RESEARCH STATEMENT
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Ricci curvature plays an important role in understanding the relationship between the geometry and topology of Riemannian manifolds. Perhaps the most notable results in this direction is Perelman’s recent breakthroughs [P2, P3, P4] in applying Hamilton’s Ricci flow to prove the Thurston Geometrization Conjecture. Many of the ideas and techniques presented in Perelman’s work have found useful applications throughout geometric analysis in understanding other aspects of Riemannian manifolds. Given the rich theory involving Ricci curvature present in smooth manifolds, much work has been done within the last five years to extend the notion of Ricci curvature to the more general setting of metric measure spaces. By employing ideas from the theory of optimal transport, Lott-Villani [LV1, LV2] and Sturm [S1, S2] have independently proven a number of exciting results in this new field. Even more recently, attention has been focused to develop an understanding of Ricci flow in metric spaces. The notions of Ricci curvature and Ricci flow in this more general setting are of great importance to the mathematical community and currently much work has been done to better understand the qualitative consequences of these ideas.

The specific aims of this proposal are two fold. First, I intend to continue my current research describing how Ricci curvature and volume growth influence the topology of Riemannian manifolds and related metric measure spaces. I have already achieved some results in this area and I intend to take advantage of the thriving environment provided by the Geometric Analysis group at the Mathematics Institute of the University of Warwick to further these results. Second, I aim to investigate Ricci curvature as it is defined on metric measure spaces and develop the theory of Ricci flow in this more general setting. Recall that the Ricci flow is a process which deforms a manifold over time and it is possible to reach a singularity in finite time. By generalizing the notion of Ricci flow to these limit spaces, one hopes then to continue the flow through the singularity and examine the resulting behavior.

**Manifolds with Nonnegative Ricci Curvature.** Much work has been done to better understand the structure of manifolds with a lower bound on Ricci curvature; see [Ch2], [Co], [SS], [W], [Z] for a survey of these results. Over the last 20 years, this field has experienced tremendous development. Using comparison theorems, researchers have obtained many interesting results that relate geometric properties of these manifolds to the corresponding quantity in the model space. A lower bound on Ricci curvature has been shown to provide good control on the topology of the underlying Riemannian manifold, most notably on the fundamental group and first Betti number; (c.f. [SS] for results specifically involving the topology of manifolds with...
nonnegative Ricci curvature). My current research aims to precisely describe how the volume growth in manifolds with nonnegative Ricci curvature influences the topology of the manifold.

From a geometric standpoint, the Ricci curvature acts as a measure of volume distortion. In the presence of nonnegative Ricci curvature, the Bishop-Gromov Volume Comparison Theorem ([BC], [Gr]) states that 

\[ r \to \frac{\text{vol}(B_p(r))}{\omega_n r^n} \]

is a non-increasing function of \( r \); here \( \omega_n r^n \) denotes the volume of the Euclidean ball of radius \( r \). Note that on any manifold this ratio converges to 1 as \( r \) decreases to 0. If \( M^n \) has \( \text{Ric} \geq 0 \), then

\[ \alpha_M = \lim_{r \to \infty} \frac{\text{vol}(B_p(r))}{\omega_n r^n} \leq 1 \]

and if the limit equals 1, then \( M^n \) isometric to Euclidean space.

If \( \alpha_M > 0 \), then \( M^n \) is said to have large volume growth or Euclidean volume growth.

It was shown independently by Anderson [A] and Li [Li] that in manifolds with Euclidean volume growth the order of \( \pi_1(M^n) \) is bounded above by \( \frac{1}{\alpha_M} \), indicating that such manifolds have finite fundamental group. Anderson uses volume comparison arguments on the universal cover of the given manifold while Li proved this by using the heat equation on \( M^n \). More recently, in 1994, Perelman [P1] showed that there exists a small constant \( \epsilon_n > 0 \), depending only on the dimension of the manifold, such that if \( \alpha_M > 1 - \epsilon_n \), then \( M^n \) is contractible. This implies that all of the homotopy groups are trivial, i.e. \( \pi_k(M^n) = 0 \) for all \( k \geq 0 \). Later, this result was improved by Cheeger-Colding [ChCo1] who showed that, with tighter restrictions on \( \epsilon_n \), \( M^n \) is actually \( C^{1,\alpha} \) diffeomorphic to \( \mathbb{R}^n \). Neither Perelman nor Cheeger-Colding investigated an actual value for \( \epsilon_n \). In my thesis, I address this issue. Namely, I determine lower bounds for \( \alpha_M \) which imply that the \( k \)-th homotopy group is trivial, meaning that \( M \) has no holes of dimension \( k \) but might have higher dimensional holes.

I have iteratively determined precise constants \( c_{k,n} > 0 \) to prove the following:

**Theorem 1.** Let \( M^n \) be a \( n \)-dimensional, complete, open Riemannian manifold with \( \text{Ric} \geq 0 \). If \( \alpha_M \geq c_{k,n} \), then \( \pi_k(M^n) = 0 \).

The constants I have found are the strongest that can be found using Perelman’s techniques to construct the homotopies. Note that Menguy [M] has examples of 4-dimensional manifolds with large volume growth that have infinitely generated \( \pi_2(M^4) \), thus providing a lower bound for the best possible \( c_{2,4} \). It is already known by the Anderson and Li results mentioned above that the optimal \( c_{1,n} = 1/2 \).

In my thesis I am also examining manifolds for which \( \pi_1(M^n) \neq 0 \). As mentioned earlier, Anderson and Li have shown that if \( \lim_{r \to \infty} \frac{\text{vol}(B_p(r))}{\omega_n r^n} > \frac{1}{k} \), then \( \pi_1(M^n) \) has at most \( k \) elements. I am exploring manifolds with \( \lim_{r \to \infty} \frac{\text{vol}(B_p(r))}{\omega_n r^n} \geq \frac{1}{k} - \epsilon_{k,n} \), for small constants \( \epsilon_{k,n} \), that are known to have \( k \) elements in their fundamental group. Note that when \( k = 1 \), we are able to
compare with Perelman’s result in [P1]. In this preliminary case, Perelman has shown that the manifold is contractible. It is reasonable to assume that when \( k > 1 \), the manifold, though not contractible, could be contracted to a circle or some other compact submanifold. Such a result can be compared to the celebrated Cheeger-Gromoll soul theorem [ChGl1] for manifolds with nonnegative sectional curvature. In particular, I am in the process of proving the following:

**Conjecture 2.** Let \( M^n \) be an open Riemannian manifold with \( \text{Ric} \geq 0 \) and with \( k \) elements in the fundamental group. There exists a constant \( \epsilon_{k,n} \), depending only on \( n \) and \( k \), such that if \( \alpha_M > \frac{1}{k} - \epsilon_{k,n} \), then \( M^n \) has a soul.

Note that the assumption on volume growth cannot be avoided due to the well-known examples of Sha-Yang [SY1, SY2] which have nonnegative Ricci curvature and do not possess a soul. The proof of this conjecture and related ones concerning the various homotopy groups of the space, involves applying the techniques from the first paper of my thesis in an equivariant manner to the universal cover of the space.

With the results obtained by Cheeger and Colding [ChCo1] that describe the Gromov-Hausdorff limits of manifolds with non-negative Ricci curvature, these results can also be extended to metric spaces that are limits of manifolds satisfying the same restrictions on volume growth. The consequences I obtain on the topology do not require a smooth structure. In fact they only require two key ingredients from the Ricci curvature: a prevalence of geodesics in the presence of large volume and the Abresch-Gromoll excess estimate [AbGl]. The latter theorem was proven using supersolutions of an elliptic partial differential equation to control distances and this has been extended by Cheeger and Colding to limit spaces [ChCo1].

A natural extension of the work in my thesis would be to examine the other extreme: manifolds with nonnegative Ricci curvature and minimal rather than maximal volume growth:

**Question 1:** Investigate the topology of a manifold with \( \text{Ric} \geq 0 \) and linear volume growth.

These spaces and their limits were studied by my advisor in [So1, So2] using Cheeger-Colding methods applied to the Busemann function rather than the distance function. I plan to apply Perelman’s techniques to restrict the topology of these spaces. These techniques involve constructing radial nets as discussed in my thesis, while in this setting it would be more natural to apply the controls on geodesics to gradient curves of Busemann functions. I have already worked with Busemann functions in an earlier paper written with Jorgenson and Dan Garbin which examines the limiting behavior of certain analytic properties of Riemann surfaces through metric degeneration. I do not envision difficulties arising when pursuing this project.

**Notions of Ricci Curvature on Measured Length Spaces.** A compact measured length space \((X, d, \nu)\) is a compact metric space \((X, d)\) with an associated Borel probability measure \(\nu\) whose metric is the metric of a length space (so that it has geodesics). The metric measure
limits of Riemannian manifolds with nonnegative Ricci curvature lie in this class of spaces and, by studying their properties, Cheeger and Colding were able to prove a number of fundamental results about the Gromov-Hausdorff limits of manifolds with nonnegative Ricci curvature [ChCo1, ChCo2, ChCo3]. Naturally it is of interest to extend the concept of nonnegative Ricci curvature to this larger class of spaces, even those which are not the limits of manifolds under Gromov-Hausdorff convergence.

Ultimately, through recent work by Lott and Villani [LV1, LV2] and independently by Sturm [S1, S2], an idea for Ricci curvature bounded below has been developed for compact metric measure spaces. Interestingly, the definition is obtained via the theory of optimal transport. This relation of Ricci curvature to optimal transport can originally be found in the work of Cordero, Erausquin, McCann, and Schmuckenschlager [CMS], Otto and Villani [OV], and Sturm and von Renesse [SvR].

We now briefly develop the theory necessary to state the definition of a lower bound on Ricci curvature for compact metric measure spaces. It is possible to formulate the definition for noncompact length spaces as well.

Let $X$ be a compact Hausdorff space and let $P(X)$ denote the set of Borel probability measures on $X$. We equip the space $P(X)$ with a metric $W_2$ called the Wasserstein distance. For measures $\mu_0, \mu_1 \in P(X)$, define

$$W_2(\mu_0, \mu_1)^2 = \inf_{\pi} \int_{X \times X} d(x_0, x_1)^2 d\pi(x_0, x_1),$$

where the infimum is taken over all possible probability measures $\pi \in P(X \times X)$. Intuitively, this metric determines the minimal 'cost' of moving points $x_0 \in (X, \mu_0)$ to the points $x_1 \in (X, \mu_1)$. Equipped with this metric, the Wasserstein space $(P(X), W_2)$ is a contractible, compact metric space and generally has infinite Hausdorff dimension. Furthermore ([LV1], [S1]), if $(X, d)$ is a compact length space, so is $(P(X), W_2)$ and a Wasserstein geodesic is a minimizing geodesic in the length space $(P(X), W_2)$. Loosely speaking, a lower bound on Ricci curvature in a metric space can be described as a convexity of a certain entropy functional along a specific Wasserstein geodesic. We now make this notion more precise.

Let $U : [0, \infty) \to \mathbb{R}$ be a continuous convex function with $U(0) = 0$. For a given probability measure $\nu \in P(X)$, define the entropy function $U_\nu$ on $P(X)$ by

$$U_\nu(\mu) = \int_X U(\rho(x)) d\nu + U'(\infty) \mu_s(X),$$

where $\mu = \rho \nu + \mu_s$ is the Lebesgue decomposition of $\mu$ so that $\rho \nu$ is an absolutely continuous part and $\mu_s$ is a singular part with respect to $\nu$, and $U'(\infty) = \lim_{r \to \infty} \frac{U(r)}{r}$. This entropy function $U_\nu$ is minimized when $\mu = \nu$.  


As mentioned earlier, one motivation for defining Ricci curvature on metric spaces is to better understand the singular spaces that may arise as the metric evolves. In fact, spaces might collapse to a lower dimension and one would like to preserve the notion of the higher dimension while passing through the singular space. Thus, when computing Ricci curvature on metric spaces, it is necessary to specify the effective dimension $N$ which plays a necessary role in the definition; thus, our definition describes a lower bound on $N$-Ricci curvature.

For $N \in [1, \infty)$, set $DC_N = \{ U \mid \lambda^N U(\lambda^{-N}) \text{ is convex in } \lambda \text{ on } (0, \infty) \}$, and define

**Definition 3.** Given $N \in [1, \infty)$, a compact measured length space $(X, d, \nu)$ is said to have nonnegative $N$–Ricci curvature if for all $\mu_0, \mu_1 \in P(X)$ with $\text{supp}(\mu_0) \subset \text{supp}(\nu)$ and $\text{supp}(\mu_1) \subset \text{supp}(\nu)$, there is some Wasserstein geodesic $\{ \mu_t \}_{t \in [0,1]}$ from $\mu_0$ to $\mu_1$ so that for all $U \in DC_N$ and all $t \in [0,1],$

$$U_\nu(\mu_t) = t U_\nu(\mu_1) + (1-t) U_\nu(\mu_0).$$

Using this definition, Lott and Villani [LV1, LV2] and Sturm [S1, S2] were independently able to show that many results for Riemannian manifolds with nonnegative Ricci curvature have analogs in the more general setting of compact length spaces with nonnegative $N$-Ricci curvature. For example, they prove an analog of the Bishop-Gromov Volume Comparison Theorem which states that for a compact length space $(X, d, \nu)$ with nonnegative $N$-Ricci curvature and $x \in X$; if $0 < r_1 < r_2$, we have

$$\frac{\nu(B_x(r_2))}{\nu(B_x(r_1))} \leq \left( \frac{r_2}{r_1} \right)^N.$$

Furthermore, they also prove an analog of Gromov’s Precompactness Theorem among other fundamental theorems of classical Riemannian geometry.

It is also possible to prove analogs of results describing various analytic properties such as eigenvalue inequalities, Sobolev inequalities and local Poincare inequalities for compact length spaces with nonnegative $N$-Ricci curvature. These analytic properties provide useful tools when studying the structure of this class of length spaces.

**Question 2:** What known topological results about Riemannian manifolds with Ricci curvature bounded below by $K$ (for example, nonnegative Ricci curvature) extend to the class of measured length spaces with $N$-Ricci curvature bounded below by $K$ (resp. nonnegative $N$-Ricci curvature).

In particular, I would like to explore to what extent the Perelman techniques I’ve used in my thesis apply in this more general setting. These methods are geometrically based on elliptic supersolutions and the results are topological. It is reasonable to assume that these techniques extend to this larger class in a natural way. Peter Topping of the University of Warwick has used Perelman’s techniques in relation to optimal transport on metric measure spaces.
Ricci Flow on Measured Length Spaces. Once the notion of Ricci curvature is understood in this more general setting, it is natural to examine the possibility of applying other techniques such as Ricci flow to metric spaces. Both Hamilton and Lott have expressed interest in exploring the consequences to such a development. This was an important question discussed at the MSRI summer school program on Ricci Flow that I attended in 2005.

In a very recent paper McCann and Topping [MT] make significant progress in this direction, effectively providing the first result in the area. This paper considers a family of metrics \( g(\tau) \) on a compact, oriented \( n \)-dimensional Riemannian manifold which satisfy the backward-time Ricci flow equation given by

\[
\frac{\partial g}{\partial \tau} = 2 \text{Ric}(g).
\]

The authors examine two families of normalized \( n \)-forms \( \omega(\tau) \geq 0 \) and \( \tilde{\omega}(\tau) \geq 0 \) which evolve according to a forward-time heat equation on a compact oriented \( n \)-dimensional Riemannian manifold \( M \) defined by

\[
\frac{\partial \omega}{\partial \tau} = \Delta_{g(\tau)} \omega,
\]

where the \( \Delta_g \) denotes the connection Laplacian with respect to the metric \( g \). This technique was first introduced by Perelman in [P2] and is described in detail in Topping’s book on Ricci flow [T].

The \( n \)-forms \( \omega \) and \( \tilde{\omega} \) naturally induce measures \( \nu \) and \( \tilde{\nu} \), respectively, in the sense that \( \nu(A) = \int_A \omega \) for every Borel set \( A \subset M \), and similarly for \( \tilde{\omega} \). Naturally the definition of Wasserstein distance changes in this setting to one dependent on time \( \tau \). That is, we define

\[
W_2(\nu, \tilde{\nu}, \tau) = \inf_{\pi} \int_{M \times M} d(x,y,\tau)^2 d\pi(x,y),
\]

where \( d(x,y,\tau) \) denotes the geodesic distance between points \( x \) and \( y \) as determined by the metric \( g(\tau) \). As a consequence, of the main theorem in that paper, the authors demonstrate the monotonicity of the \( W_2 \) functional. Namely, they prove

**Proposition 4.** [MT] *On a compact oriented manifold \( M \), suppose a family of metrics \( g(\tau) \) satisfies the backward-time Ricci flow (6) on the some interval \( [\tau_1, \tau_2] \subset \mathbb{R} \) and that \( \omega(x,\tau) \geq 0 \) and \( \tilde{\omega}(x,\tau) \geq 0 \) are unit mass solutions to the diffusion equation (7). Then \( W_2(\omega(\tau), \tilde{\omega}(\tau),\tau) \) is a non-increasing function of \( \tau \in [\tau_1, \tau_2] \).*

The proof of this result relies on the notion of a supersolution to the Ricci flow. As defined, such a supersolution allows one to define precisely the notion of super Ricci flow, or even Ricci flow, on metric spaces where there exists a notion of diffusion even if the classical Ricci tensor is not defined:
Definition 5. A super Ricci flow (parameterized backwards in time) is a smooth family \( g(\tau) \) of metrics, \( \tau \in [\tau_1, \tau_2] \), such that at each \( \tau \in (\tau_1, \tau_2) \), and at each point \( M \), we have

\[
-\frac{\partial g}{\partial \tau} + 2\text{Ric}(g(\tau)) \geq 0
\]

Note that the variable \( \tau \) denotes reverse-time as compared to the time parameter \( t \). So, \( -\frac{\partial}{\partial \tau} \) is a derivative forwards in time \( t \). This reversal of time reflects the nature of entropy developed by Perelman in [P2].

**Question 3:** Can one use this new approach to define the Ricci flow through a singularity? What are the topological and geometric consequences?

**References**


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