{X^{P_{k-2}(f_1,\ldots,f_{k-1})}}{Y^{P_{k-2}(f_2,\ldots,f_{k-1})}}$ the regular parameters $(X_k, Y_k)$ satisfy the equality $(X_k, Y_k) = \left( \frac{X^r}{Y^t} \right)$.

If $k$ is odd then $R_{F_k}$ has a system of regular parameters $(X_k, Y_k) = \left( X_{k-1}, \frac{Y^r}{X^x} \right)$ where $\nu(X_k) = \frac{r_k}{q} \mu$ and $\nu(Y_k) = \frac{r_2}{q} \mu$. We notice that since

$\frac{X_{k-1}}{X^t} = \frac{X^{P_{k-2}(f_1,\ldots,f_{k-1})}}{X^{P_{k-2}(f_2,\ldots,f_{k-1})}}$ the regular parameters $(X_k, Y_k)$ satisfy the equality $(X_k, Y_k) = \left( \frac{X^r}{Y^t} \right)$.

Finally, if $N > 1$ is odd then for all $0 < j < f_N$ the ring $R_{F_{N+1}+j}$ is not free and has a system of regular parameters $(X_{N-1}, \frac{Y}{X^x})$. Moreover, $R_{F_N}$ is the first free ring after a sequence of non-free rings $R_{F_1+1} \rightarrow \cdots \rightarrow R_{F_N-1}$. If $c \in k$ is the residue of $\frac{Y^u}{X^x}$ then
form a regular system of parameters in $R_{F_n}$ with $\nu(X_N) = \frac{r_\nu}{q} \mu = \frac{1}{q} \mu = (\nu(X), \nu(\tilde{Y}))$. If $N > 1$ is even then for all $0 < j < f_N$ the ring $R_{F_{N-1} + j}$ is not free and has a system of regular parameters $(X_{N-1}, Y_{N-1})$. Moreover, $R_{F_N}$ is the first free ring after a sequence of non-free rings $R_{F_{1} + ... + R_{F_{N-1} + 1}}$. If $c \in k$ is the residue of $\frac{\tilde{Y}^q}{X^p}$ then

$$(X_N, Y_N) = \left( \frac{X_{N-1}}{Y_{N-1}}, Y_{N-1} - c \right) = \left( \frac{X_{N-1}}{Y_{N-1}}, Y_{N-1} - c \right) = \left( \frac{X_{N-1}}{Y_{N-1}}, Y_{N-1} - c \right)$$

form a regular system of parameters in $R_{F_n}$ with $\nu(X_N) = \frac{r_\nu}{q} \mu = \frac{1}{q} \mu = (\nu(X), \nu(\tilde{Y}))$.

The following lemma summarizes the above discussion.

**Lemma 3.1.** Suppose that $R$ is a free ring and $(X, Y)$ is a regular system of parameters in $R$ such that $\frac{\nu(Y)}{\nu(X)} = \frac{p}{q}$ for some coprime integers $p$ and $q$. Let $F_N = f_1 + \cdots + f_N$, where the $f_i$'s are as in the division algorithm of $p$ and $q$, and let $a$ and $b$ be nonnegative integers such that $a \leq p$, $b < q$, and $aq - bp = 1$. Then the sequence of quadratic transforms along $\nu$

$$R \to R_1 \to \cdots \to R_{f_1} \to R_{f_1 + 1} \to \cdots \to R_{F_{N-1}} \to R_{F_N} \tag{3.1}$$

has the following properties:

1) $R, R_1, \ldots, R_{f_1}$ and $R_{F_N}$ are free rings.
2) Non-free rings appear in (3.1) if and only if $N > 1$, that is if $q > 1$. In this case $R_{F_{1} + \cdots + R_{F_{N-1}}}$ are non-free.
3) \( R_{F_N} \) has a regular system of parameters \((X_{F_N}, Y_{F_N}) = \left( \frac{X^a}{\tilde{Y}^q}, \frac{Y^q}{X^p} - c \right) \), where \( c \in k \) is the residue of \( \frac{\tilde{Y}^q}{X^p} \). Moreover, \( \nu(X_{F_N}) = \frac{1}{q} \nu(X) \) and \( X = X_{F_N}^g(Y_{F_N} + c)^a \), \( \tilde{Y} = X_{F_N}^p(Y_{F_N} + c)^a \).

Proof. We only check that \( X_{F_N}^q(Y_{F_N} + c)^b = \frac{X^{aq}}{\tilde{Y}^b} \cdot \frac{Y^{qb}}{X^{pb}} = X^{aq-bp} = X \) and \( X_{F_N}^p(Y_{F_N} + c)^a \).

Remark 3.2. If we apply lemma 3.1 to \( R = R_0 = R_{c'} \) and our regular parameters \((X, \tilde{Y}) \) in \( R \), we see that \( q > 1; R_{c_1} = R_{c_1} \) and \( R_{c_2} = R_{c_2} \).

Further, \( c \neq 0 \) in 3) and \( \nu_{c_2}(X) = q \) and \( \nu_{c_2}(\tilde{Y}) = p \).

4. Theorem

**Theorem 4.1.** Let \( R \) be a regular local ring of dimension 2 with regular parameters \((X, \tilde{Y}) \) and quotient field \( K \).

Let \( k \) be an algebraically closed field of characteristic \( \geq 0 \), and let \( \nu \) be a non-discrete valuation of \( K/k \) of value group \( \Gamma \) and valuation ring \( V \) that dominates \( R \).

Then there exists \( \{ P_i \} \) a sequence of polynomials in Weierstrass form of elements of \( R = R_{c'} \) that form a minimal generating sequence of \( \nu \), such that \( P_0 = X, P_1 = \tilde{Y} \) and for \( g \geq 0, \exists \) regular parameters \( X_{\nu+1}, Y_{\nu+1} \) in \( R_{\nu+1} \) with \( X_{\nu+1} = X, Y_{\nu+1} = \tilde{Y} \) such that \( \nu_{\nu+1}(X_{\nu+1}) = 1 \) and \( P_i = X_{\nu_{\nu+1}(P_i)} \cdot u_i \) in \( R_{\nu+1} \) where \( u_i \) is a unit in \( R_{\nu+1} \) for \( 0 \leq i \leq g \), with:

\[
\begin{align*}
\nu_{\nu_{\nu+1}(P_0)} &= a_1a_2...a_g \\
\nu_{\nu_{\nu+1}(P_1)} &= a_2...a_gb_1 \\
\nu_{\nu_{\nu+1}(P_i)} &= a_{i-1}\nu_{\nu_{\nu+1}(P_{i-1})} + a_{i+1}...a_gb_i \text{ for } 2 \leq i \leq g \\
\end{align*}
\]

with the convention that \( a_j...a_g = 1 \) for \( j > g \).

so in particular \( \nu_{\nu_{\nu+1}(P_g)} = a_{g-1}\nu_{\nu_{\nu+1}(P_{g-1})} + b_g \) for \( g \geq 2 \).

\[
\nu_{\nu_{\nu+1}(P_i)} = a_g\nu_{\nu_{\nu+1}(P_i)} \text{ for } i \leq g - 1 
\]

\[
P_{\nu+1} = X_{\nu_{\nu+1}(P_{\nu+1})} \cdot \tilde{Y}_{\nu_{\nu+1}} \text{ and } \nu_{\nu_{\nu+1}(P_{\nu+1})} = a_g\nu_{\nu_{\nu+1}(P_g)}.
\]

(4.3)
(4) $\sum_{i=0}^{g} \mathbb{Z}v(P_i) = \mathbb{Z}v(X_{r_{g+1}'}).$

(5) There are positive integers $a_{g+1}$ and $b_{g+1}$ defined by $\frac{v(\tilde{r}_{g+1}')}{v(X_{r_{g+1}'})} = \frac{b_{g+1}}{a_{g+1}}$ with $(a_{g+1}, b_{g+1}) = 1, a_{g+1} > 1.$

(6) $a_{g+1} = \min \{ n \in \mathbb{N}^+ / n\nu(P_{g+1}) \in \sum_{0}^{g} \mathbb{Z}v(P_i) \}.$

We start the proof of the Theorem by stating and proving the following lemmas.

**Lemma 4.2.** Suppose that $a$ and $b$ are positive integers and that $(a, b) = 1.$ Then $\forall n \in \mathbb{N}, \exists a$ solution to the equation $ia + jb = ab + n$ with $i, j \in \mathbb{N}$ and $0 \leq j < a.$

**Proof.** $(a, b) = 1 \Rightarrow \exists \alpha, \beta \in \mathbb{Z}$ such that $a\alpha - b\beta = \pm 1$

$\Rightarrow \exists i', j' \in \mathbb{Z}$ such that $i'a + j'b = ab + n.$

Thus by the Division Algorithm we can write:

$i' = mb + s; \text{ with } 0 \leq s < b$ and $m \in \mathbb{Z}$

$j' = pa + r; \text{ with } 0 \leq r < a$ and $p \in \mathbb{Z}$

$\Rightarrow (mb + s)a + (pa + r)b = ab + n$

$\Rightarrow (mb + s + pb)a + rb = ab + n$ and $0 \leq j = r < a$

Now, we have: $(m + p)ba = ab + n - rb - sa > ab + n - ab - ab > n - ab \geq -ab$

$\Rightarrow m + p > -1$ and $m, p \in \mathbb{Z} \Rightarrow m + p > 0,$ thus $mb + s + pb = (m + p)b + s \geq 0$

Let $i = mb + s + pb$ and $j = r.$ We have $ia + jb = ab + n$ with $i, j \in \mathbb{N}$ and $0 \leq j < a.$ \hfill $\square$

**Lemma 4.3.** Assume that $\tilde{\beta}_i \in \mathbb{Q}$ for $i = 0, \ldots, k + 1$ are given so that for all $j = 1, \ldots, k$ we have that $\tilde{\beta}_{j+1} > a_j\tilde{\beta}_j$ with $a_j = \min \{ n \in \mathbb{N} ; n\tilde{\beta}_j \in \sum_{i=0}^{j-1} \mathbb{Z}\tilde{\beta}_i \}.$ Then for any $j = 1, \ldots, k$ there exists a (unique) decomposition $a_j\tilde{\beta}_j = \sum_{i=0}^{j-1} m_{j,i}\tilde{\beta}_i$ where $0 \leq m_{j,l} < a_l$ for $l = 1, \ldots, j - 1.$

**Proof.** Since the $P_i$ are in Weierstrass form, then by (4) of definition 1.6, there exists $m_l \in \mathbb{Z},$ with $a_l\tilde{\beta}_j = \sum_{i=0}^{j-1} m_{i}\tilde{\beta}_i.$ By the Euclidean division, $m_{j-1} = a_{j-1} a_{j-1} + r_{j-1}$ with $0 \leq r_{j-1} < a_{j-1},$ so since $a_{j-1}\tilde{\beta}_{j-1} \in \sum_{i=0}^{j-2} \mathbb{Z}\tilde{\beta}_i,$ we can assume that $0 \leq m_{j-1} < a_{j-1}.$ Inductively, we get $0 \leq m_l < a_l$ for $l = 1, l < j.$ We still need to show that $m_0 \geq 0.$ Let $S = \sum_{i=1}^{j-1} m_l\tilde{\beta}_i.$ We have $m_1\tilde{\beta}_1 < a_1\tilde{\beta}_1 < \tilde{\beta}_2$ and $m_2 < a_2,$ so $m_2 + 1 \leq a_2$ and $m_1\tilde{\beta}_1 + m_2\tilde{\beta}_2 < (1 + m_2)\tilde{\beta}_2 \leq a_2\tilde{\beta}_2 < \tilde{\beta}_3 \Rightarrow S < (1 + m_{j-1})\tilde{\beta}_j < a_{j-1}\tilde{\beta}_{j-1} < \tilde{\beta}_j$ so that $m_0\tilde{\beta}_0 \geq \tilde{\beta}_j - S \geq 0.$ \hfill $\square$

**Lemma 4.4.** Let notation be as in lemma 4.3.

If $x \in \sum_{i=0}^{g} \mathbb{Z}\tilde{\beta}_i$ and $x \geq a_g\tilde{\beta}_g.$ Then there exists a unique representation

$$x = \sum_{j=0}^{g} m_{j}\tilde{\beta}_j$$

(4.4)
with integer coefficients $0 \leq m_j < n_j$ for $1 \leq j \leq g$.

**Proof.** By assumption $x \in \sum_{i=0}^g \mathbb{Z} \tilde{\beta}_i \Rightarrow \exists n_i \in \mathbb{Z}/x = \sum_{i=0}^g n_i \tilde{\beta}_i$

By Division algorithm $\exists q_g, r_g \in \mathbb{Z}$ such that $n_g = a_g q_g + r_g$ with $0 \leq r_g < a_g$.

Also by lemma 4.3 $\exists \tilde{\beta}_g$ such that $a_g \tilde{\beta}_g = \sum_{i=0}^{g-1} l_i \tilde{\beta}_i \Rightarrow n_g \tilde{\beta}_g = q_g \sum_{i=0}^{g-1} l_i \tilde{\beta}_i + r_g \tilde{\beta}_g$

$\Rightarrow x = \sum_{i=0}^{g-1} m_i \tilde{\beta}_i + r_g \tilde{\beta}_g$ for some $m_i \in \mathbb{Z}$ and with $r_g < a_g$.

Inductively we get $x = \sum_{i=0}^g m_i \tilde{\beta}_i$ with $0 \leq m_i < a_i$ for $1 \leq i \leq g$. We still need to check that $m_0 \geq 0$.

Let $S = \sum_{i=1}^g m_i \tilde{\beta}_i$ then we have $m_1 \tilde{\beta}_1 < a_1 \tilde{\beta}_1 < \tilde{\beta}_2$ and $m_2 < a_2$, so $m_2 + 1 \leq a_2$ and $m_1 \tilde{\beta}_1 + m_2 \tilde{\beta}_2 < (1 + m_2) \tilde{\beta}_2 \leq a_2 \tilde{\beta}_2 < \tilde{\beta}_3 \Rightarrow S < (1 + m_{g-1}) \tilde{\beta}_{g-1} \leq a_g \tilde{\beta}_g$ so that $m_0 \tilde{\beta}_0 = x - S \geq a_g \tilde{\beta}_g - a_g \tilde{\beta}_g = 0$, since $x \geq a_g \tilde{\beta}_g$.

$\Rightarrow m_0 \geq 0.$

□

**Proof.** of Theorem 4.1

The proof is by induction on $g$

For $g = 0$ it follows by the construction of $\tilde{Y}$.

For $g = 1$; $P_0 = X$, $P_1 = \tilde{Y}$

By assumption (4) for $g = 0$ of Theorem 4.1 $\nu_{r_2}(X) = \frac{b_1}{a_1} , $ such that $(a_1, b_1) = 1$ and $a_1 > 1$ , thus by lemma 3.1 and remark 3.2, there exists an expansion:

\[
X = X_{r_2}^{a_1} (Y_{r_2} + c)^{a_1} \\
\tilde{Y} = X_{r_2}^{b_1} (Y_{r_2} + c)^{b_1} \tag{4.5}
\]

Where $b'_1 a_1 - a'_1 b_1 = \pm 1, 0 \neq c \in k$ and $(X_{r_2}, Y_{r_2})$ are regular parameters of $R_{r_2}$ with $\nu_{r_2}(X_{r_2}) = 1$.

In the following we will show that for the case $g = 1$ the respective properties (1)-(6) of Theorem 4.1 are satisfied;

(1) $\nu_{r_2}(P_0) = \nu_{r_2}(X) = a_1$

$\nu_{r_2}(P_1) = \nu_{r_2}(\tilde{Y}) = b_1$

This follows from remark 3.2.

We also see from (4.5) that $P_i = X_{r_2}^{\nu_{r_2}(P_i)} . u_i$ where $u_i$ is a unit in $R_{r_2}$ for $i = 0, 1$.

(2) $\nu_{r_2}(P_0) = a_1 = a_1 . 1 = a_1 \nu_{r_2}(X)$

(3) Construction of $P_2$: 

Let $\theta = c^{a_1 b_1'} - a_1 b_1' \in k$ where $c$ comes from (4.5).
\[ \hat{Y}^{a_1} - \theta X^{b_1} = X^{a_1 b_1'} Y_2 + (Y_2 + c)^{a_1 b_1'} - \theta (Y_2 + c)^{b_1 a_1'} \]

Set $\Psi(Y_{r_2}) = Y_{r_2} + (c)^{a_1 b_1'} - \theta (Y_{r_2} + c)^{b_1 a_1'} = c^{a_1 b_1'} - \theta c^{b_1 a_1'} + c^{a_1 b_1'-1} [a_1 b_1' - a_1' b_1] Y_{r_2} + \text{higher order terms in } Y_{r_2}.

\[ \Psi(Y_{r_2}) = \pm c^{a_1 b_1'-1} Y_{r_2} + \text{ higher order terms of } Y_{r_2} \quad (4.6) \]

since $\nu$ dominates $R_{r_2} \Rightarrow \nu(\Psi(Y_{r_2})) > 0 \Rightarrow
\[ \nu(\hat{Y}^{a_1} - \theta X^{b_1}) > \nu(X_{r_2}^{a_1 b_1}) = \nu_1(X). \quad (4.7) \]

Suppose that $\nu(\hat{Y}^{a_1} - \theta X^{b_1}) \in \mathbb{Z} \nu(X) + \mathbb{Z} \nu(Y) \Rightarrow
\exists n_1, n_2 \in \mathbb{Z}$ such that $\nu(\hat{Y}^{a_1} - \theta X^{b_1}) = n_1 \nu(X) + n_2 \nu(Y)
= (n_1 a_1 + n_2 b_1) \frac{\nu(X)}{a_1}$

By (4.7) $\Rightarrow (n_1 a_1 + n_2 b_1) \frac{\nu(X)}{a_1} > b_1 \nu(X)$
$\Rightarrow n_1 a_1 + n_2 b_1 > b_1$ $\Rightarrow$ by lemma 4.2 $\exists i_0, i_1 \in \mathbb{N}$ such that $i_1 < a_1$ and
$i_0 a_1 + i_1 b_1 = n_1 a_1 + n_2 b_2$
$\nu(P_0^{i_0} P_1^{i_1}) = i_0 \nu(P_0) + i_1 \nu(P_1) = i_0 \nu(X) + i_1 \nu(X) \frac{b_1}{a_1} = (i_0 a_1 + i_1 b_1) \frac{\nu(X)}{a_1}$
$= \frac{\nu(X)}{a_1} (n_1 a_1 + n_2 b_2) = \nu(\hat{Y}^{a_1} - \theta X^{b_1})$
Since $V/m_{\nu} \cong k \Rightarrow \exists \lambda \in k$ such that $\nu(\hat{Y}^{a_1} - \theta X^{b_1}) - \lambda P_0^{i_0} P_1^{i_1} > \nu(\hat{Y}^{a_1} - \theta X^{b_1})$

Set $m_{1,0} = b_1$
Iterate the previous construction to construct

\[ P_2 = P_1^{a_1} - \theta P_0^{m_{1,0}} - \sum_{l_1} \lambda_{11} P_0^{m_{1,0}} P_1^{m_{1,1}} \quad (4.8) \]

Such that $\nu(P_2) \notin \mathbb{Z} \nu(X) + \mathbb{Z} \nu(Y)$.

The construction terminates after a finite number of iterations since $\Gamma$ is not-discrete.

We have $P_0^{m_{1,0}} P_1^{m_{1,1}} \equiv X_{r_2}^{m_{1,0} a_1 + m_{1,1} b_1} u_0^{m_{1,0}} u_1^{m_{1,1}}$
with $m_{1,0} \nu(P_0) + m_{1,1} \nu(P_1) > \nu(\hat{Y}^{a_1}) = \nu(X^{b_1}) = a_1 b_1 \nu(X_{r_2})
\Rightarrow m_{1,0} a_1 + m_{1,1} b_1 > a_1 b_1, \forall I_1 = (m_{1,0}, m_{1,1})$ in (4.8).
A Geometric Construction of Minimal Generating Sequences

Set \( \tilde{Y}_{r_2} = \Psi(Y_{r_2}) = \sum_{i_1} \lambda_{i_1} X_{r_2}^{m_{i,0}a_1 + m_{i,1}b_1 - a_1b_1} u_{i_1} \)
then by (4.6), \( X_{r_2}, \tilde{Y}_{r_2} \) are a regular system of parameters of \( R_{r_2} \) and \( P_2 = X_{r_2} a_1b_1 \tilde{Y}_{r_2} = X_{r_2}^{a_1b_1} \tilde{Y}_{r_2} \Rightarrow (3) \) of Theorem 4.1 holds.

By construction, \( P_2 \) satisfies conditions (1), (2), (4) and (5) of definition 1.6, and since \( m_{i,0} = b_1 \) and \( (a_1, b_1) = 1 \) \( \Rightarrow \) the order of \( \nu(P_0^0) \) in \( \Gamma_0/a_1 \nu(P_0) \cong \nu(P_0)/a_1 \nu(P_0) / \mathbb{Z} \) is \( a_1 \) \( \Rightarrow \) (3) of definition 1.6 holds for \( j = 1 \). Thus \( P_0, P_1 \) and \( P_2 \) are in Weierstrass form.

(4). Notice that \( \mathbb{Z} \nu(X) + \mathbb{Z} \nu(\tilde{Y}) = \mathbb{Z} a_1 \nu(X_{r_2}) + \mathbb{Z} b_1 \nu(X_{r_2}) = \mathbb{Z} \nu(X_{r_2}) \) since \( (a_1, b_1) = 1 \).

(5). \( \nu(\tilde{Y}_{r_2}) = \nu(P_2) - a_1b_1 \nu(X_{r_2}) \) and by (4.8) \( \nu(P_2) \not\in \mathbb{Z} \nu(P_0) + \mathbb{Z} \nu(P_1) \Rightarrow \nu(P_2) \not\in \mathbb{Z} a_1 \nu(X_{r_2}) + \mathbb{Z} b_1 \nu(X_{r_2}) \Rightarrow \nu(\tilde{Y}_{r_2}) \not\in \mathbb{Z} \nu(X_{r_2}) \)
\( \Rightarrow \exists a_2, b_2 \in \mathbb{N} \) such that \( \nu(\tilde{Y}_{r_2}) = \frac{b_2}{a_2} \) with \( (a_2, b_2) = 1 \) and \( a_2 > 1 \).

(6). Also, \( a_2 \nu(P_2) = \nu(P_2^{a_2}) = \nu(X_{r_2}^{a_1b_1a_2} \tilde{Y}_{r_2}^{a_2}) = \nu(X_{r_2}^{a_1b_1}) b_2 a_2 + \nu(\tilde{Y}_{r_2}^{a_2}) = \nu(X_{r_2}^{a_1b_1}) b_2 a_2 + \nu(\tilde{Y}_{r_2}^{a_2}) \)
\( \Rightarrow a_2 \nu(P_2) = b_2 a_2 \nu(X_{r_2}) + b_2 a_1 x \nu(X_{r_2}) + b_1 y \nu(X_{r_2}) = (b_1 a_2 + b_2 x) \nu(X_{r_2}) + y \nu(\tilde{Y}_{r_2}) \)
\( \Rightarrow a_2 \nu(P_2) = b_2 a_1 x \nu(X_{r_2}) + n_2 \nu(\tilde{Y}_{r_2}) \)
\( \Rightarrow a_2 \nu(P_2) = n_2 \nu(X_{r_2}) + b_2 a_1 x \nu(X_{r_2}) + n_2 b_1 \nu(X_{r_2}) \)
\( \Rightarrow a_2 \nu(P_2) = n_2 (a_1 x + b_2 a_1 b_1) \delta_2 = \frac{b_2}{a_2} \)
\( \Rightarrow a_2 \leq \delta_2 \) since \( (a_2, b_2) = 1 \).

Suppose that the theorem holds for \( g - 1 \).
So we have \( \{P_0, P_1, \ldots, P_{g-1}\} \) a sequence of polynomials in \( R \) in Weierstrass form, \( P_0 = X, P_1 = \tilde{Y} \) and \( P_i = X_{r_i}^{a_i} \tilde{Y}_{r_i}^{b_i} \) \( \Rightarrow \) \( u_i \) in \( R_{r_i} \), where \( (X_{r_i}, \tilde{Y}_{r_i}) \) are regular parameters and \( u_i \) is a unit in \( R_{r_i} \) for \( 0 \leq i \leq g - 1 \).
Also, properties (1)-(4) hold for \( g \) i.e.:

(1)
\[ \nu_{r_i}^{a_i}(P_0) = a_1 a_2 \ldots a_{g-1} \quad (4.9) \]
\[ \nu_{\gamma_g}^{i}(P_i) = a_2...a_{g-1}b_i \]  
\[ \nu_{\gamma_g}^{i}(P_i) = a_{i-1}\nu_{\gamma_g}^{i}(P_{i-1}) + a_{i+1}...a_{g-1}b_i \text{ for } 2 \leq i \leq g-1, \]

Where we define \( a_j...a_g = 1 \) for \( j > g \).

2. \( \nu_{\gamma_g}^{i}(P_i) = a_{g-1}\nu_{\gamma_g}^{i-1}(P_i) \) for \( j \leq g - 2 \)

3. \[ P_g = X_{\gamma_g}^{\nu_{\gamma_g}^{g}}(P_g) \text{ and } \nu_{\gamma_g}^{g}(P_g) = a_{g-1}\nu_{\gamma_g}^{g-1}(P_{g-1}) \]

3. There are positive integers \( a_g \) and \( b_g \) defined by

\[ \frac{\nu(Y_{\gamma_g}^{b_g})}{\nu(X_{\gamma_g}^{a_g})} = \frac{b_g}{a_g} \text{ with } (a_g, b_g) = 1, a_g > 1. \text{ and } \sum_{i=0}^{g-1} \mathbb{Z}\nu(P_i) = \mathbb{Z}\nu(X_{\gamma_g}^{a_g}). \]

Furthermore \( a_g = \min\{n \in \mathbb{N}^+/n\nu(P_g) \in \sum_{i=0}^{g-1} \nu(P_i)\} \)

Thus by lemma 3.1 and remark 3.2 there exists an expansion:

\[ X_{\gamma_g}^{a_g} = X_{\gamma_g}^{a_g} \left( Y_{\gamma_g}^{b_g} + \alpha \right)^{a_g} \]
\[ Y_{\gamma_g}^{b_g} = X_{\gamma_g}^{b_g} \left( Y_{\gamma_g}^{a_g} + \alpha \right)^{b_g} \]

Where \( a_g b_g - a_g' b_g' = \pm 1, 0 \neq \alpha \in k \) and \( \left( X_{\gamma_g}^{b_g}, Y_{\gamma_g}^{a_g} \right) \) are regular parameters in \( R_{\gamma_g+1} \) and \( \nu_{\gamma_g+1}(Y_{\gamma_g+1}) = 1. \)

We will show that for \( g + 1 \) the conclusions of Theorem 4.1 are satisfied:

We have from (4.14):

\[ P_i = X_{\gamma_g+1}^{a_g^{\nu_{\gamma_g}^{i}}(P_i)} \cdot (Y_{\gamma_g+1}^{b_g} + \alpha)^{a_g^{\nu_{\gamma_g}^{i}}(P_i)} (c_i + X_{\gamma_g+1} \Omega_i) \]

Where \( 0 \neq c_i \in k \) and \( \Omega_i \) is in \( R_{\gamma_g+1} \) for \( 0 \leq i \leq g - 1. \)

Thus \( P_i = X_{\gamma_g+1}^{a_g^{\nu_{\gamma_g}^{i}}(P_i)} \cdot u_i' \) where \( u_i' \) is a unit in \( R_{\gamma_g+1} \)

and \( \nu_{\gamma_g+1}(P_i) = a_g^{\nu_{\gamma_g}^{i}}(P_i) \)

\[ P_g = X_{\gamma_g+1}^{a_g^{\nu_{\gamma_g}^{g}}(P_g)} \cdot (Y_{\gamma_g+1}^{b_g} + \alpha)^{a_g^{\nu_{\gamma_g}^{g}}(P_g) + b_g} \]

and thus \( \nu_{\gamma_g+1}(P_g) = a_g^{\nu_{\gamma_g}^{g}}(P_g) + b_g \)

\[ P_g = X_{\gamma_g+1}^{a_g^{\nu_{\gamma_g}^{g}}(P_g) + b_g} \cdot u_g \] where \( u_g \) is a unit in \( R_{\gamma_g+1} \)
Thus

\[ \nu_{g+1}(P_0) = a_g \nu_g(P_0) = a_g(a_1...a_{g-1}) \text{ by (4.9)} \]

Thus \( \nu_{g+1}(P_0) = a_1a_2...a_g \)

\[ \nu_{g+1}(P_1) = a_g \nu_g(P_1) = a_g(a_2...a_{g-1}b_1) \text{ by (4.10)} \]

Thus \( \nu_{g+1}(P_1) = a_2...a_gb_1 \)

\[ \nu_{g+1}(P_i) = a_g \nu_g(P_i) = a_g(a_{i-1}...a_{g-1}b_{i-1}) \text{ by (4.11)} \]

Thus \( \nu_{g+1}(P_i) = a_{i-1}a_{i-2}...a_{g-1}b_{i-1} \) for \( 2 \leq i \leq g-1 \)

\[ \nu_{g+1}(P_g) = b_g + a_g \nu_g(P_g) = b_g + a_g a_{g-1}...a_1b_1 \text{ by (4.12)} \]

Thus \( \nu_{g+1}(P_g) = a_1a_2...a_gb_1 \) for \( 2 \leq i \leq g \) with \( a_j...a_g = 1, \forall j > g \).  

(2) \( \nu_{g+1}(P_j) = a_g \nu_g(P_j) \) for \( 0 \leq j < g - 1 \) by (4.15).

(3) Constructing \( P_{g+1} \):

By (4.13), \( a_g = \min \{ n \in \mathbb{N}^+ / n \nu(P_g) \in \sum_{i=0}^{g-1} \mathbb{Z} \nu(P_i) \} \)
\( \Rightarrow \exists n_i \in \mathbb{Z} \) such that \( a_g \nu(P_g) = \sum_{i=0}^{g-1} n_i \nu(P_i) \)
\( \Rightarrow \) by lemma 4.3 (with \( \beta_j = \nu(P_j) \) so that \( n_j = a_j \exists m_{g,i} \in \mathbb{N} \) such that \( m_{g,i} < a_i \) for \( i = 1, ..., g-1; m_{g,0} \geq 0 \) and \( a_g \nu(P_g) = \sum_{i=0}^{g-1} m_{g,i} \nu(P_i) \)
\( \Rightarrow \nu(P_{g+1}^{a_g}) = \nu(\prod_{i=0}^{g-1} P_i^{m_{g,i}}). \)
\( \Rightarrow \)

\[ a_g \nu_{g+1}(P_g) = \sum_{i=0}^{g-1} m_{g,i} \nu_{g+1}(P_i), \] (4.16)
since we have verified that $\nu(P_i) = \nu(X_{r_{i+1}})\nu_{r_{i+1}}(P_i)$ for $0 \leq i \leq g$.

This implies

$$a'_g\nu_{r_{g+1}}(P_g) = \sum_{i=0}^{g-1} \frac{a'_g}{a_g} m_{g,i}\nu_{r_{g+1}}(P_i) = \sum_{i=0}^{g-1} \frac{a'_g}{a_g} m_{g,i}\nu_{r_{g+1}}(P_i) \quad (4.17)$$

Let $\lambda_g = \alpha a_{b_g} - a'_g b_g \prod_{i=0}^{g-1} c_i^{-m_{g,i}} \in k$ where $\alpha$ comes from (4.14), and the $c_i$ come from (4.15).

$$P_{g} = \lambda_g \prod_{i=0}^{g-1} P_{i}^{m_{g,i}} = X_{r_{g+1}}(P_g) \left( Y_{r_{g+1}} + \alpha \right) a'_g(P_g) + \lambda_g \prod_{i=0}^{g-1} X_{r_{g+1}}(P_i) \prod_{j=0}^{g-1} (c_i + X_{r_{g+1}}(P_i))^{m_{g,i}}$$

by (4.14)

$$P_{g} = \lambda_g \prod_{i=0}^{g-1} (c_i + X_{r_{g+1}}(P_i))^{m_{g,i}} \quad (4.18)$$

Let $\Psi(Y_{r_{g+1}}) = (Y_{r_{g+1}} + \alpha) a'_g \nu_{r_{g+1}}(P_g) \left( (Y_{r_{g+1}} + \alpha) a_{b_g} - a'_g b_g \prod_{i=0}^{g-1} c_i^{-m_{g,i}} \right)$

by (4.16) and (4.17).

$$\Psi(Y_{r_{g+1}}) = (Y_{r_{g+1}} + \alpha) a'_g \nu_{r_{g+1}}(P_g) \left( (Y_{r_{g+1}} + \alpha) a_{b_g} - a'_g b_g \prod_{i=0}^{g-1} c_i^{-m_{g,i}} \right) \prod_{j=0}^{g-1} (c_i + X_{r_{g+1}}(P_i))^{m_{g,i}}$$

Where $\Phi_g$ is a unit in $R_{r_{g+1}}$ and $\Omega_2$ is in $R_{r_{g+1}}$.

Since $\nu$ dominates $R_{r_{g+1}} \Rightarrow \nu(\Psi(Y_{r_{g+1}})) > 0$

$\Rightarrow \nu(P_{g} - \lambda_g \prod_{i=0}^{g-1} P_{i}^{m_{g,i}}) = a_g \nu_{r_{g+1}}(P_g) \nu(X_{r_{g+1}}) = a_g \nu(P_g)$. 

Suppose that $\nu(P_{g} - \lambda_g \prod_{i=0}^{g-1} P_{i}^{m_{g,i}}) \in \sum_{0}^{g} \nu(P_i)$

Thus by lemma 4.4 with $\beta_g = \nu(P_g)$, $m'_{g,i} \in \mathbb{N}, i = 0, \ldots, g$ such that $m'_{g,1} < a_i$

for $1 \leq i \leq g$ such that $\nu(P_{g} - \lambda_g \prod_{i=0}^{g-1} P_{i}^{m_{g,i}}) = \sum_{0}^{g} m'_{g,i} \nu(P_i) = \nu(\prod_{i=0}^{g} P_{i}^{m'_{g,i}})$

Since $V/\nu \cong k \Rightarrow \exists \lambda_g, 1$ such that

$\nu(P_{g} - \lambda_g \prod_{i=0}^{g-1} P_{i}^{m_{g,i}}) > \nu(P_{g} - \lambda_g \prod_{i=0}^{g-1} P_{i}^{m'_{g,i}})$
We iterate the previous construction to construct $P_{g+1}$ such that $\nu(P_{g+1}) \notin \sum_{i=0}^{g} \mathbb{Z}\nu(P_i)$.

Since $\Gamma$ is not discrete, the construction terminates after a finite number of iterations producing:

$$P_{g+1} = P_g - \lambda_g \prod_{i=0}^{g-1} P_i^{m_{g,i}} - \sum_{i=0}^{g} \lambda_g \prod_{i=0}^{g} P_i^{m_{g,i}}$$

For the following calculation, if we assume that the construction terminated after $l$ finite iterations, then we may consider the following expression for $P_{g+1}$:

$$P_{g+1} = P_g - \lambda_g \prod_{i=0}^{g-1} P_i^{m_{g,i}} - \sum_{i=0}^{l} \lambda_g \prod_{i=0}^{g} P_i^{m_{g,i}}$$

We have

$$\prod_{i=0}^{g} P_i^{m_{g,i}} = \prod_{i=0}^{g-1} [X_{r_{g+1}}^{\alpha_g \nu_{r_{g+1}}(P_i)} (Y_{r_{g+1}} + \alpha)^{a_g \nu_{r_{g+1}}(P_i)} (c_i + X_{r_{g+1}}^{\nu_{r_{g+1}}(P_i)})]^{m_{g,i}}$$

$$\times (Y_{r_{g+1}} + \alpha)^{a_g \nu_{r_{g+1}}(P_i)} (c_i + X_{r_{g+1}}^{\nu_{r_{g+1}}(P_i)})^{m_{g,i}}.$$

For all $n = 1, \ldots, l$.

We also have:

$$\nu(\prod_{i=0}^{g} P_i^{m_{g,i}}) = \nu(P_g - \lambda_g \prod_{i=0}^{g-1} P_i^{m_{g,i}} - \sum_{k=0}^{n-1} \lambda_g \prod_{i=0}^{g} P_i^{m_{g,i}}) > a_g \nu(P_g) = a_g \nu_{r_{g+1}}(P_g) \nu(X_{r_{g+1}}^{\nu_{r_{g+1}}(P_i)}).$$

Thus $\sum_{i=0}^{g} m_{g,i} \nu_{r_{g+1}}(P_i) > a_g \nu_{r_{g+1}}(P_g); \forall n = 1, \ldots, l$.

Thus $\prod_{i=0}^{g} P_i^{m_{g,i}} = X_{r_{g+1}}^{\sum_{i=0}^{g} m_{g,i} \nu_{r_{g+1}}(P_i)} u_g$ where $u_g$ is a unit in $R_{r_{g+1}}$, for all $n = 1, \ldots, l$.

Set $Y_{g+1} = \Psi(Y_{g+1}) - \sum_{n=1}^{l} \lambda_g \prod_{i=0}^{g} X_{r_{g+1}}^{\sum_{i=0}^{g} m_{g,i} \nu_{r_{g+1}}(P_i)} - a_g \nu_{r_{g+1}}(P_g) u_g$.

then by (4.18) $Y_{g+1}$, $X_{g+1}$ are regular parameters of $R_{g+1}$ and

$$P_{g+1} = X_{r_{g+1}}^{a_g \nu_{r_{g+1}}(P_g)} Y_{g+1}.$$  (4.19)

By construction, $P_{g+1}$ satisfies (1),(2),(4) and (5) of definition 1.6 and if $n \leq a_g$ is the order of $\nu(\prod_{i=0}^{g-1} P_i^{m_{g,i}})$ in $\Gamma_{g-1}/a_g \Gamma_{g-1}$, then $na_g \nu(P_g) \in a_g \Gamma_{g-1}$

$\Rightarrow n\nu(P_g) \in \Gamma_{g-1} \Rightarrow n = a_g$ by (4.13) and (3) of definition 1.6 holds, and thus
Let \( P_0, \ldots, P_{g+1} \) are in Weierstrass form.

Notice that by induction on \( g \): \( \sum_{i=0}^{g} Z \nu(P_i) = Z \nu(X_{r_{g+1}}) + Z \nu(P_g) \).

We have \( X_{r_g} = X_{r_{g+1}} + Y + a \alpha \), \( Y + b \gamma \), where \( \nu \) is the conditions of the theorem. In the following, we will show that such a sequence

\[
\nu(x_{r_{g+1}}) = a_g \nu(x_{r_{g+1}}(P_{g-1}) + b_g \nu(x_{r_{g+1}}(P_{g-1} + b_g) \nu(x_{r_{g+1}})
\]

Iterating this process we get

\[
\nu(\tilde{\gamma}^{\nu} + 1, ...) P_{g+1} = a_g \nu(\tilde{x}_{r_{g+1}}(P_{g-1}) + b_g \nu(\tilde{x}_{r_{g+1}}(P_{g-1} + b_g) \nu(\tilde{x}_{r_{g+1}})
\]

\( \nu(\tilde{\gamma}^{\nu}) = a_g \nu(\tilde{x}_{r_{g+1}}(P_{g-1}) + b_g \nu(\tilde{x}_{r_{g+1}}(P_{g-1} + b_g) \nu(\tilde{x}_{r_{g+1}})
\]

since \( (a_g, b_g) = 1 \) we have \( \sum_{i=0}^{g} Z \nu(P_i) = Z \nu(X_{r_{g+1}}) \).

(5) \( \nu(\tilde{\gamma}^{\nu}) = a_g \nu(x_{r_{g+1}}(P_{g-1}) + b_g \nu(x_{r_{g+1}}) \) and \( \nu(P_{g+1}) \notin \sum_{i=0}^{g} Z \nu(P_i) = Z \nu(X_{r_{g+1}}) \Rightarrow \nu(\tilde{\gamma}^{\nu}) \notin \nu(\tilde{x}_{r_{g+1}}). \) Thus \( a_{g+1} > 1 \).

(6) If \( \exists \delta_{g+1} \in \mathbb{N} \) such that \( \delta_{g+1} \nu(P_{g+1}) \in \sum_{i=0}^{g} Z \nu(P_i) \Rightarrow \exists n_0, ..., n_g \in \mathbb{Z} \) such that

\[
\delta_{g+1} \nu(P_{g+1}) = \sum_{i=0}^{g} n_i \nu(P_i)
\]

\( \Rightarrow \delta_{g+1} \nu(P_{g+1}) \in (0) \nu(x_{r_{g+1}}(P_{g-1}) + \delta_{g+1} \nu(\tilde{x}_{r_{g+1}}) = \sum_{i=0}^{g} n_i \nu(P_i)
\]

\( \Rightarrow \delta_{g+1} \nu(\tilde{x}_{r_{g+1}}) = \sum_{i=0}^{g} n_i \nu(x_{r_{g+1}}(P_{g-1}) + \delta_{g+1} \nu(\tilde{x}_{r_{g+1}}(P_{g-1}) + b_g \nu(\tilde{x}_{r_{g+1}}(P_{g-1} + b_g) \nu(\tilde{x}_{r_{g+1}})
\]

\( \Rightarrow \nu(\tilde{x}_{r_{g+1}}) = \sum_{i=0}^{g} n_i \nu(x_{r_{g+1}}(P_{g-1}) + \delta_{g+1} \nu(\tilde{x}_{r_{g+1}}) \nu(x_{r_{g+1}})
\]

\( \Rightarrow \delta_{g+1} \geq a_{g+1} \) since \( (a_{g+1}, b_{g+1}) = 1 \).

Thus we have obtained a sequence of polynomials \( \{P_j\} \) in Weierstrass form, satisfying the conditions of the theorem. In the following, we will show that such a sequence is a generating sequence of \( \nu \), and thus obtain the full statement of the theorem.

Let \( \{P_j\}_{j \geq 0} \) be the (infinite) set of polynomials obtained from our construction, and let \( \nu(P_j) = \tilde{\beta}_j \in \mathbb{Q} \), for \( 0 \leq j < \infty \).

Since \( \Gamma \) is not discrete we have that \( a_k > 1 \) for infinitely many \( k \); and thus \( \text{deg}_y P_k \to \infty \) as \( k \to \infty \).

Let \( \phi \in R \) then by lemma 1.7 \( \exists k \) such that \( \text{deg}_y \phi < d_{k+1} \).

Then there exists an expansion in \( \hat{R}, \phi = \sum_{i=1}^{d_k} \alpha_i P_0^i \ldots P_k^i \) with \( \alpha_i \in k, i_j < a_j, \forall j = 1, \ldots, k, \) \( i_0 > 0 \) and, \( \nu(\phi) = \min I \{ \sum_{j=0}^{k} i_j \tilde{\beta}_j \} \).

Proof. Since \( P_k \) is unitary in \( y \), write \( \phi = \sum_{i} \phi_i P_k^i \) with \( \text{deg}_y \phi_i < d_k \) for all \( i \).

Iterating this process we get \( \phi = \sum_{i} \alpha_i P_0^i \ldots P_k^i \); with \( \alpha_i \in k, i_j < a_j, \) for \( 1 \leq j \leq k \) and \( i_0 \geq 0 \).

Since \( \nu(P_0^i \ldots P_k^i) = \sum_{j=0}^{k} i_k \tilde{\beta}_j \), to show that \( \nu(\phi) = \min I \{ \sum_{j=0}^{k} i_j \tilde{\beta}_j \} \), it is enough to show that
if $\sum_{l=0}^{k} i_l \tilde{\beta}_l = \sum_{l=0}^{k} j_l \tilde{\beta}_l$ then $i_l = j_l, \forall l$

Assume that $\sum_{l=0}^{k} i_l \tilde{\beta}_l = \sum_{l=0}^{k} j_l \tilde{\beta}_l \Rightarrow |i_k - j_k| \tilde{\beta}_{k-1} \in \sum_{l=0}^{k-1} Z \tilde{\beta}_l \Rightarrow a_{k-1} |i_{k-1} - j_{k-1}|

but $i_{k-1}, j_{k-1} < a_{k-1} \Rightarrow i_{k-1} = j_{k-1}$

Iterating this process we show that $i_l = j_l$, for $1 \leq l \leq k$ and thus $i_0 = j_0$ and we are done.

It also follows by (4.20) that $I_{\gamma} := \{ \phi \in R/\nu(\phi) \geq \gamma \} = \langle \prod P_j^{a_j} / \sum a_j \nu(P_j) \geq \gamma \rangle$.

And thus by definition 1.1 $\{P_j\}_{j \geq 0}$ is a minimal generating sequence of $\nu$.

Remark 4.5. with the notation of Theorem 4.1 and by and by (3) of lemma 3.1 we have that $\nu(X_{r_{k+1}}') = \frac{\nu(P_0)}{a_1...a_k}$ and thus $\nu_{r_{k+1}}(\phi) = \frac{a_1...a_k}{\nu(P_0)} \nu(\phi)$, if $\phi \in R$ and $\text{deg}_y(\phi) < d_{k+1}$.
A GEOMETRIC CONSTRUCTION OF MINIMAL GENERATING SEQUENCES

REFERENCES