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# O-MINIMALISM

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**Abstract.** This paper is devoted to o-minimalism, the study of the first-order properties of o-minimal structures. The main protagonists are the pseudo-o-minimal structures, that is to say, the models of the theory of all o-minimal $L$-structures, but we start with a more in-depth analysis of the well-known fragment DCTC (Definable Completeness/Type Completeness), and show how it already admits many of the properties of o-minimal structures: dimension theory, monotonicity, Hardy structures, and quasi-cell decomposition, provided one replaces finiteness by discreteness in all of these. Failure of cell decomposition leads to the related notion of a eukaryote structure, and we give a criterium for a pseudo-o-minimal structure to be eukaryote.

To any pseudo-o-minimal structure, we can associate its Grothendieck ring, which in the non-o-minimal case is a non-trivial invariant. To study this invariant, we identify a third o-minimalistic property, the Discrete Pigeonhole Principle, which in turn allows us to define discretely valued Euler characteristics. As an application, we study certain analytic subsets, called Taylor sets.

## CONTENTS



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§**1. Introduction.** O-minimality has been studied extensively (see [21] for some of the literature). It also has been generalized in many ways (weak o-minimality [4, 11], quasi-o-minimality [1], d-minimality [8], local o-minimality [20, 22], o-minimal open cores [6, 7, 15], etc.) These generalizations attempt to bring into the fold certain ordered structures that fail some of the good finiteness properties of o-minimality, but still behave "tamely". We offer a different perspective in this paper, where our point of departure is the observation that, in contrast to an ultra*power*, an ultra*product* of o-minimal structures need no longer be o-minimal; let us call it *ultra-o-minimal*. This leads to two natural questions: (i) under which conditions on the o-minimal components is an ultra-o-minimal structure again o-minimal? And (ii), what properties do ultra-o-minimal structures have? In Part 1, we attempt to answer (ii); in Theorem 11.20, we give an answer to (i) in terms of Euler characteristics.

Let $L$ be a language containing a binary predicate $<$, to be interpreted as a dense linear ordering. We call an $L$-structure $\mathcal{M}$ *pseudo-o-minimal*, if it is a model of $T^{\text{omin}} := T^{\text{omin}}(L)$, the collection of $L$-sentences that hold true in every o-minimal $L$-structure. A structure is pseudo-o-minimal if and only if it is an elementary substructure of an ultra-o-minimal structure. O-minimalism is in essence a non-standard feature, as any pseudo-o-minimal expansion of the reals is already o-minimal (Corollary 2.4). In the first part of this paper, we will focus on two elementary properties, *definable completeness* (=every definable subset has an infimum) and *type completeness*[1] (=every one-sided type of a point, including the ones at infinity, is complete). We denote by DCTC these axiom schemes on one-variable definable sets (where the dependence on parameters has to be quantified out to get sentences in the language $L$). In a recent preprint [17], Rennet shows that $T^{\text{omin}}$ is not recursive, whence it cannot be equal to DCTC, as the latter is recursive. It is not clear if we can axiomatize o-minimalism by first-order conditions on one-variable formulae only. For instance, another o-minimalistic property, the Discrete Pigeonhole Principle (DPP=any definable, injective map from a discrete set to itself is bijective; see §11.5 for details), is a priori a multi-variable condition, although Fornsasiero [7] has conjectured that it follows from DCTC.

We show that the weaker theory DCTC proves already many properties that resemble those of o-minimal structures, such as the Monotonicity Theorem (Theorem 3.2), Fiber Dimension Theorem (Corollary 7.2), Quasi-Cell Decomposition (Theorem 8.13), Hardy structures on germs at infinity (Theorem 6.10), etc. Since some of these have already been treated by others, we often only give details for two variables, leaving higher arities to the reader, commenting on it occasionally. As we will argue in detail in §12.7, *o-finitism*, that is to say, the first-order properties of finite sets in o-minimal structures, includes discreteness, boundedness, and closedness. Moreover, under the DCTC assumption, discrete always implies bounded and closed. So, we will have to substitute 'discrete' for 'finite' in any of the above properties of o-minimal structures. Nonetheless, this program does not always pan out. For instance, while decomposing into cells, we seem to run into infinite disjunctions, leading to the notion of a *quasi-cell*, which is only locally a cell. However, there is a large class of definable subsets, called *eukaryote* subsets, that have a 'definable' cell decomposition, that is to say, loosely speaking, they admit a cell decomposition in 'discretely' many cells (see §9 for the precise definition).

[1] This is a slightly stronger version of what is called in the literature *local o-minimality*, but which agrees with it in the case of an expansion of an ordered field.

A *eukaryote* structure is then one in which every definable subset is eukaryote, and we show that it is always at least a model of DCTC. Any model of DCTC which is an expansion of a field by one-variable functions is eukaryote. Although we do not yet know whether every pseudo-o-minimal structure is eukaryote, we can show that every pseudo-o-minimal structure has a eukaryote, pseudo-o-minimal reduct (Theorem 10.4).

In Part 2, we turn to the study of pseudo-o-minimal structures. Therefore, whereas most papers on generalizing o-minimality are searching for weakenings that would include certain tamely behaving structures, our hands are tied and we have to obey by the laws of o-minimalism. Thus, to the chagrin of some of my esteemed colleagues, we have to discard the structure $(\mathbb{Q}, <, +, \mathbb{Z})$ as it is not pseudo-o-minimal, although it is definably complete and locally o-minimal. However, it fails to have the type completeness property at infinity, which forces every discrete set to be bounded.

In §11, we study the Grothendieck ring of a pseudo-o-minimal structure. It will follow from the DDP that this Grothendieck ring is equal to the ring of integers if and only if the structure is o-minimal (in which case it corresponds to the Euler characteristic). Using Grothendieck rings, we can also formulate a condition for an *ultra-o-minimal* structure $\mathcal{M}_\natural$, that is to say, an ultraproduct of o-minimal structures $\mathcal{M}_i$, to be o-minimal (no such criterion seemed to have existed before): this is the case if for each $L$-formula $\varphi$, there is a bound $N_\varphi$ on the absolute value of the Euler characteristic of $\varphi(\mathcal{M}_i)$, independent of $i$ (Theorem 11.20).

In §12, we study expansions of a pseudo-o-minimal structure by a predicate so that the expansion is again pseudo-o-minimal. For discrete subsets, we get the notion of a *pseudo-o-finite* set, that is to say, a set enjoying all first-order properties of an arbitrary finite set in an o-minimal structure. This notion is particularly interesting when it comes to classifying definable subsets up to 'virtual' isomorphism, that is to say, definable in some pseudo-o-minimal expansion; the corresponding Grothendieck ring is called the *virtual Grothendieck ring* and studied in §13. However, a priori, the treatment depends on a choice of 'context', that is to say, of an ultra-o-minimal elementary extension. Using this technology, we associate in §14 to each infinite, definable, discrete subset of $M$ a (discretely valued) Euler characteristic defined on its virtual Grothendieck ring. This allows us to calculate explicitly this virtual Grothendieck ring in the special case of a eukaryote, pseudo-o-minimal expansion of an ordered field admitting a power dominant discrete subset (Corollary 14.6).

The last section is an application to the study of analytic sets. In the o-minimal context, (sub)analytic sets are normally understood to be given by analytic functions supported on the unit box (often simply called *restricted analytic functions*), as the corresponding structure $\mathbb{R}_{\text{an}}$ is o-minimal, and admits quantifier elimination in an appropriate language by the seminal work of [3]. There is a good reason to restrict to compact support, as the global sine function defines $\mathbb{Z}$, and hence can never be part of an o-minimal expansion. Our approach here is to look at subsets of $\mathbb{R}^k$ that can be uniformly approximated on compact sets by $\mathbb{R}_{\text{an}}$-definable subsets. More precisely, we call a subset $X \subseteq \mathbb{R}^k$ a *Taylor* set, if the ultraproduct over all $n$ of the truncations $X_{\rfloor n} := \{\mathbf{x} \in X \,|\, |\mathbf{x}| \leq n\}$ is definable in $\mathcal{R}^{\text{an}}_\natural$, where the latter structure is obtained as the ultraproduct of the scalings of $\mathbb{R}_{\text{an}}$ by a factor $n$ (that is to say, for each $n$, the expansion of $\mathbb{R}$ by power series converging on $|\mathbf{x}| \leq n$). It follows from aforementioned work of Denef and van den Dries that $\mathcal{R}^{\text{an}}_\natural$ is pseudo-o-minimal. Any subset definable by a quantifier free formula using convergent power series, whence in particular, any globally analytic variety, is Taylor. A discrete subset is Taylor if and only if it is closed, and any such set satisfies the Discrete Pigeonhole Principle with respect to Taylor maps. However, we can now also define sets by analytic parameterization, like the spiral with polar coordinates $R = \exp\theta$, for $\theta > 0$ (in contrast, the spiral obtained by allowing $\theta$ to be negative as well is not Taylor!). We use our pseudo-o-minimal results to give a geometric treatment of the class of Taylor sets: to a Taylor set $X$, we associate an $\mathcal{R}^{\text{an}}_{\natural}$-definable subset $X_\flat$, called its *protopower*, given as the ultraproduct of its truncations. We obtain a good dimension theory, a Monotonicity Theorem, a (partly conjectural, locally finite) cell-decomposition, and a corresponding Grothendieck ring, all indicative of the tameness of the class of Taylor sets, albeit not first-order.

There is some overlap of the first part of the present paper with [7], independently written by Fornasiero. Apart from the additional assumption of an underlying field (which, as pointed out by the author, is not always necessary), he also derives a Monotonicity Theorem, develops a good dimension theory, and achieves a decomposition into what he calls *multi-cells*. It would be of interest to combine this with our quasi-cell decomposition theorem, and see whether this eliminates the need for quasi-cells. I am also grateful for Fornasiero's comments on an earlier version of this paper, and for the many valuable remarks by an anonymous referee.

**Notations and conventions.** Definable always will mean definable with parameters, unless stated explicitly otherwise. Throughout this paper, $L$ denotes some language containing a distinguished binary relation symbol $<$ and any $L$-structure $\mathcal{M}$ will be a dense linear order without endpoints. We introduce two new symbols $-\infty$ and $\infty$, and, given an $L$-structure $\mathcal{M}$, we let $M_\infty := M \cup \{\pm\infty\}$, viewing $-\infty$ as the least element, and $\infty$ as the largest. When needed, $\mathtt{U}$ denotes some predicate (often unary), and we will write $(\mathcal{M}, X)$ for the $L(\mathtt{U})$-structure in which $X$ is the interpretation of $\mathtt{U}$.

We will use the following ISO convention for intervals: *open* $]a, b[$ (which we always assume to be non-empty, that is to say, $a < b$), *closed* $[a, b]$ (including the singleton $\{a\} = [a, a]$), *half-open* $]a, b]$ or $[a, b[$, and their infinite variants like $]-\infty, a[$, $]-\infty, a]$, $]a, \infty[$, and $[a, \infty[$, with $a, b \in M$. Note that the usage of $\infty$ here is only informal since these are definable subsets in the language without the extra constants $\pm\infty$ by formulae of the form $x < a$, etc.: any interval is definable (with parameters). The union and the intersection of two non-disjoint intervals are again intervals. Note that in $\mathbb{Q}$ the set of all rational numbers $q$ with $3 < q < \pi$ is not an interval, as it is only an infinite conjunction of definable subsets. Given a subset $Y \subseteq M$ and a point $b \in M$, we will sometimes use notations like $Y_{\leq b} := Y \cap\ ]-\infty, b]$ or $Y_{<b} := Y \cap\ ]-\infty, b[$.

When taking ultraproducts, we rarely ever mention the underlying index set or (non-principal) ultrafilter. We use the notation introduced in [18], denoting ultraproducts with a subscript $\natural$. Thus, we write $\mathbb{N}_\natural$, $\mathbb{Z}_\natural$, and $\mathbb{R}_\natural$ for the ultrapower of the set of natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, and reals $\mathbb{R}$ respectively. On occasion we need the (countable) ultraproduct of the diagonal sequence $(n)_n$ in $\mathbb{N}_\natural$, which we denote suggestively by $\omega_\natural$.

## Part 1. Definable completeness and type completeness: DCTC

§**2. The theory DCTC.** Recall that an (ordered) $L$-structure $\mathcal{M}$ is called *o-minimal*, if every definable subset $Y \subseteq M$ is a finite union of open intervals and points. It is called *definably complete*, if every definable subset in $M$ has an infimum (possibly $\pm\infty$); taking complements then yields that every definable subset also has a supremum.

Since intervals have this property, every definable subset in an o-minimal structure is definably complete, that is to say, definable completeness is an o-minimalistic property. By [12, Corollary 1.5], it is equivalent with $M$ being definably connected, and also with the validity of the *Intermediate Value Theorem* (IVT) for one-variable, definable continuous functions. Recall that $\mathcal{M}$ is called *locally o-minimal*, if for every definable subset $Y \subseteq M$ and every point $x \in M$, there exists an open interval $I$ containing $x$ such that $I \cap Y$ is a finite union of open intervals and points, and by shrinking $I$ even further, we may even take $I \cap Y$ to be an interval (for more on local o-minimality, see [7, 20, 22]). However, it appears to me a flaw of the definition, that one only requires $x$ to be a point in $M$, that is to say, excluding the case $x = \pm\infty$. In case $\mathcal{M}$ has also the structure of a field—the most studied case—we can take reciprocals, bringing $\infty$ to $0$, and so there is no need for the more general concept, but in general, one should include infinite points in this definition. We can reformulate local o-minimality as follows. Given $Y \subseteq M$ and $a \in M$, we say that $a^-$ *belongs to* $Y$, if there exists an open interval $]b, a[\ \subseteq Y$ (similarly, $a^+$ *belongs to* $Y$, if $]a, b[\ \subseteq Y$ for some $b > a$). Thinking of $a^-$ as a partial type (that is to say, consisting of all formulae $b < x < a$ in the variable $x$, where $b$ runs over all elements less than $a$), if $Y$ is defined by a formula $\varphi$, then $a^-$ belongs to $Y$ if and only if any realization of the type of $a^-$ in any elementary extension of $\mathcal{M}$ satisfies $\varphi$. Therefore, local o-minimality says that $a^-$ is a complete type, meaning that if $Y$ is definable, then $a^-$ belongs either to $Y$ or to $M \setminus Y$, for any $a \in M_\infty$. By taking complements, $a^+$ is then also complete. As mentioned, it is important to include the infinite points, where the two types $(-\infty)^+$ and $\infty^-$ are defined in the obvious way. For this reason, we will refer to this property as *type completeness* instead of local o-minimality.

2.1. DEFINITION. For a fixed language $L$ with a binary order symbol $<$, we define the theory DCTC as the extension of the theory of dense linear orders without endpoints by the two axiom schemes (one axiom for every formula) given by definable completeness and type completeness. In other words, in a model $\mathcal{M}$ of DCTC, any definable subset $Y \subseteq M$ has an infimum and its characteristic function has a left limit at each point. We will call an ultraproduct of o-minimal $L$-structures an *ultra-o-minimal* structure. It follows that such a structure is a model of DCTC, and more generally, any pseudo-o-minimal structure is a model of DCTC. Rennet showed in [17] that DCTC is a strictly weaker theory than $T^{\text{omin}}$.

2.2. EXAMPLE. Let $L$ be the language of ordered fields together with a unary predicate $\mathtt{U}$. Each $L(\mathtt{U})$-structure $(\mathbb{R}, \{0, 1, \dots, n\})$ is o-minimal, but their ultraproduct $(\mathbb{R}_\natural, (\mathbb{N}_\natural)_{\leq\omega_\natural})$ is not: indeed the set $(\mathbb{N}_\natural)_{\leq\omega_\natural}$ is discrete but not finite.

2.3. PROPOSITION. *Given a model $\mathcal{M}$ of DCTC, let $K \subseteq M$ be compact and $Y \subseteq M$ definable. If either $K$ is open or $Y$ is contained in $K$, then $K \cap Y$ is a finite union of intervals.*

PROOF. Given a definable subset $Y \subseteq M$, by assumption, we can find in the open case, for each $x \in K$, an open interval $I_x \subseteq K$ such that $Y \cap I_x$ is an interval. Since $K$ is compact and the $I_x$ cover $K$, there exist finitely many points $x_1, \dots, x_n \in K$ such that $K = I_{x_1} \cup \dots \cup I_{x_n}$ and hence $K \cap Y$ is a finite union of intervals. If $K$ is arbitrary, then we cannot arrange for all $I_x$ to be contained in $K$, and so we only get $K \subseteq I_{x_1} \cup \dots \cup I_{x_n}$. But since $Y \subseteq K$, the same conclusion can be drawn. $\dashv$

The next corollary improves [5, 2.13(3)] as it does not assume any underlying field structure.

2.4. COROLLARY. *If $\mathcal{M} \models$ DCTC with underlying order that of the reals, then it is o-minimal.*

PROOF. Identify $M$ with $\mathbb{R}$, and let $Y \subseteq \mathbb{R}$ be definable. Depending whether $(-\infty)^+$ or $\infty^-$ belong to $Y$ or not, we may assume after possibly removing one or two unbounded intervals that $Y$ is bounded, whence contained in some closed, bounded interval $K := [a, b]$. Hence $Y = Y \cap K$ is a finite union of intervals by Proposition 2.3. $\dashv$

2.5. *Remark.* From the proof it is clear that we have the following more general result: if a model of DCTC has the Heine-Borel property, meaning that any closed bounded set is compact, then it is o-minimal.

2.6. PROPOSITION. *For a definable subset $Y \subseteq M$ in a model $\mathcal{M} \models$ DCTC, we have:*

- 2.6.i. *The infimum of $Y$ is either infinite or a topological boundary point of $Y$.*
- 2.6.ii. *If $a, b \in \partial Y$ and $]a, b[\, \cap\, \partial Y = \emptyset$, then $]a, b[$ is either disjoint from or entirely contained in $Y$.*
- 2.6.iii. *If $Y$ is definably connected, then it is an interval.*
- 2.6.iv. *$Y$ either has a non-empty interior or is discrete.*
- 2.6.v. *If $Y$ is discrete, then it has a minimum and a maximum, and it is closed and bounded.*
- 2.6.vi. *The topological boundary $\partial Y$ is discrete, closed, and bounded.*

PROOF. To prove (2.6.i), let $l \in M$ be the infimum of $Y$. By type completeness, $l^-$ either belongs to $Y$ or to $M \setminus Y$. The former case is excluded since $l$ is the infimum of $Y$. In particular, $l$ is not an interior point of $Y$. If $l^+$ does not belong to $Y$, then $l$ is an isolated point of $Y$, and hence belongs to the (topological) boundary. In the remaining case, $l$ lies in the closure of $Y$, since some open interval $]l, x[$ lies inside $Y$. To prove (2.6.ii), suppose there exists $x \in\, ]a, b[\, \cap Y$. By type completeness, either $x^-$ belongs to $Y$ or to $M \setminus Y$. In the latter case, there exists $z < x$ such that $]z, x[$ is disjoint from $Y$. However, $x$ is then not an interior point of $Y$, whence must belong to its topological boundary, contradiction. So $x^-$ belongs to $Y$, which means that the set of all $z \in\, ]a, x[$ such that $]z, x[\, \subseteq Y$ is non-empty. The infimum of this set must be a topological boundary point of $Y$ by (2.6.i), and hence must be equal to $a$, showing that $]a, x] \subseteq Y$. Arguing the same with $x^+$, then shows that also $[x, b[\, \subseteq Y$, as we needed to prove.

To prove (2.6.iii), let $Y \subseteq M$ be definably connected. Let $l$ and $h$ be its respective infimum and supremum (including the case that these are infinite). The case $l = h$ is trivial, so assume $l < h$. If there were some $x \in\, ]l, h[$ not belonging to $Y$, then $Y$ would be the union of the two definable, non-empty, disjoint open subsets $Y_{<x}$ and $Y_{>x}$, contradiction, Hence, $Y$ is an interval with endpoints $l$ and $h$. To prove (2.6.iv), assume $Y$ is not discrete. Hence there exists $a \in Y$ which is not isolated, that is to say, such that any open interval containing $a$ has some other point in common with $Y$. If both $a^-$ and $a^+$ belong to $M \setminus Y$, then there are $x < a < y$ such that $]x, a[, ]a, y[$ are disjoint from $Y$, contradicting that $a$ is not isolated. Hence, say, $a^-$ belongs to $Y$ and $Y$ has non-empty interior.

Assume next that $Y$ is discrete and let $l$ be its infimum (including possibly the case $l = -\infty$). If $l^+$ belongs to $Y$, then $]l, z[\ \subseteq Y$ for some $z > l$, contradicting discreteness. So $l^+$ does not belong to $Y$, which forces $l \in Y$. In particular, $l$ is finite, proving the first part of (2.6.v), and in particular, that $Y$ is bounded. To show that $Y$ is closed, suppose it is not. Let $x \notin Y$ be a point in its closure. Since $Y \cup \{x\}$ is definable but not discrete, it must have interior by (2.6.iv), so that some open interval $I$ is contained in $Y \cup \{x\}$. But then $I \cap Y = I \setminus \{x\}$ is not discrete, contradiction. To see (2.6.vi), it suffices by (2.6.v) to show that $\partial Y$ is discrete. Let $b \in \partial Y$. We have to show that $b$ is an isolated point of $\partial Y$, and this will clearly hold if $b$ is an isolated point of either $Y$ or $M \setminus Y$. In the remaining case, exactly one from among $b^-$ and $b^+$ belongs to $Y$, say, $b^-$. Hence, there exist $x < b < y$ so that $]x, b[\ \subseteq Y$ and $]b, y[\ \cap Y = \emptyset$. Since any point in $]x, b[$ is interior to $Y$ and any point in $]b, y[$ exterior to $Y$, we get $\partial Y \cap\ ]x, y[\ = \{b\}$, as we needed to prove. $\dashv$

2.7. *Remark.* For any definable, discrete subset $Y \subseteq M$ in a model of DCTC, we can therefore define a successor function $\sigma_Y$ on $Y$ by letting $\sigma_Y(b)$ be the minimum of (the definable subset) $Y_{>b}$, for any non-maximal $b$ in $Y$.

2.8. COROLLARY. *The theory DCTC is equivalent with type completeness and discrete definable completeness, where in the latter we only require that definable, discrete sets have a minimum.*

PROOF. Let $Y \subseteq M$ be a definable set in a model $\mathcal{M}$ of the weaker system from the assertion. Inspecting the argument in the proof of (2.6.vi), type completeness already implies that $\partial Y$ is discrete. Hence $\partial Y$ has a minimum $b$, and it is now not hard to show that $b$ is also the infimum of $Y$. $\dashv$

2.9. COROLLARY. *In a model of DCTC, any finite union of one-variable definable, discrete subsets is discrete.* $\dashv$

Using (2.6.vi), (2.6.ii) and Remark 2.7, we get immediately the following structure theorem for one-variable definable subsets (compare this with the notion of *d-minimality* from [13], where one needs finitely many discrete subsets):

2.10. THEOREM. *An $L$-structure $\mathcal{M}$ is a model of DCTC if and only if every definable subset $Y \subseteq M$ is a disjoint union of open intervals and a single closed, bounded, discrete set.*

PROOF. We only need to prove the converse. Let us first show that $M$ is definably connected. Indeed, if $U_1$ and $U_2$ are disjoint definable open sets covering $M$, then this would yield a covering of $M$ by disjoint open intervals. However, considering what the endpoints would be, this can only be the trivial covering, showing that one of the $U_i$ must be empty. By [12, Corollary 1.5], definable connectedness implies definable completeness. To prove type completeness, let $Y \subseteq M$ be definable and $b \in \partial Y$ (boundary points are the only points in which it can fail). There is nothing to prove if $b$ is an isolated point of $Y$ or of $M \setminus Y$, so assume it is not. Decompose $Y = U \cup D$ into definable subsets with $U$ a disjoint union of open intervals and $D$ closed, bounded, and discrete. Let $]p, q[$ and $]u, v[$ be the open interval in $U$ immediately to the left and to the right of $b$ respectively. Since $b$ is not isolated but in the boundary, it must be equal to exactly one of $q$ or $u$, that is to say, either $q = b < u$ or $q < u = b$. Say the latter holds, so that $b^+$ belongs to $Y$. Since $D \cup \{b\}$ is discrete, we can find an open interval

$I$ containing $b$ such that $I \cap (D \cup \{b\}) = \{b\}$. Shrinking $I$ if necessary, we can make it disjoint from $]p, q[$, and hence $I_{<b} \cap Y = \emptyset$, showing that $b^-$ belongs to $M \setminus Y$. We need to verify this also at $b = \pm\infty$, where the same argument works in view of the boundedness of $D$. $\dashv$

2.11. COROLLARY. *A model $\mathcal{M}$ of DCTC is o-minimal if and only if every definable discrete subset of $M$ is finite.*

PROOF. Let $Y \subseteq M$ be definable. Hence $Y$ is a disjoint union of open intervals $I_i$ and a single definable discrete subset $F$ by Theorem 2.10. By assumption, $F$ is finite. By (2.6.vi), the boundary points of $Y$ form a discrete, whence finite subset, and hence, there can only be finitely many open intervals $I_i$, proving o-minimality. $\dashv$

§**3. Definable maps.** For the remainder of Part 1, we will work in a model $\mathcal{M}$ of DCTC, unless noted explicitly otherwise. Next we study definable maps, where we call a map $f\colon Y \subseteq M^n \to M^k$ *definable* if its graph $\Gamma(f) \subseteq M^{n+k}$ is a definable subset. Note that since its domain $Y \subseteq M^n$ is the projection of its graph, it too is definable. Similarly, the set $\Gamma^*(f)$ consisting of all $(f(x), x) \in M^{k+n}$ is definable and is called the *reverse graph* of $f$. If $k = n = 1$, we speak of *one-variable* maps.

3.1. LEMMA. *Given a definable map $f\colon Y \to M$,*

3.1.i. *if $Y$ is discrete, then so is its image $f(Y)$;*

3.1.ii. *if $f(Y)$ and each fiber of $f$ is discrete, then so is $Y$.*

PROOF. Suppose (3.1.i) does not hold, so that $Y \subseteq M$ is discrete but not $f(Y)$. Let $H$ be the (non-empty, definable, discrete) subset of all $x \in Y$ such that $f(Y_{\geq x})$ is non-discrete, and let $h$ be its maximum. Since $h$ cannot be the maximal element of $Y \subseteq M^n$ lest $f(Y_{\geq h})$ be a singleton, we can find its successor $\sigma(h) \in Y$ by (4.1.iii). But $f(Y_{\geq h}) = \{f(h)\} \sqcup f(Y_{\geq \sigma(h)})$, so that $f(Y_{\geq \sigma(h)})$ is non-discrete by Corollary 2.9, contradicting maximality.

Assume next that (3.1.ii) is false, so that $Y$ is non-discrete, but $Z := f(Y)$ and all $f^{-1}(u)$ are discrete. This time, let $H$ be the subset of all $x \in Z$ such that $f^{-1}(Z_{\geq x})$ is non-discrete, and let $h$ be its maximum. Again $h$ must be non-maximal in $Z$, and so admits an immediate successor $\sigma(h) \in Z$. Since both subsets on the right hand side of

$$f^{-1}(Z_{\geq h}) = f^{-1}(h) \sqcup f^{-1}(Z_{\geq \sigma(h)})$$

are discrete, the first by assumption and the second by maximality, so must their union be by Corollary 2.9, contradiction. $\dashv$

3.2. THEOREM (Monotonicity). *The set of discontinuities of a one-variable definable map $f\colon Y \to M$ is discrete, closed, and bounded, and consists entirely of jump discontinuities. Moreover, there is a definable discrete, closed, bounded subset $D \subseteq Y$ so that in between any two consecutive points of $D \cup \{\pm\infty\}$, the map is* monotone*, that is to say, either strictly increasing, strictly decreasing, or constant.*

PROOF. We start with proving that all discontinuities are jump discontinuities. By symmetry, it suffices to show that the left limit of $f$ at each point $a \in Y$ exists. For each $y < a$, let $w(y)$ be the supremum of $f(\,[y, a[\,)$ and let $b$ be the infimum of $w(Y_{<a})$. I claim that $b$ is the left limit of $f$ at $a$. To this end, choose $p < b < q$, and we need to show that there is some $x < a$ with $p < f(x) < q$. If $b^+$ does not belong $w(Y_{<a})$, and

therefore belongs to its complement, then $b$ is an isolated point of $w(Y_{<a})$, implying that $f$ takes constant value $b$ on some interval $]y, a[$, so that $b$ is indeed the left limit at $a$. In the remaining case, we can find $u > b$ such that $]b, u] \subseteq w(Y_{<a})$. We may choose $u < q$. In particular, $u = w(y)$ for some $y < a$. Since $b$ is strictly less than the supremum $u = w(y)$, we can find $x \in [y, a[$ such that $b < f(x) \leq u$, whence $p < f(x) < q$, as required.

Let $C \subseteq Y$ be the definable subset given as the union of the interior of all fibers, that is to say, $x \in C$ if and only if $x$ is an interior point of $f^{-1}(f(x))$. Being an open set, $C$ is a disjoint union of open intervals, and $f$ is constant, whence continuous on each of these open intervals. Every fiber of the restriction of $f$ to $Y \setminus C$ must have empty interior, whence is discrete by (2.6.iv). So upon replacing $f$ by this restriction, we may reduce to the case that $f$ has discrete fibers. There is nothing to show if $Y$ is discrete, and so without loss of generality, we may assume $Y$ is an open interval. For fixed $a \in Y$, let $L_a$ (respectively, $H_a$) be the set of all $x \in Y$ such that $f(x) < f(a)$ (respectively, $f(a) < f(x)$). Since $Y$ is the disjoint union of $L_a$, $H_a$, and $f^{-1}(f(a))$ with the latter being discrete, $a^-$ must belong to one of the first two sets by Corollary 2.9, and depending on which is the case, we will denote this symbolically by writing respectively $f(a^-) < f(a)$ or $f(a^-) > f(a)$ (with a similar convention for $a^+$). Let $L_-$ (respectively, $H_-$, $L_+$, and $H_+$) be the set of all $a \in Y$ such that $f(a^-) < f(a)$ (respectively, $f(a^-) > f(a)$, $f(a^+) < f(a)$, and $f(a^+) > f(a)$), so that $Y$ is the disjoint union of these four definable subsets. Let $D$ be the union of the topological boundaries of these four sets, a discrete set by Corollary 2.9. If $b < c$ are consecutive elements in $D$, then $]b, c[$ must belong to exactly one of these four sets by (2.6.ii), say, to $L_-$. It is now easy to see that in that case $f$ is strictly increasing on $]b, c[$. This then settles the last assertion.

Let $S$ be the (definable) subset of all discontinuities of $f$. To prove that $S$ is discrete, we need to show by (2.6.iv) that it has empty interior, and this will follow if we can show that any open interval $I \subseteq Y$ contains a point at which $f$ is continuous. By what we just proved, by shrinking $I$ if necessary, we may assume $f$ is monotone on $I$, say, strictly increasing. Note that $f$ is then in particular injective. By Lemma 3.1, the image $f(I)$ contains an open interval $J$. Since $f$ is strictly increasing, $f^{-1}(J)$ is also an open interval, and $f$ restricts to a bijection between $f^{-1}(J)$ and $J$. We leave it to the reader to verify that any strictly increasing bijection between intervals is continuous. $\dashv$

3.3. *Remark.* The above result, and some others in this section, can also be proven by the same techniques used in the o-minimal setting, as, for instance, in [21, Chapt. 3].

3.4. *Remark.* Given a definable map $f$ and a point $a$, we denote its left and right limit simply by $f^-(a)$ or $f^+(a)$ respectively, even if these values are infinite (to be distinguished from the symbol $f(a^-)$ which occurred above in formulae of the form $f(a^-) < f(a)$). Note that we even have this property at $\pm\infty$, so that we can define $f^+(-\infty)$ and $f^-(\infty)$, which we then simply abbreviate as $f(-\infty)$ and $f(\infty)$ respectively.

3.5. COROLLARY. *A definable map $f\colon I \to M$ with domain an open interval $I$ is continuous if and only if its graph is definably connected.*

PROOF. Let $C$ be the graph of $f$. If $f$ is not continuous, then it has a jump discontinuity at some point $a \in I$ by Theorem 3.2. If $f^-(a)$ and $f^+(a)$ are equal, and hence different from $f(a)$, then the point $(a, f(a))$ is easily seen to be an isolated point of $C$, contradiction. Without loss of generality, we may therefore assume $f^-(a) < f^+(a)$. Let $c$ be some element between these two limits and different from $f(a)$. By definition of one-sided limit, there exist $p < a < q$ such that $f(x) < c$ whenever $p < x < a$, and $f(x) > c$ whenever $a < x < q$. Consider the two open subsets

$$U_- := (I_{<a} \times M) \cup (I_{<q} \times M_{<c})$$
$$U_+ := (I_{>a} \times M) \cup (I_{>p} \times M_{>c}).$$

It is not hard to check that $C$ is contained in their union but disjoint from their intersection, showing that it is not definably connected.

Conversely, assume $f$ is continuous but $C$ is not definably connected, so that there exist definable open subsets $U$ and $U'$ whose union contains $C$ but whose intersection is disjoint from $C$. Since the projections $\pi(C \cap U)$ and $\pi(C \cap U')$ onto the first coordinate are definable subsets partitioning $I$, they must have a common boundary point $b \in I$ by Proposition 2.6. Since $(b, f(b))$ belongs to either $U$ or $U'$, say, to $U$, there exists a box $J \times J' \subseteq U$ containing $(b, f(b))$. By continuity, we may assume $f(J) \subseteq J'$. This implies that $(x, f(x)) \in U$, for all $x \in J$, and hence that $J \subseteq \pi(C \cap U)$, contradicting that $b$ is a boundary point of the latter. ⊣

3.6. *Remark.* Without proof, we claim that the above results extend to arbitrary dimensions: given a definable map $f\colon X \subseteq M^n \to M^k$, the set of discontinuities of $f$ is nowhere dense in $X$. For instance (with terminology to be defined below), if $X \subseteq M^2$ has dimension two, for each $a, b \in M$, let $D_a$ and $E_b$ be some discrete sets, as given by Theorem 3.2, such that between any two consecutive points the respective maps $y \mapsto f(a, y)$ and $x \mapsto f(x, b)$ are continuous and monotone. Let $D$ and $E$ be the respective union of all $\{a\} \times D_a$ and all $E_b \times \{b\}$. By Corollary 7.2 and Proposition 5.1 below, both $D$ and $E$ are one-dimensional, closed subsets, and hence $X' := X \setminus (D \cup E)$ is open and dense in $X$. It is now not hard to show that $f$ is continuous on $X'$ (see [21, Chapt. 3, Lemma 2.16]).

We can also strengthen this for expansions of fields by proving the same results with 'continuous' replaced by 'differentiable', or more generally, by $\mathcal{C}_n$ (see, for instance, [7, §7.4]).

Recall that a function of topological spaces $f\colon X \to Y$ is called *locally constant*, if around every point, we can find an open interval on which $f$ is constant. Since a locally constant function has open fibers, it must be constant if its domain is connected, and the same holds true in the definable category. Let us call $f$ a *step function*, if there exists a discrete subset $F \subseteq X$ such that the restriction of $f$ to $X \setminus F$ is locally constant.

3.7. COROLLARY. *Let $\mathcal{M}$ be a model of DCTC. For a definable function $f\colon M \to M$, the following are equivalent:*

3.7.i. *$f$ is a step function;*

3.7.ii. *there exists a definable, discrete subset $F \subseteq M$, such that $f$ is constant on any open interval which is disjoint from $F$;*

3.7.iii. *the image of $f$ is discrete.*

PROOF. By the Monotonicity Theorem (Theorem 3.2), we can find a closed, discrete, bounded, definable subset $G \subseteq M$, such that $f$ is monotone or constant on any intermediate interval. Clearly (3.7.ii) implies (3.7.i). As for the converse, let $I$ be an open interval disjoint from $G$, and take any point $t \in I$. By choice of $G$, the function

$f$ is either monotone or constant on $I$. Since it is constant in a neighborhood of $t$, the former is excluded, showing that $f$ is constant on $I$, and so we may take $F = G$ in (3.7.ii).

Suppose (3.7.ii) holds and let $u$ be some element strictly less than any element in $F$ (note that $F$ is bounded by (2.6.v)). Define a map $g$ from $F \cup \{u\}$ to $M$ by sending $a \in F \cup \{u\}$ to the constant value of $f$ on some open interval with left endpoint $a$ and disjoint from $F$. This is well-defined, and it is now easy to see that the image of $f$ is equal to the union of $f(F)$ and the image of $g$, where both of the latter sets are discrete by (3.1.i), and whence so is their union by Corollary 2.9. For the converse, assume $f$ has discrete image. Let $I$ be an open interval in $X \setminus G$. If $f$ is monotone on $I$, then $f(I)$ is an interval, contradicting that the image is discrete, so $f$ must be constant on $I$ by choice of $G$. $\dashv$

§**4. Discrete sets.** As before, $\mathcal{M} \models$ DCTC. We start our analysis of multi-variable definable subsets, with a special emphasis on definable subsets of the plane $M^2$, called *planar* subsets, and only address the general case through some sporadic remarks. Since projections play an important role, we introduce some notation. Fix $n$ and let $\sigma \subseteq \{1, \dots, n\}$ of size $|\sigma| := e$. We let $\pi_\sigma \colon M^n \to M^e$ be the projection $(a_1, \dots, a_n) \mapsto (a_{i_1}, \dots, a_{i_e})$, where $\sigma = \{i_1 < i_2 < \dots < i_e\}$. When $\sigma$ is a singleton $\{i\}$, we just write $\pi_i$ for the projection onto the $i$-th coordinate. Given a tuple $\mathbf{a} = (a_1, \dots, a_e) \in M_e$, the ($\sigma$-)*fiber* of $X$ above $\mathbf{a}$ is the set

$$X_\sigma[\mathbf{a}] := \pi_{\sigma^c}\left(\pi_\sigma^{-1}(\mathbf{a}) \cap X\right),$$

where $\sigma^c$ is the complement of $\sigma$. In other words, $X_\sigma[\mathbf{a}]$ is the set of all $\mathbf{b} \in M^{n-e}$ such that $\tilde{\mathbf{b}} \in X$, where $\tilde{\mathbf{b}}$ is obtained from $\mathbf{b}$ by inserting $a_{i_k}$ at the $k$-th spot. In case $\sigma$ is of the form $\{1, \dots, e\}$, for some $e$, we omit $\sigma$ from the notation, since the length of the tuple $\mathbf{a}$ then determines the projection, and we refer to it as a *principal projection*, with a similarly nomenclature for fibers. Thus, for example, the *principal fiber* $X[a]$ is the set $X_1[a]$ of all $n-1$-tuples $\mathbf{b}$ such that $(a, \mathbf{b}) \in X$. Recall that by (2.6.v) any definable discrete subset of $M$ is closed and bounded. The same is true in higher dimensions, for which we first prove:

4.1. THEOREM. *A definable subset $X \subseteq M^n$ is discrete if and only if all projections $\pi_1(X), \dots, \pi_n(X)$ are discrete.*

PROOF. Suppose all projections are discrete and let $(a_1, \dots, a_n) \in X$. Hence we can find open intervals $I_k$, for $k = 1, \dots, n$, such that $I_k \cap \pi_k(X) = \{a_k\}$. The open box $I_1 \times \dots \times I_n$ then intersects $X$ only in the point $(a_1, \dots, a_n)$, proving that $X$ is discrete. To prove the converse, we will induct on $n$, proving simultaneously the following three properties for $X \subseteq M^n$ discrete:

4.1.i. $\pi_1(X), \dots, \pi_n(X)$ are discrete;
4.1.ii. $X$ with the induced lexicographical ordering has a minimal element;
4.1.iii. for this ordering, there exists a definable map $\sigma_X$ of $X$, sending every non-maximal element in $X$ to its immediate successor.

All three properties have been established by Proposition 2.6 when $n = 1$, so assume they hold for $n - 1$. Assume towards a contradiction that $\pi_1(X)$ is not discrete. For each $a \in \pi_1(X)$, the fiber $X[a]$ (that is to say, the set of all $\mathbf{b} \in M^{n-1}$ such that $(a, \mathbf{b}) \in X$), is discrete since $a \times X[a] \subseteq X$. By the induction hypothesis for (4.1.ii), in its lexicographical order, $X[a]$ has a minimum, denoted $f(a)$, yielding a definable map $f\colon \pi_1(X) \to M^{n-1}$ whose graph lies in $X$. By Theorem 3.2, each composition $\pi_i \circ f\colon \pi_1(X) \to M$, for $i = 1, \dots, n-1$, is continuous outside a discrete set. The union of these discrete sets is again discrete by Corollary 2.9, and hence, since $\pi_1(X)$ is assumed non-discrete, there is a common point $a$ at which all $f_i$ are continuous, whence also $f$. By the discreteness of $X$, we can find an open interval $I$ and an open box $U \subseteq M^{n-1}$ containing respectively $a$ and $f(a)$ such that $(I \times U) \cap X = \{(a, f(a))\}$. By continuity, we can find an open interval $J \subseteq I$ containing $a$ such that $f(J) \subseteq U$. However, this means that for any $u \in J$ different from $a$, we have $f(u) \in U$, whence $(u, f(u)) \in (I \times U) \cap X = \{(a, f(a))\}$, forcing $u = a$, contradiction.

To prove (4.1.ii), we now have established that $\pi_1(X)$ is discrete, whence has a minimum $l$. The minimum of $X$ in the lexicographical ordering is then easily seen to be $(l, \min(X[l]))$. To define $\sigma_X$, let $\mathbf{a} = (a, \mathbf{b}) \in X$. For $\mathbf{a} \neq \max(X)$, either $\mathbf{b}$ is not the maximum of $X[a]$ and we set $\sigma(\mathbf{a}) := (a, \mathbf{b}')$ where $\mathbf{b}' := \sigma_{X[a]}(\mathbf{b})$; or otherwise, $a$ is not the maximum of $\pi_1(X)$ and we set $\sigma(\mathbf{a}) := (a', \min(X[a']))$, where $a' := \sigma_{\pi_1(X)}(a)$. Note that the existence of $a'$ and $\mathbf{b}'$ follow from the induction hypothesis on (4.1.iii). We leave it to the reader to verify that $\sigma_X$ has the required properties. $\dashv$

4.2. COROLLARY. *An $\mathcal{M}$-definable, discrete subset is closed and bounded.*

PROOF. Let $X \subseteq M^n$ be a definable, discrete subset. By Theorem 4.1, all $\pi_i(X)$ are discrete, whence bounded and closed by (2.6.v). It is now easy to deduce from this that so is then $X$. $\dashv$

The following are now routine corollaries, the proof of which we leave to the reader.

4.3. COROLLARY. *The image under a definable map of an $\mathcal{M}$-definable discrete subset is again discrete.* $\dashv$

4.4. COROLLARY. *A definable subset $X \subseteq M^n$ is discrete if and only if for some (equivalently, for all) $\sigma \subseteq \{1, \dots, n\}$, the projection $\pi_\sigma(X)$ as well as each fiber $X_\sigma[\mathbf{a}]$ is discrete.* $\dashv$

Suppose $\mathcal{M} \models$ DCTC is an expansion of an ordered group—which is therefore Abelian and divisible by [12, Proposition 2.2]. We call a map $f\colon X \to X$, for $X \subseteq M$, *contractive*, if

(1) $$|f(x) - f(y)| < |x - y|\,,$$

for all $x \neq y \in X$. We say that $f$ is *weakly contractive*, if instead we have only a weak inequality in (1). Recall that a *fixed point* of $f$ is a point $x \in X$ such that $f(x) = x$. If $f$ is contractive, it can have at most one fixed point.

4.5. THEOREM (Fixed Point Theorem). *Suppose $\mathcal{M} \models$ DCTC expands an ordered group, and let $f\colon D \to D$ be a definable map on a discrete, definable subset $D \subseteq M$. If $f$ is contractive, it has a unique fixed point. If $f$ is weakly contractive, then $f^2$ has a fixed point.*

PROOF. We treat both cases simultaneously. Assume $f$ does not have a fixed point. In particular, $f(l) > l$, where $l$ is the minimum of $D$. Hence the set of $x \in D$ such that $x < f(x)$ is non-empty, whence has a maximal element $u$. Clearly, $u < h$, where $h$ is the maximum of $D$, and hence $u$ has an immediate successor $v := \sigma_D(u)$ by

(4.1.iii). By maximality, we must have $f(v) < v$. Hence $v \leq f(u)$ and $u \leq f(v)$, and therefore $v - u \leq |f(u) - f(v)|$, leading to a contradiction in the contractive case with (1), showing that $f$ must have a fixed point, necessarily unique. In the weak contractive case, we must have an equality in the latter inequality, whence also in the former two, that is to say, $f(u) = v$ and $f(v) = u$. Hence, $u$ and $v$ are fixed points of $f^2$. $\dashv$

§**5. Sets with non-empty interior.** We continue to work in a model $\mathcal{M}$ of DCTC. Shortly, we will introduce the notion of dimension, and whereas the discrete sets are those with minimal dimension (=zero), the sets with non-empty interior will be those of maximal dimension. Note that the non-empty definable subsets of $M$ are exactly of one of these two types by (2.6.iv).

5.1. PROPOSITION. *A definable subset $X \subseteq M^n$ has non-empty interior if and only if the set of points $a \in M$ such that the fiber $X[a]$ has non-empty interior is non-discrete.*

PROOF. If $X$ has non-empty interior, it contains an open box, and the assertion is clear. For the converse, note that, since we can pick definably the first open interval inside a definable non-discrete subset of $M$ by the properties proven in Proposition 2.6, we may reduce to the case that $\pi(X)$ is an open interval and each fiber $X[a]$ for $a \in \pi(X)$ is an open box, where $\pi\colon M^n \to M$ is the projection onto the first coordinate. The proof for $n > 2$ is practically identical to that for $n = 2$, and so, for simplicity, we assume $n = 2$. Let $l(a)$ and $h(a)$ be respectively the infimum and supremum of $X_a$, so that $l, h\colon \pi(X) \to M_\infty$ are definable maps. The subset of $\pi(X)$ where either function takes an infinite value is definable, whence it or its complement contains an open interval, so that we can either assume that $l$ is either finite everywhere or equal to $-\infty$ everywhere, and a similar dichotomy for $h$. The infinite cases can be treated by a similar argument, so we will only deal with the case that they are both finite (this is a practice we will follow often in our proofs). By Theorem 3.2, there is a point $a \in \pi(X)$ at which both $l$ and $h$ are continuous. Fix some $c < l(a) < p < q < h(a) < d$, so that by continuity, we can find $u < a < v$ so that $l(\,]u,v[\,) \subseteq \,]c,p[$ and $h(\,]u,v[\,) \subseteq \,]q,d[$. I claim that $]u,v[ \,\times\, ]p,q[$ is entirely contained in $X$. Indeed, if $u < x < v$ and $p < y < q$, then from $c < l(x) < p < y < q < h(x) < d$, we get $y \in X[x]$, that is to say, $(x,y) \in X$. $\dashv$

By a simple inductive argument, we get the following analogue of Corollary 4.4:

5.2. COROLLARY. *A definable subset $X \subseteq M^n$ has non-empty interior if and only if for some (equivalently, for all) $\sigma \subseteq \{1,\dots,n\}$, the set of points* $\mathbf{a}$ *for which $X_\sigma[\mathbf{a}]$ has non-empty interior, has non-empty interior.* $\dashv$

5.3. COROLLARY. *A finite union of definable subsets of $M^n$ has non-empty interior if and only if one of the subsets has non-empty interior.*

PROOF. One direction is immediate, and to prove the other we may by induction reduce to the case of two definable subsets $X_1, X_2 \subseteq M^n$ whose union $X := X_1 \cup X_2$ has non-empty interior. We induct on $n$, where the case $n = 1$ follows from Corollary 2.9 and (2.6.iv). Let $W \subseteq M$ be the subset of all points $a \in M$ for which the fiber $X[a] \subseteq M^{n-1}$ has non-empty interior. By Proposition 5.1, the interior of $W$ is non-empty. Since $X[a] = (X_1)[a] \cup (X_2)[a]$, our induction hypothesis implies that for $a \in W$, at least one of $(X_i)[a]$ has non-empty interior, in which case we put $a$ in $W_i$.

In particular, $W = W_1 \cup W_2$ so that at least one of the $W_i$ has non-empty interior, say, $W_1$. By Proposition 5.1, this then implies that $X_1$ has non-empty interior. $\dashv$

§**6. Planar cells and germs.** For the remainder of our analysis of multi-variable definable sets, apart from separate remarks, we restrict to planar subsets. Given an ordered structure $\mathcal{O}$, let us define a 2-*cell* in $O^2$ as a definable subset $C$ of the following form: suppose $I$ is an open interval, called the *domain* of the cell, and $f, g\colon I \to O$ are definable, continuous maps such that $f < g$ (meaning that $f(x) < g(x)$ for all $x \in I$). Let $C$ be the subset of all $(x, y) \in O^2$ with $x \in I$ and $f(x) \diamond_1 y \diamond_2 g(x)$, where $\diamond_i$ is either no condition or a strict inequality (when we only have at most one inequality, we get an example of an *unbounded* cell; the remaining ones are call *bounded*, and in arguments we often only treat the latter case and leave the former with almost identical arguments to the reader). Any 2-cell is open. By a 1-*cell* $C \subseteq O^2$, we mean either the graph of a continuous definable map $f$ with domain an open interval $I$, or a Cartesian product $x \times I$. We call the former *horizontal* and the latter *vertical*. Finally, by a 0-*cell*, we mean a point. We may combine all these definitions into a single definition: a cell $C$ is determined by elements $a < b$ and definable, continuous maps $f < g\colon O \to O$, as the set of all pairs $(x, y)$ such that $a \diamond_1 x \diamond_2 b$ and $f(x) \diamond_3 y \diamond_4 g(x)$, where each $\diamond_i$ is either no condition, equality or strict inequality. Moreover, if $C$ is non-empty, then it is a $d$-cell, where $d$ is equal to two minus the number of equality signs among the $\diamond_i$. We sometimes use some suggestive notation like $C(I; f < g)$ to denote, for instance, the cell given by $x \in I$ and $f(x) < y < g(x)$. If $C$ is a cell with domain $I$ and $J \subseteq I$ is an open interval, then we call $C \cap (J \times O)$ the *restriction* of $C$ to $J$. Any restriction of a cell is again a cell, and so is any principal projection.

6.1. *Remark.* For higher arity, we likewise define cells inductively: we say that $C \subseteq O^n$ is a $d$-*cell* if either $C$ is the graph of a definable, continuous function with domain some $d$-cell in $O^{n-1}$, or otherwise, is the region strictly between two such graphs with common domain some $(d-1)$-cell in $O^{n-1}$. As we shall see below in Remark 7.3, the $d$ in $d$-cell refers to the dimension of the cell.

**6.2. Germs.** As before, let $\mathcal{M} \models$ DCTC. Given a definable subset $X \subseteq M^2$, a point $P = (a, b) \in M^2$, and a definable map $h\colon Y \subseteq M \to M$ such that $a \in Y$ and $h^-(a) = b$, we will say that $P_h^-$ *belongs to* $X$, if there exists an open interval $]u, a[\ \subseteq I$ so that the graph of the restriction of $h$ lies inside $X$. By Theorem 3.2, we may shrink $]u, a[$ so that $h$ is continuous on that interval, and so we could as well view this as a property of the horizontal 1-cell $C$ defined by $h$. Note that $P$ lies in the closure of $C$. Moreover, we only need $a$ to lie in the closure of $Y$ to make this work. So, given a 1-cell $C$ such that $P$ lies in its closure, we say that $P_C^-$ *belongs to* $X$ if $P_h^-$ does, where $h$ is the definable, continuous map determining $C$, in case $C$ is a horizontal cell, or if $b^-$ belongs to $X[a]$ in case $C$ is a vertical cell. Of course, we can make a similar definition for $P_h^+$ or $P_C^+$. The following result essentially shows that viewed as a type, $P_C^-$ is complete:

6.3. LEMMA. *Given a planar subset $X \subseteq M^2$, a 1-cell $C \subseteq M^2$, and a point $P$ in the closure of $C$, either $P_C^-$ belongs to $X$ or it belongs to its complement.*

PROOF. Let $P = (a, b)$ be in the closure of $C$. If $C$ is a vertical cell, then $P_C^-$ belongs to $X$ if and only if $b^-$ belongs to $X[a]$, and so we are done in this case by type completeness. In the horizontal case, there exists a definable, continuous map $h\colon\ ]u,a[\ \to M$ whose graph is contained in $C$. By type completeness, either $a^-$ belongs to $\pi(X \cap C)$ or to its complement. In the former case, after increasing $u$ if necessary, we have $]u,a[\ \subseteq \pi(X \cap C)$, whence $(x, h(x)) \in X$ for every $x \in\ ]u,a[$. In the latter case, $]u,a[$ is disjoint from $\pi(X \cap C)$, and hence $(x, h(x)) \notin X$ for every $x \in\ ]u,a[$. $\dashv$

Using this, it is not hard to show that the following is an equivalence relation (and in particular symmetric): given a point $P \in M^2$ and 1-cells $V, W \subseteq M^2$ such that $P$ lies in each of their closures, we say that $V \equiv_{P^-} W$, if $P_V^-$ belongs to $W$. By a *left germ* at $P$, we mean an $(\equiv_{P^-})$-equivalence class of 1-cells whose closure contains $P$; and a similar definition for $V \equiv_{P^+} W$ and *right germ*. It is now easy to see that $P_V^-$ belongs to some definable subset $X \subseteq M^2$ if and only if $P_W^-$ belongs to it, for any $W \equiv_{P^-} V$, so that we may make sense of the expression $P_\alpha$ *belongs to* $X$, for any left (or right) germ $\alpha$ at $P$. There are two unique equivalence classes containing a vertical cell, called respectively the *lower* and *upper vertical germ*; the remaining ones are called *horizontal*. Given two left horizontal germs $\alpha$ and $\beta$ at $P$, we can find a common domain $I = \ ]u,a[$ and definable continuous functions $f$ and $g$ on $I$, such that $\alpha$ and $\beta$ are the respective equivalence classes of the graphs of $f$ and $g$. Let $I_-$, $I_=$ and $I_+$ be the subsets of all $x \in I$ such that $f(x)$ is less than, equal to, or bigger than $g(x)$ respectively. If $\alpha \neq \beta$, then $a$ cannot be in the closure of $I_=$, so that upon shrinking even further, we may assume $I_=$ is empty. Hence $a^-$ belongs either to $I_-$ or $I_+$ and we express this by saying that $\alpha <_{P^-} \beta$ and $\alpha >_{P^-} \beta$ respectively. This yields a well-defined total order relation $<_{P^-}$ on the left horizontal germs at a point $P$. To include the vertical germs, we declare the lower one to be smaller than any horizontal left germ and the upper one to be bigger than any.

6.4. PROPOSITION. *Let $X \subseteq M^2$ be a definable subset, and $P \in M^2$ a point. The set of all left germs $\alpha$ at $P$ such that $P_\alpha$ belongs to $X$ has an infimum $\beta$ (with respect to the order $<_{P^-}$). If $\beta$ is not vertical, then $P_\beta$ belongs to $\partial X$.*

PROOF. Since a point is either interior, exterior or a boundary point, we may upon replacing $X$ by its complement, reduce to the case that $P = (a, b)$ is either interior or a boundary point. In the former case, the lower vertical germ is clearly minimal, so assume $P \in \partial X$. In what follows, $\alpha$ always denotes a left germ at $P$. Consider the set $L_\emptyset$ of all $x < a$ such that $X[x] \cap J$ is empty for some open interval $J$ containing $b$. If $a^-$ belongs to $L_\emptyset$, then no $P_\alpha$ belongs to $X$ so that the upper vertical germ is the minimum. So we may assume that the $X[x] \cap J$ are non-empty for $x$ close to $a$ from the left. If $b^-$ belongs to $X[a]$, then the lower vertical germ is the infimum, so assume $b^-$ belongs to $M \setminus X[a]$. Hence we may shrink $J$ so that $J \cap (X[a])_{<b}$ is empty. For each $x < a$, let $f(x)$ be the infimum of $X[x] \cap J$. On a sufficiently small open interval $]u,a[$, the function $f$ is continuous, whence defines a 1-cell $V$. Since $J \cap (X[a])_{<b} = \emptyset$, the left limit $f^-(a)$ must be equal to $b$, showing that $(a, b)$ lies in the closure of $V$, and hence the equivalence class of $V$ at $P^-$ is a left germ $\beta$. It is now easy to show that $\beta$ is the required infimum, and that it is contained in the boundary $\partial X$. $\dashv$

6.5. COROLLARY. *If $C \subseteq M^2$ is a definable subset without interior, then so is its closure, that is to say, $C$ is nowhere dense.*

PROOF. Suppose $P = (a, b)$ is an interior point of the closure $\bar{C}$, so that there exists an open box $U \subseteq \bar{C}$ containing $P$. By Proposition 5.1, the fibers $C[x]$ for $x$ close to $a$ must be discrete. By Proposition 6.4, the infimum $\alpha$ of all left germs at $P$ belonging to $C \setminus C[x]$ exists, and by discreteness of the surrounding fibers, it must be a minimum, whence also belong to $C$. Similarly, the infimum $\beta$ of all left germs at $P$ belonging to $C$ and strictly bigger than $\alpha$ is also a minimum. Choose an open interval $]u, a[$ such that $\alpha$ and $\beta$ are represented by the respective continuous, definable maps $f, g\colon\ ]u, a[\ \to M$. Enlarging $u$ if needed, we may assume $f < g$, so that the 2-cell $S := C(\,]u, a[\,; f < g)$ is disjoint from $C$. Since $S$ is open and $P$ lies in its closure, $S \cap U$ is non-empty. Since $(S \cap U) \cap C = \emptyset$, no point of $S \cap U$ can lie in the closure of $C$, contradiction. $\dashv$

**6.6. Hardy structures.** We now extend this to infinity in the obvious way: given two horizontal cells $V$ and $W$ with domain an interval unbounded to the right, we say that $V \equiv_\infty W$ if their restrictions to some interval $]u, \infty[$ are equal. Let $\mathbf{H}(\mathcal{M})$ be the set of all *germs at infinity*, that is to say, equivalence classes of cells defined on an open interval unbounded to the right. Note that any definable map $f\colon Y \to M$ whose domain is unbounded to the right yields an equivalence class in $\mathbf{H}(\mathcal{M})$, denoted $[f]$, since $f$ is continuous by Theorem 3.2 on some open interval $]u, \infty[\ \subseteq Y$. Given a definable subset $X \subseteq M^2$, we can say, as before, that $\infty_\alpha$ *belongs to* $X$, if $\infty^-$ belongs to the set of all $x \in Y$ such that $(x, f(x)) \in X$, for some $f$ with germ $\alpha$. However, in this case we can do more and make $\mathbf{H}(\mathcal{M})$ into an $L$-structure: if $\underline{c}$ is a constant symbol, then we interpret it in $\mathbf{H}(\mathcal{M})$ as the class of the constant function with value $c := \underline{c}^{\mathcal{M}}$; if $\underline{F}$ is an $n$-ary function symbol, and $\alpha_1, \dots, \alpha_n \in \mathbf{H}(\mathcal{M})$, then $\underline{F}(\alpha_1, \dots, \alpha_n)$ is the class given by the definable map $\underline{F}(g_1, \dots, g_n)$, where the $g_i$ are definable functions with domain $I := \ ]u, \infty[$ such that $[g_i] = \alpha_i$; if $\underline{R}$ is an $n$-ary predicate symbol, then $\underline{R}(\alpha_1, \dots, \alpha_n)$ holds in $\mathbf{H}(\mathcal{M})$ if and only if $\infty^-$ belongs to the set of all $x \in I$ such that $\underline{R}(g_1(x), \dots, g_n(x))$ holds in $\mathcal{M}$.

6.7. DEFINITION. We call this $L$-structure on $\mathbf{H}(\mathcal{M})$ the *Hardy structure* of $\mathcal{M}$. In particular, by the same argument as above, $<$ interprets a total order on $\mathbf{H}(\mathcal{M})$, making it into a densely ordered structure without endpoints (note that the notion of vertical germ makes no sense in this context).

By induction on the complexity of formulae, we easily can show:

6.8. LEMMA. *Let $\varphi(x_1, \dots, x_n)$ be a formula with parameters from $M$ and let $X \subseteq M^n$ be the set defined by it. For given germs $\alpha_1, \dots, \alpha_n \in \mathbf{H}(\mathcal{M})$, we have $\mathbf{H}(\mathcal{M}) \models \varphi(\alpha_1, \dots, \alpha_n)$ if and only if there is a $u \in M$ such that $(g_1(x), \dots, g_n(x)) \in X$, for all $x > u$, where each $g_i$ is some continuous function defined on $]u, \infty[$ representing the germ $\alpha_i$.* $\dashv$

Since a continuous function with values in a discrete set must be constant, Lemma 6.8 yields:

6.9. COROLLARY. *If a discrete subset $D \subseteq \mathbf{H}(\mathcal{M})^n$ is definable with parameters in $\mathcal{M}$, then $D \subseteq M^n$.* $\dashv$

6.10. THEOREM. *There is a canonical elementary embedding $\mathcal{M} \to \mathbf{H}(\mathcal{M})$. In particular, $\mathbf{H}(\mathcal{M}) \models$ DCTC.*

PROOF. The map $M \to \mathbf{H}(\mathcal{M})$ sending an element $a \in M$ to the class of the corresponding constant function is easily seen to be an elementary embedding. $\dashv$

These two results together show that if $\mathcal{M}$ is a non-o-minimal model of DCTC, then $(\mathcal{M}, \mathbf{H}(\mathcal{M}))$ is a Vaughtian pair (see, for instance, [16, Proposition 9.3]). In particular, DCTC has Vaughtian pairs.

6.11. *Remark.* We can think of $\mathbf{H}(\mathcal{M})$ as a sort of *protoproduct*, in the meaning of a 'controlled' subring of an ultraproduct as studied in [18, Chapter 9]. Namely, endowing the set $M$ with an ultrafilter containing all right unbounded open intervals, then $\mathbf{H}(\mathcal{M})$ consists of all elements in the ultrapower $\mathcal{M}_\natural$ given by definable maps (whereas an arbitrary element is given by any map).

We also can define a *standard part* operator, at least on the subset $\mathbf{H}^{\text{fin}}(\mathcal{M})$ of all *finite* elements, that is to say, the set of all germs $\alpha$ at infinity represented by some definable, continuous map $f\colon\ ]u,\infty[\ \to M$ such that $f(\infty) \in M$ (see Remark 3.4 for the definition). Indeed, the value of $f(\infty)$ only depends on $\alpha$, thus yielding a standard part map $\mathbf{H}^{\text{fin}}(\mathcal{M}) \to M$. Note, however, that as $\mathcal{M}$ is not definable in $\mathbf{H}(\mathcal{M})$, neither is $\mathbf{H}^{\text{fin}}(\mathcal{M})$.

6.12. *Remark.* Hardy structures play an important role in o-minimality; for instance, they were used by Miller to prove the growth dichotomy. For an overview, see [14] and the references therein.

## §7. Planar curves.

As always, $\mathcal{M}$ is a model of DCTC.

**7.1. Dimension.** Let us say that a non-empty definable subset $X \subseteq M^2$ has *dimension zero* if it is discrete, and *dimension two*, if it has non-empty interior. In the remaining case, we will put $\dim(X) = 1$ and call $X$ a (generalized) *planar curve*. We will assign to the empty set dimension $-\infty$, in order to make the following formula work (with the usual conventions that $-\infty + n = -\infty$):

7.2. COROLLARY. *Given a definable subset $X \subseteq M^2$, let $F_e$ be the set of all $a \in M$ for which the fiber $X[a]$ has dimension $e$, for $e = 0, 1$. Then each $F_e$ is definable and the dimension of $X$ is equal to the maximum of all $e + \dim(F_e)$.*

PROOF. Being discrete and having interior are definable properties, whence so is being a planar curve, showing that each $F_e$ is definable. The formula then follows by inspecting the various cases by means of Corollary 4.4 and Proposition 5.1. ⊣

7.3. *Remark.* There are several ways of extending this definition to larger arity, and the usual one is to define the dimension of a definable subset $X \subseteq M^n$ as the largest $d$ such that the image of $X$ under some projection $\pi\colon M^n \to M^d$ has non-empty interior. It follows that a $d$-cell has dimension $d$.

We may rephrase the previous result as a trichotomy theorem for planar definable subsets:

7.4. THEOREM (Planar Trichotomy). *Any planar definable subset of $\mathcal{M}$ either*

7.4.i. *is discrete, closed, and bounded;*

7.4.ii. *is nowhere dense, but at least one projection onto a coordinate axis has non-empty interior;*

7.4.iii. *has non-empty interior.*

PROOF. We only need to show that (7.4.ii) is equivalent with having dimension one. The converse is clear from Corollaries 7.2 and 6.5, and for the direct implication, we must show that a definable subset satisfying (7.4.ii) cannot be discrete, and this follows from Theorem 4.1. ⊣

Immediate from the definitions and Corollary 5.3, we have:

7.5. COROLLARY. *The dimension of a union $X_1 \cup \dots \cup X_n \subseteq M^2$ of definable subsets is the maximum of the dimensions of the $X_i$.* ⊣

**7.6. Nodes.** Let $S \subseteq M^2$ be an arbitrary subset. We call a point $P \in S$ a *node*, if for every open box $B$ containing $P$, there is an open sub-box $I \times J \subseteq B$ containing $P$ and some point $x \in I$ such that $S[x] \cap J$ is not a singleton. We denote the set of nodes of $S$ by $\mathrm{Node}(S)$. We call a node an *edge*, if in the above condition $S[x] \cap J$ can be made empty. By an argument similar to the one proving Corollary 3.5, one shows that a function on an open interval $h$ is continuous if and only if its graph has no edges (since it is a graph, it cannot have any other type of nodes). Note that the closure of a 1-cell $C$ has at most two edges: indeed, if $C$ is given as the graph of a definable, continuous function $h$ on an interval $\,]a,b[\,$, then $\bar{C} \setminus C$ consists of those points among $(a, h^+(a))$ and $(b, h^-(b))$ that are finite (in the notation of Remark 3.4), and these are then the edges of $\bar{C}$.

Assume now that $C$ is a planar curve. The isolated points of $C$ are edges, and they form a discrete, closed, and bounded subset. Another special case of an edge is any point lying on an open interval inside a vertical fiber $C[a]$. Let $\mathrm{Vert}(C)$ be the set of all such edges, called the *vertical component* of $C$. Note that $\mathrm{Vert}(C)$ is equal to the union of the interiors of all fibers, that is to say, $\mathrm{Vert}(C) = \bigcup_a (C[a])^\circ$, and hence in particular is definable.

7.7. PROPOSITION. *The set of nodes of a planar curve in $\mathcal{M}$ is the union of its vertical component and a discrete set.*

PROOF. Let $C \subseteq M^2$ be a planar curve. Replacing $C$ by $C \setminus \mathrm{Vert}(C)$, we may assume its vertical component is empty. Assume towards a contradiction that $N := \mathrm{Node}(C)$ is not discrete. Therefore, $\pi(N)$ cannot be discrete by Corollary 4.4, and hence contains an open interval $I$. For each $x \in I$, let $h(x)$ be the minimal $y \in C[x]$ such that $(x, y) \in N$. By Theorem 3.2, we may shrink $I$ so that $h$ becomes a continuous function on $I$. In particular, its graph $V$ is a 1-cell contained in $N$. For each $x \in I$, let $l(x)$ and $u(x)$ be the respective predecessor and successor in $C[x]$ of $h(x)$ (if $h(x)$ is always an extremal element of $C[x]$ then we can adjust the argument accordingly, and so we just assume that $l(x) < h(x) < u(x)$ always exist). Since $(x, h(x))$ is a node and $h$ is continuous, for $y < x$ sufficiently close to $x$, and $J$ an open interval such that $J \cap C[x] = \{h(x)\}$, the intersection $J \cap C[y]$ contains at least one other element besides $h(y)$, necessarily either $l(y)$ or $u(y)$. By type completeness, either $l(y)$ belongs to all $J \cap C[y]$, for all $y$ sufficiently close to the left of $x$, or otherwise $u(y)$ does. In particular, for a fixed $x \in I$, we have $h(x) = l^-(x)$ or $h(x) = u^-(x)$. Shrinking $I$ if necessary, type completeness then reduces to the case that one of these alternatives happens for every $x \in I$, say, $h(x) = l^-(x)$ for all $x \in I$. Shrinking $I$ even further, we may assume that $l$ is continuous on $I$, and hence $l = h$ on $I$, contradiction. ⊣

7.8. LEMMA. *A point $P$ on a planar curve $C \subseteq M^2$ is not a node if and only if there is some open box $B$ containing $P$ such that $C \cap B$ is a horizontal 1-cell. On the other hand, $P$ is an edge if and only if it does not belong to any horizontal cell inside $C$.*

PROOF. If $P = (a, b) \notin \mathrm{Node}(C)$, there exist open intervals $I$ and $J$ containing respectively $a$ and $b$ such that $C[x] \cap J$ is a singleton $\{f(x)\}$, for every $x \in I$, and this property is preserved for any sub-box of $I \times J$ containing $P$. Hence $f \colon I \to J$ is a definable map with $f(a) = b$. Shrinking $I$ if necessary, we may assume by Theorem 3.2 that $f$ is continuous on $I$ with a possible exception at $a$. As already observed, $f$ is also continuous at $a$ lest $(a, f(a))$ be a node. Hence the graph of $f$ is a cell equal to $(I \times J) \cap C$. If $P$ is not a node, then by definition, no intersection with an open box around $P$ can be a cell. The second assertion is obvious. ⊣

7.9. *Remark.* In [21], non-nodal points are called *normal* points. We may generalize this to higher arity: let us say that a point $P$ on a definable subset $X \subseteq M^n$ is *strongly* $e$*-normal*, for some $e \leq n$, if there exists an open box $B$ containing $P$ such that $B \cap X$ is an $e$-cell. When $n = 2$, a point is strongly 2-normal if and only if it is interior, and strongly 0-normal if and only if it is isolated. The previous result then says that on a planar curve, a point is strongly 1-normal if and only if it is not a node. As with cells, this definition of normality has a directional bias: nodes are really critical points with respect to projection onto the first coordinate. To break this bias, just taking permutations of the variables does not give enough transformations to turn some point on a curve in a non-nodal position, as for instance the origin on the curve given by $(t, t)$ if $t \leq 0$ and $(-t, -2t)$ if $t \geq 0$. However, if we assume that there is an underlying ordered group, then we say that a point $x \in X \subseteq M^n$ is $e$*-normal*, if after a translation bringing $x$ to the origin $O$, we can find a rotation $\rho$ such that $\rho(O)$ is strongly $e$-normal in $\rho(X)$, where by a *rotation* of $M^n$, we mean a linear map $\rho \colon M^n \to M^n$ given by an invertible matrix of determinant one over $\mathbb{Q}$ (by [12, Proposition 2.2], any model of DCTC expanding a group is divisible, whence admits a natural structure of a $\mathbb{Q}$-vector space).

7.10. PROPOSITION. *A definable subset $X \subseteq M^2$ has the same dimension as that of its closure $\bar{X}$, whereas the dimension of its frontier $\mathrm{fr}(X)$ is strictly less.*

PROOF. If $X$ is discrete, then it is closed by Corollary 4.2, and so $\mathrm{fr}(X) = \emptyset$, proving the assertion in this case. If $X$ has dimension one, then so does $\bar{X}$ by Corollary 6.5. Let $V := \mathrm{Vert}(C)$ be the vertical component of $C$ and let $\pi(V)$ be its projection. Since $\pi(V)$ is discrete by Proposition 5.1, the boundary $\partial V$ is equal to the union of all $\partial(X[a])$, whence is discrete by Corollary 4.4. Hence, upon removing $V$ from $X$, we may reduce to the case that $X$ has no vertical components. Suppose towards a contradiction that $\mathrm{fr}(X)$ is a planar curve. By Proposition 7.7, the set of nodes on $\bar{X}$ and on $\mathrm{fr}(X)$ are both discrete sets, and so, there exists a $P \in \mathrm{fr}(X)$ which is not a node on $\mathrm{fr}(X)$ nor on $\bar{X}$. By Lemma 7.8, there exists an open box $B$ containing $P$ such that both $B \cap \mathrm{fr}(X)$ and $B \cap \bar{X}$ are cells, and therefore the inclusion $B \cap \mathrm{fr}(X) \subseteq B \cap \bar{X}$ must be an equality. In particular, $B \cap X$ is empty, contradicting that $P$ lies in the closure of $X$.

Finally, if $X$ has dimension two, then so must $\bar{X}$. Let $Y := X^\circ$ and $Z := X \setminus Y$. Since $\bar{X} = \bar{Y} \cup \bar{Z}$, we have $\mathrm{fr}(X) = (\bar{Y} \setminus X) \cup (\bar{Z} \setminus X)$, so that it suffices to show that neither of these two differences has interior. The first one, $\bar{Y} \setminus X$, is equal to $\partial Y$ whence has no interior, being the boundary of an open set. By construction, $Z$ has no interior, and hence by Corollary 6.5, neither does its closure. ⊣

Recall that a *constructible* subset is a finite Boolean combination of open subsets, and hence every one-variable definable subset is constructible. This is still true in higher dimensions: by an easy induction on the dimension, and using that the closure is obtained by adjoining the frontier, Proposition 7.10 yields:

7.11. COROLLARY. *Every $\mathcal{M}$-definable subset is constructible.* ⊣

7.12. COROLLARY. *The boundary of a two-dimensional, planar subset has dimension at most one.*

PROOF. Let $X \subseteq M^2$ have dimension two. Its boundary $\partial X$ is the union of its frontier $\mathrm{fr}(X)$ and $X \setminus X^\circ$. The former has dimension at most one by Proposition 7.10 and the latter has no interior. The result now follows from Corollary 5.3. ⊣

Recall that a subset in a topological space is called *codense* if its complement is dense.

7.13. COROLLARY. *If $Y$ is a codense definable subset of a non-empty definable subset $X \subseteq M^2$, then $\dim(Y) < \dim(X)$.*

PROOF. If $X$ is discrete, then it is closed by Corollary 4.2, and hence its only codense subset is the empty set. If $X$ and $Y$ both have dimension two, then $Y^\circ$ is disjoint from the closure of $X \setminus Y$, contradicting that $Y$ is codense in $X$. So remains the case that $X$ is a curve. If $Y$ is codense in $X$, then it must be contained in the frontier of $X \setminus Y$, and the latter has dimension strictly less than one by Proposition 7.10. ⊣

§**8. Planar cell decomposition.** In o-minimality, *cell decomposition* is the property that we can partition any given definable subset $X$ into a disjoint union of cells. Every point is a 0-cell but writing $X$ as a union of its points should not qualify as a cell decomposition. Slightly less worse, if $X$ is planar, then each fiber $X[a]$ is a disjoint union of intervals and points, so that we can partition $X$ into points and vertical cells. Of course, in the o-minimal context these pathologies are avoided by demanding the partition be finite. For arbitrary models of DCTC, however, we can no longer enforce finiteness, and so to exclude any unwanted partitions, we must impose some weaker restrictions. Moreover, at present, I do not see how to avoid—but see §9 below—, the use of quasi-cells:

**8.1. Quasi-cells.** We again work in a fixed model $\mathcal{M}$ of DCTC.

8.2. LEMMA. *We call a subset $S \subseteq M^2$ a (horizontal)* 1-quasi-cell *if it satisfies one of the following equivalent conditions:*

8.2.i. *$S$ is a union of mutually intersecting 1-cells in $M^2$ and has no nodes;*

8.2.ii. *$S$ is the graph of a continuous map $h\colon \pi(S) \to M$ which is* locally definable*, meaning that its restriction to any open interval in its domain is definable.*

*Moreover, $\pi(S)$ is then open and convex, and $S$ is a 1-cell if and only if $\pi(S)$ is definable.*

PROOF. The implication (8.2.ii) ⇒ (8.2.i) is easy, since the graph of a continuous function has no nodes. To show (8.2.i) ⇒ (8.2.ii), suppose $S$ has no nodes, so that in particular, no vertical cell lies inside $S$. Fix $a_1, a_2 \in \pi(S)$ and choose non-disjoint 1-cells $C_1 \subseteq S$ and $C_2 \subseteq S$ containing $a_1$ and $a_2$ respectively. Let $I_k := \pi(C_k)$ and let $h_k$ be the definable (continuous) function on $I_k$ whose graph is $C_k$. Let $I := I_1 \cup I_2$. Since

$C_1 \cap C_2$ is non-empty, so is $I_1 \cap I_2$, showing that $I$ is an interval. Let $H$ be the subset of $I_1 \cap I_2$ on which $h_1$ and $h_2$ agree, that is to say, $H = \pi(C_1 \cap C_2)$. For $a \in H$ with common value $b$, since $(a, b)$ is not a node, there exist open intervals $U$ and $V$ containing respectively $a$ and $b$ such that $S[x] \cap V$ is a singleton, for all $x \in U$. Shrinking $U$ if necessary, continuity allows us to assume that $h_k(U) \subseteq V$, so that $(x, h_k(x))$ both lie in $S[x] \cap V$ for $x \in U$, whence must be equal. This shows that $U \subseteq H$, and hence that $H$ is open. Let $a \in \partial H$. Since $H$ is open, $a$ does not belong to $H$ whereas either $a^-$ or $a^+$ does. If $a$ lies in $I_1 \cap I_2$, then $a \notin H$ implies $h_1(a) \neq h_2(a)$, and by continuity, this remains the case on some open interval around $a$, contradicting that either $a^-$ or $a^+$ belongs to $H$. Hence $a \notin I_1 \cap I_2$. This means that the only boundary points of $H$ are the endpoints of the interval $I_1 \cap I_2$, proving that $h_1$ and $h_2$ agree on this interval. Let $h(x)$ be equal to $h_1(x)$ if $x \in I_1$ and to $h_2(x)$ otherwise. Since the graph of $h$ is then equal to $C_1 \cup C_2 \subseteq S$, whence contains no nodes, $h$ is continuous. It is not hard to see that $\pi(S)$ is open and convex. The last assertion then follows since $\pi(S)$ is a disjoint union of open intervals by Theorem 2.10, whence, being also convex, a single open interval, if definable. ⊣

8.3. *Remark.* Condition (8.2.i) is equivalent with: $S$ is a direct union of 1-cells and all points are 1-normal.

Although we should also entertain the notion of vertical quasi-cells (see Definition 8.5 below), they do not occur in the analysis of planar subsets. Given a curve $C$ without nodes and a quasi-cell $S \subseteq C$, we say that $S$ is *optimal* in $C$, if no quasi-cell inside $C$ strictly contains $S$.

8.4. COROLLARY. *Any point on a planar curve $C \subseteq M^2$ without nodes lies on a (uniquely determined) optimal quasi-cell in $C$. In particular, $C$ is a disjoint union of quasi-cells.*

PROOF. Fix $P \in C$. By Lemma 7.8, there exists a cell $V \subseteq C$ containing $P$. Let $S$ be the union of all cells inside $C$ containing $P$. Since $S \subseteq C$, has no nodes, it is a quasi-cell by Lemma 8.2. Suppose $S' \subseteq C$ is a quasi-cell containing $S$ and let $P' \in S'$. By Lemma 8.2, there exists a cell $V' \subseteq S'$ containing both $P$ and $P'$. By construction, we then have $V' \subseteq S$, whence $P' \in S$, showing that $S = S'$ is optimal. ⊣

If the curve has nodes, then to preserve uniqueness of optimal quasi-cells, we have to amend this definition as follows: for an arbitrary planar curve $C$, a horizontal 1-quasi-cell $S$ is called *optimal* if $S \subseteq C$ contains no node of $C$ and is maximal with this property. Finally, we define the notion of a 2-*quasi-cell* $S \subseteq M^2$ given as the region between two continuous, locally definable maps defined on an open, convex subset of $M$ (again called the *domain* of the quasi-cell), or an unbounded variant as in the case of 2-cells. More precisely, let $V \subseteq M$ be an open convex subset, $f, g\colon V \to M$ continuous and locally definable with $f < g$, then $S$ consist of all $(x, y)$ such that $x \in V$ and $f(x) \diamond y \diamond' g(x)$, with $\diamond, \diamond'$ strict inequality or no condition. By definition of local definability, the restriction of a quasi-cell $S$ to an open interval $I \subseteq V$, that is to say, $S \cap (I \times M)$ is a cell, and hence every 2-quasi-cell is the union of 2-cells and therefore open. Moreover, we can arrange for all these cells in this union to contain a given fixed point of the quasi-cell.

8.5. DEFINITION. The definition of an arbitrary $d$-quasi-cell is entirely similar: simply replace in the recursive definition from Remark 6.1 'cell' by 'quasi-cell' and 'definable, continuous map' by 'locally definable, continuous map' everywhere.

**8.6. Locally definable subsets.** Quasi-cells are particular instances of locally definable subsets, which we now briefly study. In an arbitrary ordered structure $\mathcal{O}$, we call a subset $X \subseteq O^n$ *locally definable* if for each point $P \in O^n_\infty$, there exists an open box $B$ containing $P$ such that $B \cap X$ is definable. It is important to include in this definition also the *infinite points* of $O^n_\infty$, that is to say, points with at least one coordinate equal to $\pm\infty$, where, just as an example, we mean by an *open box around an infinite point* like $(0, \infty, -\infty)$ one of the form $]u, v[\ \times\ ]p, \infty[\ \times\ ]-\infty, q[$, with $u < 0 < v$. It is also important to note that the definition applies to all points, not just to those belonging to $X$. In fact, the condition is void if $P$ is either an interior or an exterior point, since then some open box is entirely contained in or entirely disjoint from $X$. So we only need to verify local definability at boundary points and at infinite points. Therefore, any clopen is locally definable. It is not hard to show that a finite Boolean combination of locally definable sets is again locally definable. Moreover, the interior, closure and exterior of a locally definable subset are again locally definable. Using (8.2.ii), it is easy to see that 1-quasi-cells are locally definable, and using a higher dimensional version of Lemma 8.2, one can extend this to any quasi-cell.

8.7. PROPOSITION. *A discrete set is locally definable in $\mathcal{M}$ if and only if it is closed and bounded.*

PROOF. Let $D \subseteq M^n$ be discrete. If $D$ is not closed, then there is a $P \in \partial D$ not belonging to $D$. But then the intersection $D \cap B$ with any open box $B$ containing $P$ will have $P$ in its closure, so that $D \cap B$ is not closed, whence cannot be definable by Corollary 4.2. Similarly, if $D$ is not bounded, say, in the first coordinate on the right, then its intersection with any open box of the form $]p, \infty[\, \times B'$ will still be unbounded, whence not definable by Corollary 4.2. Suppose therefore $D$ is closed and bounded. To check local definability at a boundary point $P$, as it belongs to $D$ by closedness, there is an open box $B$ such that $D \cap B = \{P\}$. To check at an infinite point, we can find an open box around $P$ which is disjoint from $D$. ⊣

8.8. COROLLARY. *The topological boundary of a locally definable subset $Y \subseteq M$ is a discrete, closed, bounded set.*

PROOF. If the locally definable set $\partial Y$ has non-empty interior, it would contain an open interval $I$ and we may shrink this so that $F := I \cap Y$ is definable. Since $\partial F = I \cap \partial Y = I$, we get a contradiction with (2.6.vi). Hence $\partial Y$ has no interior, and so, for $b \in \partial Y$ and an open interval $I$ containing $b$ such that $I \cap \partial Y$ is definable, the latter set, having no interior, must be discrete by (2.6.iv), and hence, shrinking $I$ further if necessary, $I \cap \partial Y = \{b\}$. Hence $\partial Y$ is discrete, whence bounded and closed by Proposition 8.7. ⊣

Given an arbitrary ordered structure $\mathcal{M}$, let $\mathcal{M}^{\text{loc}}$ be the structure generated by the locally definable subsets of $\mathcal{M}$ (formally, we have a language with an $n$-ary predicate $\mathtt{X}$ for any locally definable subset $X \subseteq M^n$, and we interpret $\mathtt{X}(\mathcal{M}^{\text{loc}})$ as the subset $X$). Since the class of locally definable subsets is closed under projection, fibers, and finite Boolean combinations, the definable subsets of $\mathcal{M}^{\text{loc}}$ are precisely the locally $\mathcal{M}$-definable subsets.

8.9. COROLLARY. *The reduct $\mathcal{M}^{loc}$ is type complete.*

PROOF. Given a one-variable definable subset of $\mathcal{M}^{\mathrm{loc}}$, whence a locally definable subset $Y \subseteq M$, and a point $a \in M_\infty$, we may choose an open interval $I$ around $a$ such that $Y \cap I$ is definable. Since $a^-$ belongs to $Y \cap I$ or to its complement, the same is true with respect to $Y$, proving type completeness. $\dashv$

Since bounded clopens are locally definable but have no infimum, definable completeness usually fails and $\mathcal{M}^{\mathrm{loc}}$ is in general not a model of DCTC.

**8.10. Planar cell decomposition.** Let us introduce the following terminology, which we give only for the planar case (but can easily be extended to larger arity, see Remark 8.12 below). First we extend the definition of dimension to arbitrary subsets of the plane (which is not necessarily well-behaved if the subset is not definable) by the same characterization: a non-empty subset $B \subseteq M^2$ has *dimension* 2, if it has non-empty interior; *dimension* 1, if it has empty interior but is non-discrete; and *dimension zero* if it is discrete. We can also define the *local dimension* $\dim_P(B)$ of $B$ at a point $P \in M^2$ as the minimal dimension of $B \cap U$ where $U$ runs over all open boxes containing $P$. Note that $\dim_P(B) \geq 0$ if $P \in \bar{B}$. It follows that the dimension of $B$ is the maximum of its local dimensions at all points. It is not hard to see that the dimension of $B$ is the largest $e$ for which it contains an $e$-cell. In particular, a 2-quasi-cell has dimension 2, whereas a 1-quasi-cell has dimension one. More generally, by Corollaries 2.9 and 5.3 and the local nature of dimension, we showed that:

8.11. LEMMA. *The dimension of a finite union of $e_i$-quasi-cells is equal to the maximum of the $e_i$.* $\dashv$

Given a collection $\mathcal{B}$ of (not necessarily definable) subsets of $M^2$, we say that a definable subset $X \subseteq M^2$ has a *$\mathcal{B}$-decomposition*, if there exists a partition $X = \bigsqcup_{i \in I} B_i$ with all $B_i \in \mathcal{B}$, with the additional property that if $X^{(e)}$ denotes the union of all $e$-dimensional $B_i$ in this partition, then $X^{(e)}$ is definable and has dimension at most $e$, for $e = 0, 1, 2$ (whence of dimension $e$ if and only if it is non-empty). Put simply, in a decomposition there cannot be too many lower dimensional subsets. By a *cell* decomposition (respectively, a *quasi-cell* decomposition) we mean a $\mathcal{B}$-decomposition where $\mathcal{B}$ is the collection of all (quasi-)cells. By Lemma 8.11, any finite partition into quasi-cells is a cell decomposition (as there can be no quasi-cell in a finite decomposition since each subset in the partition is then definable).

8.12. *Remark.* For higher arities, we define the dimension of a subset $B \subseteq M^n$ to be the largest $d$ such that it contains a $d$-cell (in case $B$ is not definable, this might be different from the largest $d$ such that the projection of $B$ onto some $M^d$ has non-empty interior, but both are equal in the definable case). The definition of $\mathcal{B}$-decomposition for a definable set $X \subseteq M^n$ now easily generalizes: it is a partition of $X$ into sets from $\mathcal{B}$ such that the union of all sets in this partition of a fixed dimension $e$ is a definable subset of dimension at most $e$.

8.13. THEOREM. *In $\mathcal{M}$, any planar definable subset has a quasi-cell decomposition.*

PROOF. Let $X \subseteq M^2$ be a definable subset. There is nothing to show if $X$ has dimension zero. If $X$ is a curve, then $\mathrm{Vert}(X)$ is a disjoint union of vertical cells (see the proof of Proposition 7.10). So after removing it from $X$, we may assume $X$ has no vertical components. In that case, $\mathrm{Node}(X)$ is discrete by Proposition 7.7, and so after removing it, we may assume $X$ has no nodes, and so we are done by Corollary 8.4.

So remains the case that $X$ is 2-dimensional. Let $C := \partial X$ be its boundary. Since $C$ has dimension at most one by Corollary 7.12, and so can be decomposed into disjoint quasi-cells by what we just argued, we may assume, after removing it, that $X$ is moreover open. The projection $\pi(N)$ of the set $N := \mathrm{Node}(C)$ of all nodes is discrete by Proposition 7.7 and Theorem 4.1, and therefore $X \cap (\pi(N) \times M)$ can be partitioned into vertical cells. So remains to deal with points $(a, b) \in X$ such that $a \notin \pi(N)$. Since $X$ is open, $b$ is an interior point of $X[a]$. Let $l$ and $h$ be respectively the maximum of $(C[a])_{\leq b}$ and the minimum of $(C[a])_{\geq b}$, so that the open interval $]l, h[$ lies inside $X[a]$ and contains $b$ (we leave the case that one of these endpoints is infinite to the reader). By choice, neither $(a, l)$ nor $(a, h)$ is a node of $C$, so that by Corollary 8.4, there are (unique) optimal 1-quasi-cells $L, H \subseteq C$ containing $(a, l)$ and $(a, h)$ respectively, say, given as the graphs of locally definable, continuous maps $f\colon V \to M$ and $g\colon W \to M$. Consider all open intervals $I \subseteq V \cap W$ containing $a$ such that the 2-cell $C(I;\, f|_I < g|_I)$ lies entirely inside $X$, and let $Z \subseteq M$ be the union of all these intervals. Hence $Z$ is open and convex, and $C(Z;\, f|_Z < g|_Z)$ is an (optimal) 2-quasi-cell inside $X$, by Lemma 8.2. To show that this construction produces a disjoint union of quasi-cells, we need to show that if $(a', b')$ is any point in $S$, then the above procedure yields exactly the same quasi-cell containing $(a', b')$. Indeed, by convexity, we can find an open interval $I \subseteq V$ containing $a$ and $a'$. Since the intersections of $F$ and $G$ with $I \times M$ are 1-cells, $C(I;\, f|_I < g|_I) = S \cap (I \times M)$ is a 2-cell contained in $X$, whence must lie inside $S$ by construction.

To show that this is a decomposition, we induct on the dimension $d$ of $X$, where the case $d = 0$ is trivial. In the above, at various stages, we had to remove some subsets of $X$ of dimension strictly less than $d$, and partition each separately. Since each of these finitely many exceptional sets was definable, so is their union and by Lemma 8.11, has dimension strictly less than $d$. Hence the complement $X^{(d)}$, consisting of all $d$-quasi-cells in the partition, is also definable. After removing $X^{(d)}$, we are left with a definable subset of strictly less dimension, and so we are done by induction. ⊣

The proof gives in fact some stronger results, where for the sake of brevity, we will view any point as a 0-quasi-cell:

8.14. *Remark.* Keeping track of the various (quasi-)cells, we actually showed that we may partition $X$ in quasi-cells $S_i$, such that each $\bar{S}_i \cap X$ is a disjoint union of $S_i$ and some of the other $S_j$.

§**9. Tameness.** The quasi-cell decomposition version given by Theorem 8.13 is not very useful in applications. Moreover, the non-definable nature of quasi-cells is a serious obstacle. Perhaps quasi-cells never occur, but in the absence of a proof of this, we make the following definitions, for $\mathcal{O}$ any (ordered) $L$-structure. Let us call a definable map $c\colon X \to Y$ *pre-cellular*, if every fiber $c^{-1}(y)$ is a cell. Note that the non-empty fibers of $c$ then constitute a partition of $X$ into cells. Injective maps are pre-cellular, but the resulting partition in cells is clearly not a decomposition if $X$ has positive dimension. To guarantee that we get a cell decomposition, we require moreover that the image of $c$ be discrete, bounded, and closed, and we call such a map then *cellular*. In particular, we may assume, if we wish to do so, that the cellular map $c\colon X \to D$ is surjective, where $D$ is discrete, bounded, and closed.

Assume now that $\mathcal{O} \models$ DCTC, and let $c\colon X \to D$ be cellular. The collection $X^{(e)}$ of all fibers $c^{-1}(y)$ of dimension $e$ is a definable subset, for each $e$, since we can express in a first-order way whether a fiber $c^{-1}(y)$ has dimension $e$ (for instance, if $X$ is planar, then having interior or being discrete are elementary properties). If $X^{(e)}$ is non-empty, then its dimension is equal to $e$ by Corollary 7.2 and the fact that $D$ is discrete, showing that we have indeed a cell decomposition. Note that in particular, the graph of $c$ and $X$ have the same dimension. We have the following converse: given a definable subset $X \subseteq O^k$ and a cellular map $c\colon X \to Y$ such that $X = \bigsqcup_{y \in c(X)} c^{-1}(y)$ is a cell decomposition, then $c$ is cellular, that is to say, $c(X)$ is discrete. It suffices to show that each $c(X^{(e)})$ is discrete, for $e \leq k$, and this follows from Corollary 7.2. It is not clear whether every definable subset admits a cell decomposition of this type, and so we make the following definition:

9.1. DEFINITION. A definable subset $X \subseteq O^n$ in an ordered structure $\mathcal{O}$ is called *eukaryote* if it is the domain of a cellular map. If every definable subset in $\mathcal{O}$ is eukaryote, then we call $\mathcal{O}$ *eukaryote*.

9.2. LEMMA. *Any eukaryote structure is a model of DCTC.*

PROOF. Let $\mathcal{O}$ be eukaryote. By Theorem 2.10, it suffices to show that any definable subset $Y \subseteq O$ is a disjoint union of open intervals and a single discrete, bounded, and closed subset. Let $c\colon Y \to D$ be cellular, with $D$ discrete, bounded, and closed. Hence, each fiber $c^{-1}(a)$ must be a one-variable cell, that is to say, either a point or an open interval. Let $E$ be the subset of all $a \in D$ for which $c^{-1}(a)$ is a point. Hence $Y \setminus c^{-1}(E)$ is a disjoint union of open intervals, so that upon removing them, we may assume $E = D$, so that $c$ is a bijection. Since $D$ is discrete, $c^{-1}$ is continuous. Therefore, $Y = \operatorname{Im}(c^{-1})$ is closed and bounded by [12, Prop. 1.10] (which we may invoke by definable completeness). Moreover, by the same result, any closed subset is again mapped to a closed subset, showing that $c^{-1}$, whence also $c$, is a homeomorphism. In particular, $Y$ is discrete, as we needed to show. ⊣

Clearly any cell is eukaryote. Since a (principal) fiber of a cell is again a cell, the same holds for eukaryote subsets. Since a principal projection of a cell is a cell, the collection of eukaryote subsets is closed under principal projections (we will generalize this in Corollary 9.13 below). Any finite cell decomposition is easily seen to be given by a cellular map, and hence in particular, any o-minimal structure is eukaryote.

9.3. PROPOSITION. *Suppose $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are elementary equivalent $L$-structures. If $\mathcal{O}$ is eukaryote, then so is $\tilde{\mathcal{O}}$.*

PROOF. Since both structures have isomorphic ultraproducts, we only need to show that eukaryoteness is preserved under elementary substructures and extensions. The former is easy, so assume $\mathcal{O}$ is a eukaryote elementary substructure of $\tilde{\mathcal{O}}$ and let $\tilde{X}$ be a definable subset in $\tilde{O}^n$. Since eukaryoteness is preserved under fibers, we may assume that $\tilde{X}$ is definable without parameters, say $\tilde{X} = \varphi(\tilde{\mathcal{O}})$. By assumption, there exists a cellular map $c\colon \varphi(\mathcal{O}) \to D$, that is to say, formulae $\gamma$ and $\delta$, with $\gamma(\mathcal{O})$ the graph of a map all of whose fibers are cells of dimension at most $n$ and whose image is the discrete, closed, bounded subset $\delta(\mathcal{O})$. Since all this is first-order, it must also hold in $\tilde{\mathcal{O}}$, so that $\gamma(\tilde{\mathcal{O}})$ is the graph of a cellular map $\tilde{c}\colon \tilde{X} \to \delta(\tilde{\mathcal{O}})$. ⊣

By Theorem 6.10, the associated Hardy structure of a eukaryote structure is therefore again eukaryote.

9.4. PROPOSITION. *In a reduct of a eukaryote structure, every definable set admits a cell decomposition.*

PROOF. Let $\mathcal{O}$ be a eukaryote structure, $\bar{\mathcal{O}}$ some reduct, and $X$ an $\bar{\mathcal{O}}$-definable subset. By assumption, there exists an $\mathcal{O}$-definable cellular map $c\colon X \to D$, the fibers of which yield an $\mathcal{O}$-cell decomposition of $X$. We will need to show how we can turn this into an $\bar{\mathcal{O}}$-cell decomposition. As always, we only treat the planar case, $X \subseteq O^2$. There is nothing to show if $X$ is discrete, so assume it is a curve. We already argued that its vertical component $\mathrm{Vert}(X)$ admits a cell decomposition, and so we may remove it. The remaining set of nodes is discrete, and hence may be removed as well, so that we are left with the case that $X$ has no nodes. By Corollary 8.4, every point of $X$ lies on a unique optimal quasi-cell. Hence if $C := c^{-1}(d)$ is one of the cells in the above decomposition, then it is contained in a unique $\bar{\mathcal{O}}$-quasi-cell $S$. As $C$ is then the restriction of $S$ to $I$, it is $\bar{\mathcal{O}}$-definable by Lemma 8.2.

If $X$ has dimension two, we may assume it is open after removing its boundary, as we already dealt with curves. By Theorem 8.13, there exists an $\bar{\mathcal{O}}$-quasi-cell decomposition of $X$. Following that proof, we may assume, after removing all points lying on a vertical line containing a node of $\partial X$, that any quasi-cell $S$ in this decomposition is open, and its boundary consists of quasi-cells of $\partial X$. By the one-dimensional case, the latter decompose into $\bar{\mathcal{O}}$-cells, whence so does $S$. $\dashv$

As before, we work again a model $\mathcal{M}$ of DCTC.

9.5. LEMMA. *Every one-variable $\mathcal{M}$-definable subset is eukaryote.*

PROOF. Most proofs involving eukaryoteness will require some coding of disjoint unions, and as we will gloss over this issue below, let me do the proof in detail here. For ease of discussion, let us assume $Y \subseteq M$ is bounded (the unbounded case is only slightly more complicated and left to the reader). Assume $0$ and $1$ are distinct elements in $M$. Define $c\colon Y \to M^2$ by letting $c(y)$ be equal to $(y, 0)$, in case $y \in \partial Y$; and equal to $(x, 1)$ where $x$ is the maximum of $(\partial Y)_{<y}$, in the remaining case. The fiber $c^{-1}(d, e)$ is either a point in $\partial Y$ (when $e = 0$), or the interval $]d, \sigma_{\partial Y}(d)[\ \subseteq Y$. Since its image is contained in $\partial Y \times \{0, 1\}$, the map $c$ is cellular by (2.6.v). $\dashv$

9.6. *Remark.* As it will be of use later, note that by the above argument, we can refine the cell decomposition given by $c$ as follows: for any discrete subset $D$ containing $\partial Y$, we can construct a cellular map $c_D\colon Y \to M^2$ whose cells have endpoints in $D$.

9.7. PROPOSITION. *Let $g\colon X \to M^n$ be a definable map with finite image. If every fiber $g^{-1}(a)$ is eukaryote, then so is $X$.*

PROOF. Let $A := g(X)$, so that $A$ is finite. By assumption, there exists for each $a \in A$, a cellular map $c_a\colon g^{-1}(a) \to D_a$ with $D_a \subseteq M^e$ discrete (for some large enough $e$). Let $D$ be the union of all $\{a\} \times D_a \subseteq M^{n+e}$, for $a \in A$. It follows from Corollary 4.4 that $D$ is discrete. Define $c\colon X \to D$ by the rule $c(x) = (g(x), c_{g(x)}(x))$. To see that this is cellular, note that the fiber over a point $(a, d) \in D$ is equal to $c_a^{-1}(d)$ for $d \in D_a$, whence is by assumption a cell. $\dashv$

9.8. THEOREM. *The collection of eukaryote $\mathcal{M}$-definable subsets is closed under (finite) Boolean combinations.*

PROOF. For simplicity, we only prove this for planar subsets, and leave the general case to the reader (by an induction on the arity). Since the complement of a cell $V \subseteq M^2$ is a finite union of cells, it is eukaryote. For instance, if $V = C(\,]a,b[\,; f < g)$, then its complement consists of the four 2-cells $]-\infty, a[\times M$, $]b, \infty[\times M$, $C(I; -\infty < f)$ and $C(I; g < \infty)$, and the four 1-cells, $a \times M$, $b \times M$ and the graphs of $f$ and $g$. Since any union can be written as a disjoint union by taking complements, an application of Proposition 9.7 then reduces to showing that the intersection of two cells $V_1$ and $V_2$ in $M^2$ is eukaryote. This is trivial if either one is discrete, whence a singleton. Suppose $V_1$ is a 1-cell, given by the definable, continuous map $f_1 \colon I_1 \to M$. Let $Y$ be the subset of all $x \in I_1$ such that $(x, f_1(x))$ belongs to $V_2$. Choose a cellular map $c \colon Y \to D$ (by Lemma 9.5, or, for higher arities, by induction). Its composition with the (bijective) projection $V_1 \cap V_2 \to Y$ is then also cellular.

Suppose next that $V_i = C(I_i; f_i < g_i)$ are both 2-cells, assumed once more for simplicity to be bounded. Let $I := I_1 \cap I_2$ and for $x \in I$, let $f(x)$ be the maximum of $f_1(x)$ and $f_2(x)$, and let $g(x)$ be the minimum of $g_1(x)$ and $g_2(x)$. Note that $f$ and $g$ are continuous on $I$. Let $Y$ consist of all $x \in I$ for which $f(x) < g(x)$, and let $c \colon Y \to D$ be cellular. The composition of $c$ with the projection $V_1 \cap V_2 \to Y$ is again cellular, since its fibers are the cells $C(c^{-1}(a); f < g)$. $\dashv$

9.9. EXAMPLE. It is important in this result that the structure is already a model of DCTC. For instance, let $D$ be the subset of the ultrapower $\mathbb{R}_\natural$ of the reals (viewed as an ordered field) consisting of all elements of the form $n$ or $\omega_\natural - n$, for $n \in \mathbb{N}$. Note that $D$ is closed, bounded, and discrete, and hence eukaryote. However, $(\mathbb{R}_\natural, D)$ is not eukaryote, since $\mathbb{N} = D_{<\omega_\natural/2}$ is definable but fails to have a supremum, and so, $(\mathbb{R}_\natural, D)$ is not even a model of DCTC.

It is not hard to show that the product of two cells is again a cell. Therefore, the product of two eukaryote subsets is again eukaryote. Similarly, the fiber of a cell is again a cell, and hence if $X \subseteq M^n$ is eukaryote, then so is each fiber $X[\mathbf{a}]$. Together with Theorem 9.8 and the fact that a principal projection of a eukaryote subset is again eukaryote, we showed that the collection of eukaryote subsets determines a first-order structure on $M$ (in the sense of [21, Chapt. 1, 2.1], with a predicate for every eukaryote subset of $M^n$). Calling this induced structure on $M$ the *eukaryote reduct* of $\mathcal{M}$ and denoting it $\mathcal{M}^{\mathrm{eu}}$, is justified by:

9.10. COROLLARY. *If $\mathcal{M} \models$ DCTC, then $\mathcal{M}^{eu}$ is eukaryote, whence in particular a model of DCTC.*

PROOF. The definable subsets of $\mathcal{M}^{\mathrm{eu}}$ are precisely the eukaryote definable subsets of $\mathcal{M}$, so that in particular, $\mathcal{M}$ and $\mathcal{M}^{\mathrm{eu}}$ have the same cells. So remains to show that if $c \colon X \to D$ is cellular in $\mathcal{M}$, then it is also cellular in $\mathcal{M}^{\mathrm{eu}}$. Discrete sets are eukaryote by definition, and the graph $\Gamma(c)$ of $c$ is eukaryote via the projection $\Gamma(c) \to D$. In particular, $c$ is $\mathcal{M}^{\mathrm{eu}}$-definable, and since its fibers are cells, we are done. The last assertion then follows from Lemma 9.2. $\dashv$

We also have the following joint cell decomposition:

9.11. COROLLARY. *Given eukaryote subsets $Y_1, \dots, Y_n$ of a eukaryote set $X$ in $\mathcal{M}$, there exists a cellular map $c \colon X \to D$, such that for each $i$, the restriction of $c$ to $Y_i$ is also cellular.*

PROOF. Since any Boolean combination of eukaryote subsets is again eukaryote by Theorem 9.8, we may reduce first to the case that all $Y_i$ are disjoint, and then by induction, that we have a single eukaryote subset $Y \subseteq X$. Since $X \setminus Y$ is eukaryote too, we have cellular maps $d\colon Y \to D$ and $d'\colon X \setminus Y \to D'$, and their disjoint union is then the desired cellular map. ⊣

We call a definable map *eukaryote*, if its graph is. Note that its domain then must also be eukaryote. As already observed in the previous proof, cellular maps are eukaryote. To characterize eukaryote maps, we make the following observation/definition: given a cellular map $c\colon X \subseteq M^n \to D$, for $e \leq n$, let $X_c^{(e)} = X^{(e)}$ be the union of all $e$-dimensional fibers $c^{-1}(a)$. Since dimension is definable, so is each $X^{(e)}$, and hence the restriction of $c$ to $X^{(e)}$ is also cellular, proving in particular that each $X^{(e)}$ is eukaryote.

9.12. THEOREM. *A definable map $f\colon X \to M^k$ is eukaryote if and only if $X$ is eukaryote, and the restriction of $f$ to the set of its discontinuities is also eukaryote. In particular, a definable, continuous map with eukaryote domain is eukaryote.*

PROOF. If $f$ is eukaryote, then $f$ is $\mathcal{M}^{\mathrm{eu}}$-definable, and hence so is its set of discontinuities $X'$, proving that $X'$ is eukaryote. Since the graph of $f|_{X'}$ is $\Gamma(f) \cap (X' \times M^k)$, the restricted map is again eukaryote by Theorem 9.8. For the converse, $U := X \setminus X'$ is eukaryote by Theorem 9.8, so that we have a cellular map $c\colon U \to D$. Since $f$ is continuous on $U$, the composition of $c$ with the principal projection $\Gamma(f|_U) \to U$ is also cellular, showing that $\Gamma(f|_U)$ is eukaryote. Since by assumption, the graph of the restriction to $X'$ is eukaryote, so is $\Gamma(f) = \Gamma(f|_U) \cup \Gamma(f|_{X'})$ by Theorem 9.8, showing that $f$ is eukaryote. ⊣

A eukaryote map is $\mathcal{M}^{\mathrm{eu}}$-definable, and hence so its its image, proving:

9.13. COROLLARY. *If the domain of a definable, continuous map in $\mathcal{M}$ is eukaryote, then so is its image. More generally, the image of a eukaryote subset under a eukaryote map is again eukaryote.* ⊣

Let us call a definable map $f\colon X \subseteq M^n \to M^k$ *almost continuous*, if its set of discontinuities is discrete. By Theorem 3.2, any one-variable definable map is almost continuous. Given a definable map $f\colon X \to M^k$, let us inductively define $D_i(f) \subseteq X$, by setting $D_0(f) := X$, and by setting $D_i(f)$, for $i > 0$, equal to the set of discontinuities of the restriction of $f$ to $D_{i-1}(f)$. By Remark 3.6, each $D_i(f)$ has strictly lesser dimension than $D_{i-1}(f)$, and hence $D_n(f)$ is empty for $n$ bigger than the dimension of $X$. Hence $f$ is (almost) continuous if $D_1(f)$ is empty (respectively, discrete). Since the domain of a eukaryote function is eukaryote, an easy inductive argument using Theorem 9.12 immediately yields:

9.14. COROLLARY. *An almost continuous map in $\mathcal{M}$ (for instance, a one-variable map) with eukaryote domain is eukaryote. In particular, a definable map $f$ is eukaryote if and only if all $D_i(f)$ are eukaryote.* ⊣

Let us say that an ordered structure is *almost continuous*, if apart from a binary predicate denoting the order, all other symbols represent almost continuous functions.

9.15. COROLLARY. *If $\mathcal{M}$ is almost continuous, then it is eukaryote.*

PROOF. Since the collection of eukaryote subsets is closed under Boolean operations, projections, and products by Theorem 9.8, we only have to verify that the ones defined by unnested atomic formulae are eukaryote. Since by assumption the only predicate is the inequality sign, and the set it defines is a cell, we only have to look at formulae of the form $f(x) = g(x)$ or $f(x) < g(x)$, with $f, g$ function symbols. Since $f$ and $g$ are total functions representing almost continuous maps, their graphs are eukaryote by Corollary 9.14, whence so is their intersection by Theorem 9.8. The projection of the latter is the set defined by $f(x) = g(x)$, proving that is a eukaryote subset. Let $F$ and $G$ be the subsets of $M^{n+2}$ of all $(a, f(a), c)$ and all $(a, b, g(a))$ respectively, with $a \in M^n$ and $b, c \in M$. Since these are just products of the respective graphs and $M$, both are eukaryote, and so is the subset $E$ of all $(a, b, c) \in M^{n+2}$ with $a < b$. Therefore, by another application of Theorem 9.8, the intersection $F \cap G \cap E$ is eukaryote, and so is its projection, which is just the set defined by the relation $f(a) < g(a)$. ⊣

9.16. *Remark.* More generally, by the same argument, if $\mathcal{M} \models$ DCTC is an expansion of a eukaryote structure by eukaryote functions and by predicates defining eukaryote subsets, then $\mathcal{M}$ itself is eukaryote.

## Part 2. Pseudo-o-minimal structures

§**10. O-minimalism.** Recall that an $L$-structure $\mathcal{M}$ is called *pseudo-o-minimal*, if it is a model of the theory $T^{\text{omin}}$, that is to say, the intersection of the theories $\mathrm{Th}(\mathcal{O})$, where $\mathcal{O}$ runs over all o-minimal $L$-structures.

10.1. LEMMA. *Any reduct of a pseudo-o-minimal structure is pseudo-o-minimal.*

PROOF. Let $L \subseteq L'$ be languages, let $\mathcal{M}'$ be a pseudo-o-minimal $L'$-structure, and let $\mathcal{M} := \mathcal{M}'|_L$ be its $L$-reduct. To show that $\mathcal{M}$ is pseudo-o-minimal, take a sentence in $T_L^{\text{omin}}$ and let $\mathcal{N}'$ be any o-minimal $L'$-structure. Since its reduct $\mathcal{N}'|_L$ is also o-minimal, $\sigma$ holds in the latter, whence also in $\mathcal{N}'$ itself. As this holds for all o-minimal $L'$-structures, $\sigma$ also holds in $\mathcal{M}'$. Since $\sigma$ only mentions $L$-symbols, it must therefore already hold in the reduct $\mathcal{M}$, as we needed to show. ⊣

We will call an ultraproduct of o-minimal $L$-structures an *ultra-o-minimal* structure. Using a well-known elementarity criterion via ultraproducts, we have:

10.2. COROLLARY. *An $L$-structure is pseudo-o-minimal if and only if it is elementary equivalent with (equivalently, an elementary substructure of) an ultra-o-minimal structure.* ⊣

A pseudo-o-minimal field (with no additional structure), being definably complete, is o-minimal by [12, Corollary 1.5]. Any pseudo-o-minimal structure whose underlying order is that of the reals, or more generally, admits the Heine-Borel property, must be o-minimal by Corollary 2.4 and Remark 2.5.

10.3. PROPOSITION. *In an ultra-o-minimal structure $\mathcal{M}$, a definable set has dimension $e$ if and only if it is an ultraproduct of $e$-dimensional definable sets.*

PROOF. Suppose $\mathcal{M}$ is the ultraproduct of o-minimal structures $\mathcal{M}_i$, and let $X = \varphi(\mathcal{M})$ be a definable subset. By Łos' Theorem, $X$ is the ultraproduct of the definable sets $X_i := \varphi(\mathcal{M}_i)$. The result now follows from the definability of dimension: we leave the general case to the reader, but for the *planar* case (=definable subsets of $M^2$), observe that both being discrete or having non-empty interior are first-order definable properties, and hence pass through the ultraproduct by Łos' Theorem. ⊣

10.4. THEOREM. *If $\mathcal{M}$ is pseudo-o-minimal, then so is $\mathcal{M}^{eu}$.*

PROOF. Let $\bar{L}$ be the language with a predicate for each eukaryote subset of $\mathcal{M}$, so that $\mathcal{M}^{\mathrm{eu}}$ is an $\bar{L}$-structure (see the paragraph before Corollary 9.10 for the definition). Viewing $\mathcal{M}$ as a structure in the language having a predicate for each definable subset yields again a eukaryote structure, since we added no new definable subsets (see Lemma 12.1 below). Therefore, upon replacing $L$ by the latter language, we assume from the start that $\bar{L} \subseteq L$, and the result now follows from Lemma 10.1. $\dashv$

Since the Hardy structure $\mathbf{H}(\mathcal{M})$ (see §9.10) is an elementary extension of $\mathcal{M}$, it is pseudo-o-minimal, if $\mathcal{M}$ is. As argued above, this gives rise to plenty of Vaughtian pairs, showing that o-minimalism has Vaughtian pairs.

10.5. *Remark.* We have the following puzzling fact that at least one among the following three statements holds:

10.5.i. there is a eukaryote structure which is not pseudo-o-minimal;
10.5.ii. there is a pseudo-o-minimal structure which is not eukaryote;
10.5.iii. any reduct of a eukaryote structure is again eukaryote.

Indeed, suppose both (10.5.i) and (10.5.iii) fail. So, by the latter, there is a eukaryote structure $\mathcal{M}$ with a non-eukaryote reduct $\bar{\mathcal{M}}$, and by the former, $\mathcal{M}$ is pseudo-o-minimal, whence so is $\bar{\mathcal{M}}$ by Lemma 10.1. Hence $\bar{\mathcal{M}}$ is pseudo-o-minimal but not eukaryote. I do not know whether any ultraproduct of eukaryote structures is eukaryote. If so, then (10.5.ii) fails, that is to say, any pseudo-o-minimal structure is eukaryote, since it is elementary equivalent by Corollary 10.2 with an ultra-o-minimal structure, and the latter would then be eukaryote, whence so would the former be by Proposition 9.3.

§**11. The Grothendieck ring of a pseudo-o-minimal structure.** Given any first-order structure $\mathcal{N}$, we define its *Grothendieck ring* $\mathbf{Gr}(\mathcal{N})$ as follows. Given two formulae $\varphi(x)$ and $\psi(y)$ with parameters, with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, we say that $\varphi$ and $\psi$ are *($\mathcal{N}$-)definably isomorphic*, if there exists a definable bijection $f\colon \varphi(\mathcal{N}) \to \psi(\mathcal{N})$. Let $\mathbf{Gr}(\mathcal{N})$ be the quotient of the free Abelian group generated by $\mathcal{N}$-definable isomorphism classes $\langle\varphi\rangle$ of formulae $\varphi$ modulo the *scissor relations*

$$\text{(sciss)} \qquad \langle\varphi\rangle + \langle\psi\rangle - \langle\varphi \wedge \psi\rangle - \langle\varphi \vee \psi\rangle$$

where $\varphi, \psi$ range over all pairs of formulae in the same free variables. See for instance [9, 10] for more details.

We will write $[\varphi]$ or $[Y]$ for the image of the formula $\varphi$, or the set $Y$ defined by it, in $\mathbf{Gr}(\mathcal{N})$. Since we can always replace a definable subset with a definable copy that is disjoint from it, the scissor relations can be simplified, by only requiring them for disjoint unions: $[X \sqcup Y] = [X] + [Y]$. In particular, combining all terms with a positive sign as well as all terms with a negative sign by taking disjoint unions, we see that every element in the Grothendieck ring is of the form $[X] - [Y]$, for some definable subsets $X$ and $Y$. To make $\mathbf{Gr}(\mathcal{N})$ into a ring, we define the product of two classes $[\varphi]$ and $[\psi]$ as the class of the *product* $\varphi(x) \wedge \psi(y)$ where $x$ and $y$ are disjoint sets of variables. One checks that this is well-defined and that the class of a point is the unit for multiplication, therefore denoted 1. Note that in terms of definable subsets, the product corresponds to the Cartesian product and the scissor relation to the usual inclusion/exclusion relation.

Variants are obtained by restricting the class of formulae/definable subsets. For our purposes, that is to say, working in an ordered $L$-structure $\mathcal{M}$, we will only do this for discrete subsets. Call a formula *discrete* if it defines a discrete subset. In a model $\mathcal{M} \models$ DCTC, discrete formulae are closed under Boolean combinations and products by Corollary 2.9, and if two discrete definable subsets are definably isomorphic, then the graph of this isomorphism is also given by a discrete formula. Therefore, the Grothendieck ring on discrete formulae is well-defined and will be denoted $\mathbf{Gr}_0(\mathcal{M})$. We have a canonical homomorphism $\mathbf{Gr}_0(\mathcal{M}) \to \mathbf{Gr}(\mathcal{M})$ with image the subring generated by classes of discrete formulae. The following is useful when dealing with Grothendieck rings:

11.1. LEMMA. *Two definable subsets $X$ and $Y$ in a first-order structure $\mathcal{N}$ have the same class in $\mathbf{Gr}(\mathcal{N})$ if and only if there exists a definable subset $Z$ such that $X \sqcup Z$ and $Y \sqcup Z$ are definably isomorphic.*

PROOF. One direction is immediate, for if $X \sqcup Z$ and $Y \sqcup Z$ are definably isomorphic, then $[X] + [Z] = [X \sqcup Z] = [Y \sqcup Z] = [Y] + [Z]$ in $\mathbf{Gr}(\mathcal{N})$, from which it follows $[X] = [Y]$. Conversely, if $[X] = [Y]$, then by definition of scissor relations, there exist mutually disjoint, definable subsets $A_i, B_i, C_i, D_i \subseteq N^{p_i}$ such that

$$\langle X\rangle + \sum_i \langle A_i\rangle + \langle B_i\rangle - \langle A_i \sqcup B_i\rangle = \langle Y\rangle + \sum_i \langle C_i\rangle + \langle D_i\rangle - \langle C_i \sqcup D_i\rangle$$

in the free Abelian group on isomorphism classes. Bringing the terms with negative signs to the other side, we get an expression in which each term on the left hand side must also occur on the right hand side, that is to say, the collection of all isomorphism classes $\{\langle X\rangle, \langle A_i\rangle, \langle B_i\rangle, \langle C_i \sqcup D_i\rangle\}$ is the same as the collection of all isomorphism classes $\{\langle Y\rangle, \langle C_i\rangle, \langle D_i\rangle, \langle A_i \sqcup B_i\rangle\}$. By properties of disjoint union, we therefore get $\langle X \sqcup Z\rangle = \langle Y \sqcup Z\rangle$, where $Z$ is the disjoint union of all definable subsets $A_i, B_i, C_i, D_i$. $\dashv$

If $\mathcal{M}$ is an expansion of an ordered, divisible Abelian group, then we have the following classes of open intervals. If $I = \,]a, b[\,$, then $I$ is definably isomorphic to $]0, b-a[$ via the translation $x \mapsto x - a$. Moreover, $]0, a[$ is definably isomorphic to $]0, 2a[$ via the map $x \mapsto 2x$. Hence the class $\mathbb{i}$ of $]0, a[$ is by (sciss) equal to the sum of the classes of $]0, a[\,$, $\{a\}$, and $]a, 2a[\,$. In other words, $\mathbb{i} = 2\mathbb{i} + 1$, whence $\mathbb{i} = -1$ (the additive inverse of 1). Let $\mathbb{h}$ be the class of the unbounded interval $]0, \infty[\,$. By translation and/or the involution $x \mapsto -x$, any half unbounded interval is definably isomorphic with $]0, \infty[\,$. Finally, we put $\mathbb{L} := [M]$ (the so-called *Lefschetz class*). Since $M$ is the disjoint union of $]-\infty, 0[\,$, $\{0\}$, and $]0, \infty[\,$, we get

$$\mathbb{L} = 2\mathbb{h} + 1. \tag{lef}$$

If $M$ is moreover an ordered field, then taking the reciprocal makes $]0, 1[$ and $]1, \infty[$ definably isomorphic, so that $\mathbb{h} = \mathbb{i} = -1$, and hence also $\mathbb{L} = -1$.

Under the assumption of an underlying ordered structure, whence a topology, we can also strengthen the definition by calling two definable subsets *definably homeomorphic*, if there exists a definable (continuous) homeomorphism between them, and then build the Grothendieck ring, called the *strict Grothendieck ring* of $\mathcal{M}$ and denoted $\mathbf{Gr}^{\mathrm{s}}(\mathcal{M})$, on the free Abelian group generated by homeomorphism classes of definable subsets. Note that there is a canonical surjective homomorphism $\mathbf{Gr}^{\mathrm{s}}(\mathcal{M}) \to \mathbf{Gr}(\mathcal{M})$. In the o-minimal case, the monotonicity theorem implies that both variants are equal, but this might fail in the pseudo-o-minimal case, since cell decompositions are no longer finite (but see Corollary 11.13 below). In fact, in the o-minimal case, the Grothendieck ring is extremely simple, as observed by Denef and Loeser ([21, Chap. 4, §2]):

11.2. PROPOSITION. *The Grothendieck ring of an o-minimal expansion of an ordered field is canonically isomorphic to the ring of integers $\mathbb{Z}$.*

PROOF. By the previous discussion, the class of any open interval is equal to $-1$. The graph of a function is definably isomorphic with its domain, and so the class of any 1-cell is equal to $-1$. Since a bounded planar 2-cell lies in between two 1-cells, it is definably isomorphic to an open box, and by definition of the multiplication in $\mathbf{Gr}(\mathcal{M})$, therefore its class is equal to $\mathbb{L}^2 = 1$. The unbounded case is analogous, and so is the case that the 2-cell lies in a higher Cartesian product. This argument easily extends to show that the class of a $d$-cell in $\mathbf{Gr}(\mathcal{M})$ is equal to $\mathbb{L}^d = (-1)^d$. By Cell Decomposition, every definable subset is a finite union of cells, and hence its class in $\mathbf{Gr}(\mathcal{M})$ is an integer (multiple of 1). ⊣

We denote the canonical homomorphism $\mathbf{Gr}(\mathcal{M}) \to \mathbb{Z}$ by $\chi_{\mathcal{M}}(\cdot)$ and call it the *Euler characteristic* of $\mathcal{M}$. Inspired by [2], we define the *Euler measure* of a definable subset $X$ in an o-minimal structure $\mathcal{M}$ as the pair $\mu_{\mathcal{M}}(X) := (\dim(X), \chi_{\mathcal{M}}(X)) \in (\mathbb{N} \cup \{-\infty\}) \times \mathbb{Z}$, where we view the latter set in its lexicographical ordering.

In an arbitrary first-order structure, let us say, for definable subsets $X$ and $Y$, that $X \preceq Y$ if and only if there exists a definable injection $X \to Y$. In general, this relation, even up to definable isomorphism, will fail to be symmetric (take for instance in the reals the sets $X = [0,1]$ and $Y = X \cup \{3/2\}$, where $x \mapsto x/2$ sends $Y$ inside $X$), and therefore is in general only a partial pre-order. As we will discuss below in §11.14, it does induce a partial order on isomorphism classes of discrete, definable subsets in a pseudo-o-minimal structure. In the o-minimal case, $\preceq$ is a total pre-order by the following (folklore) result. In some sense, the rest of the paper is an attempt to extend this result to the pseudo-o-minimal case.

11.3. THEOREM. *In an o-minimal expansion of an ordered field, two definable sets $X$ and $Y$ are definably isomorphic if and only if $\mu_{\mathcal{M}}(X) = \mu_{\mathcal{M}}(Y)$. Moreover, $X \preceq Y$ if and only if $\dim(X) \leq \dim(Y)$ with the additional condition that $\chi_{\mathcal{M}}(X) \leq \chi_{\mathcal{M}}(Y)$ whenever both are finite.*

PROOF. The first statement is proven in [21, Chap. 8, 2.11]. So, suppose $X \preceq Y$. Since $X$ is definably isomorphic with a subset of $Y$, its dimension is at most that of $Y$. If both are zero-dimensional, that is to say, finite, then the pigeonhole principle gives $\chi_{\mathcal{M}}(X) = |X| \leq |Y| = \chi_{\mathcal{M}}(Y)$.

Conversely, assume $\dim(X) \leq \dim(Y)$. If both are finite, the assertion is clear by the same argument, so assume they are both positive dimensional. Without loss of generality, by adding a (disjoint) cell of the correct dimension, we may then assume that they have both the same dimension $d \geq 1$. Let $e := \chi_{\mathcal{M}}(Y) - \chi_{\mathcal{M}}(X)$ and let $F$ consist of $e$ points disjoint from $X$ if $e$ is positive and of $-e$ open intervals disjoint from $X$ if $e$ is negative. Since $\chi(F) = e$, the Euler measure of $X \sqcup F$ and $Y$ are the same, and hence they are definably isomorphic by the first assertion, from which it follows that $X \preceq Y$. ⊣

Let $\mathcal{M}$ be an ultra-o-minimal structure, say, realized as the ultraproduct of o-minimal structures $\mathcal{M}_i$. We define its *ultra-Euler characteristic* $\chi_{\mathcal{M}}(\cdot)$ as follows. Let $Y \subseteq M^n$ be a definable subset, say given by a formula $\varphi(x, \mathbf{b})$ with $\mathbf{b}$ a tuple of parameters realized as the ultraproduct of tuples $\mathbf{b}_i$ in each $M_i$. Let $Y_i := \varphi(\mathcal{M}_i, \mathbf{b}_i)$, so that $Y$ is the ultraproduct of the $Y_i$, and let $\chi_{\mathcal{M}}(Y)$ now be the ultraproduct of the $\chi_{\mathcal{M}_i}(Y_i)$, viewed as an element of $\mathbb{Z}_\natural$. If $X$ is definably isomorphic with $Y$, via a definable bijection with graph $G$, choose as above definable subsets $X_i$ and $G_i$ in $\mathcal{M}_i$ with ultraproduct equal to $X$ and $G$ respectively. By Łos' Theorem, almost each $G_i$ is the graph of a definable bijection between $X_i$ and $Y_i$, and therefore $\chi_{\mathcal{M}_i}(X_i) = \chi_{\mathcal{M}_i}(Y_i)$ for almost all $i$, showing that $\chi_{\mathcal{M}}(X) = \chi_{\mathcal{M}}(Y)$. Similarly, we define the *ultra-Euler measure* $\mu_{\mathcal{M}}(X) := (\dim(X), \chi_{\mathcal{M}}(X))$. Since the ultra-Euler characteristic is easily seen to be also compatible with the scissor relations (sciss), we showed:

11.4. COROLLARY. *For an ultra-o-minimal structure $\mathcal{M}$, the ultra-Euler characteristic induces a homomorphism $\mathbf{Gr}(\mathcal{M}) \to \mathbb{Z}_\natural$.* ⊣

**11.5. The Discrete Pigeonhole Principle.** Before we proceed, we identify another o-minimalistic property, that is to say, a first-order property of o-minimal structures. For the remainder of this section, $\mathcal{M}$ will be a pseudo-o-minimal structure.

11.6. PROPOSITION (Discrete Pigeonhole Principle). *If a definable map $f\colon Y \to Y$, for some $Y \subseteq M^n$, is injective and its image is co-discrete, meaning that $Y \setminus f(Y)$ is discrete, then it is a bijection. In particular, any definable map from a discrete subset $D$ to itself is injective if and only if it is surjective.*

PROOF. For each formula $\varphi(x, y, \mathbf{z})$, we can express in a first-order way that if $\varphi(x, y, \mathbf{c})$, for some tuple $\mathbf{c}$ of parameters, defines the graph of an injective map $f\colon Y \to Y$ then

(DPP) $$Y \setminus f(Y) \text{ discrete implies } Y = f(Y).$$

It remains to show that (DPP) holds in any o-minimal structure $\mathcal{M}$. Indeed, if $D = Y \setminus f(Y)$, then $\chi_{\mathcal{M}}(Y) = \chi_{\mathcal{M}}(f(Y)) + \chi_{\mathcal{M}}(D)$. Since $f$ is injective, $Y$ and $f(Y)$ are definably isomorphic, whence have the same Euler characteristic, and so $\chi_{\mathcal{M}}(D) = 0$. But a discrete subset in an o-minimal structure is finite and its Euler characteristic is then just its cardinality, showing that $D = \emptyset$. One direction in the last assertion is immediate, and for the converse, assume $f\colon D \to D$ is surjective. For each $x \in D$, define $g(x)$ as the (lexicographical) minimum of $f^{-1}(x)$, so that $g\colon D \to D$ is an injective map, whence surjective by the above, and therefore necessarily the inverse of $f$. ⊣

At present, I do not know how to derive (DPP) from DCTC.

11.7. COROLLARY. *A pseudo-o-minimal expansion of an ordered field is o-minimal if and only if its Grothendieck ring is isomorphic to $\mathbb{Z}$.*

PROOF. One direction is Proposition 11.2, so assume $\mathbf{Gr}(\mathcal{M}) = \mathbb{Z}$. Let $D$ be a definable, discrete subset. By assumption, $[D] = n$ for some integer $n$. After removing $n$ points, if $n$ is positive, or adding $-n$ points, if negative, we may suppose $[D] = 0$. By Lemma 11.1, there exists a definable subset $X$ such that $X$ and $X \sqcup D$ are definably isomorphic. By (DPP), this forces $D = \emptyset$. ⊣

11.8. COROLLARY. *A monotone map $f\colon D \to D$ on an $\mathcal{M}$-definable, discrete subset $D$ is either constant or an involution.*

PROOF. Suppose $f$ is non-constant and hence $f^2$ is strictly increasing. So upon replacing $f$ by its square, we may already assume that $f$ is increasing, and we need to show that it is then the identity. Since $f$ is injective, it is bijective by Proposition 11.6. Let $h$ be the maximum of $D$, and suppose $f(d) = h$. If $d < h$, then $h = f(d) < f(h) \in D$, contradiction, showing that $f(h) = h$. If $f$ is not the identity, then the set $Q$ of all $d \in D$ for which $f(d) \neq d$ is non-empty, whence has a maximum, say, $u < h$. In particular, if $v := \sigma_D(u)$ is its immediate successor, then $f(u) < f(v) = v$, since $v \notin Q$, whence $f(u) < u$, since $u \in Q$. Since $u = f(a)$ for some $a \neq u$, then either $a < u$ or $v \leq a$, and hence $u = f(a) < f(u) < u$ or $v = f(v) \leq f(a) = u$, a contradiction either way. $\dashv$

11.9. *Remark.* Note that the map sending $h$ to the minimum of $D$, and equal to $\sigma_D$ otherwise is a definable permutation of $D$, but it obviously fails to be monotone. The map $x \mapsto \omega_\natural - x$ on $D = (\mathbb{N}_\natural)_{\leq \omega_\natural}$ as in Example 2.2 is a strictly decreasing involution. It is not hard to see that if an involution exists, it must be unique: indeed, if $f$ and $g$ are both decreasing, let $a$ be the maximal element at which they disagree (it cannot be $h$ since $f(h) = l = g(h)$), and assume $f(a) < g(a)$. Since $f(\sigma(a)) = g(\sigma(a)) < f(a) < g(a)$, it is now easy to see that $f(a)$ does not lie in the image of $g$, contradicting that $g$ must be a bijection by (DPP).

11.10. PROPOSITION. *If $\mathcal{M}$ expands an ordered field, then there exists for every definable subset $Y \subseteq M$, two definable, discrete subsets $D, E \subseteq Y$ such that $[Y] = [D] - [E]$ in $\mathbf{Gr}(\mathcal{M})$.*

PROOF. Since the boundary $\partial Y$ is discrete, we may remove it and assume $Y$ is open, whence a disjoint union of open intervals by Theorem 2.10. Let us introduce some notation that will be useful later too, assuming $Y$ is open. For $y \in Y$, let $l(y)$ and $r(y)$ be respectively the maximum of $(\partial Y)_{<y}$ and the minimum of $(\partial Y)_{>y}$ (allowing $\pm\infty$). Hence $]l(y), r(y)[$ is the maximal interval in $Y$ containing $y$, and we denote its barycenter by $m(y)$, where, in general, we define the *barycenter* of an interval $]a, b[$ as the midpoint $(a + b)/2$ if $a$ and $b$ are finite, or the point $a + 1$ (respectively, $b - 1$, or $0$) if $a$ (respectively, $b$, or both) is infinite. Let $L(Y)$ be the subset of all $y \in Y$ such that $y < m(y)$; and similarly, let $M(Y)$ and $R(Y)$ be the subsets for which respectively $m(y) = y$ and $m(y) < y$. Removing a maximal unbounded interval from $Y$ if necessary (whose class is equal to $-1$ as already observed above), we may assume $Y$ is bounded, so that $l(y)$ and $r(y)$ are always finite. Since the maps $f_Y : L(Y) \to Y : y \mapsto 2y - l(y)$ and $g_Y : L(Y) \to R(Y) : y \mapsto y + m(y)$ are bijections, $[Y] = [L(Y)] = [R(Y)]$. Since the scissor relations yield $[Y] = [L(Y)] + [M(Y)] + [R(Y)]$, we get $[Y] = -[M(Y)]$. By construction $M(Y)$ is discrete, and so we are done. $\dashv$

The proof gives the following more general result: given any definable discrete subset $D_0 \subseteq Y$, we can find disjoint definable discrete subsets $D, E \subseteq Y$ such that $D_0 \subseteq D$ and $[Y] = [D] - [E]$. Indeed, let $D := D_0 \cup (Y \cap \partial Y)$ and $E := M(Y \setminus D)$. If $\mathcal{M}$ merely expands an ordered group, then we have to also include the class $\mathbb{h}$ of $]0, \infty[$, that is to say, in that case we can write $[Y] = e\mathbb{h} + [D] - [E]$, where $e \in \{0, 1, 2\}$ is the number of unbounded sides of $Y$. For higher arities, we need to make a eukaryoteness assumption:

11.11. COROLLARY. *Let $X$ be a definable subset in a pseudo-o-minimal expansion $\mathcal{M}$ of an ordered field. If $\partial X$ is eukaryote, then there exist definable, discrete subsets*

*$D, E \subseteq X$ such that $[X] = [D] - [E]$ in $\mathbf{Gr}(\mathcal{M})$. In fact, the class of any eukaryote subset in $\mathbf{Gr}(\mathcal{M})$ is of the form $[D] - [E]$, for some definable discrete subsets $D, E \subseteq M$.*

PROOF. We again give the proof only for $X$ planar. There is nothing to show if $X$ is discrete. Assume next that it has dimension one. Let $V := \mathrm{Vert}(X)$ be the vertical component of $X$. Since $\pi(V)$ is discrete, as we argued before, we can carry out the argument in the proof of Proposition 11.10 on each fiber separately to write $[V]$ as the difference of two discrete classes (we leave the details to the reader, but compare with the two-dimensional case below). Removing $V$ from $X$, we may assume $X$ has no vertical components. In particular, the set $N := \mathrm{Node}(X)$ of nodes of $X$ is discrete by Proposition 7.7. Removing it, we may assume $X$ has no nodes, so that every point lies on a unique optimal quasi-cell by Corollary 8.4. However, by assumption, $X$ is eukaryote, and hence there exists a cellular map $c\colon X \to D$. Given $x \in X$, let $I_x$ be the domain $\pi(c^{-1}(c(x)))$ of the unique cell $c^{-1}(c(x))$ containing $x$. Let $L(X)$, $M(X)$, and $R(X)$ consist respectively of all $x \in X$ such that $\pi(x)$ lies in $L(I_x)$, $M(I_x)$, and $R(I_x)$ respectively (in the notation of the proof of Proposition 11.10). Define $f_X : L(X) \to X$ and $g_X : L(X) \to R(X)$ by sending $x$ to the unique point on $c^{-1}(c(x))$ lying above respectively $f_{I_x}(\pi(x))$ and $g_{I_x}(\pi(x))$, showing that $X$, $L(X)$, and $R(X)$ are definably isomorphic. Since $M(X)$ is discrete and $[X] = [L(X)] + [R(X)] + [M(X)]$, we are done in this case.

If $X$ has dimension two, its boundary has dimension at most one, and so we have already dealt with it by the previous case. Upon removing it, we may assume $X$ is open. This time, we let $L(X)$, $M(X)$ and $R(X)$ be the union of respectively all $L(X[a])$, $M(X[a])$, and $R(X[a])$, for all $a \in \pi(X)$. The maps $(a, b) \mapsto f_{X[a]}(b)$ and $(a, b) \mapsto g_{X[a]}(b)$ put $L(X)$ in definable bijection with respectively $X$ and $R(X)$ (with an obvious adjustment left to the reader if the fiber $X[a]$ is unbounded), and hence $[X] = -[M(X)]$. Since $M(X)$ has dimension at most one by Proposition 5.1, we are done by induction. Without providing the details, we can extend this argument to higher dimensions, proving the last claim, where we also must use the fact proven below in Lemma 11.15 that definable discrete subsets are univalent in an ordered field. ⊣

11.12. *Remark.* We actually proved that if $c\colon X \to D$ is a cellular surjective map, then

$$[X] = \sum_{e=0}^{d} (-1)^e [D_e] \tag{2}$$

where $D_e = c(X^{(e)})$ consist of all $a \in D$ with $e$-dimensional fiber $c^{-1}(a)$, and where $d$ is the dimension of $X$. We may reduce to the case that all fibers have the same dimension, and the assertion is then clear in the one-dimensional case, since the restriction of $c$ to $M(X)$ is a bijection. Repeating the argument therefore to $X$, we get $[X] = -[M(X)] = [M(M(X))]$, and now $M(M(X))$ is definably isomorphic with $D$ via $c$. Higher dimensions follow similarly by induction.

In particular, if $\mathcal{M}$ is a eukaryote expansion of an ordered field, then its Grothendieck ring is generated by the definable discrete subsets of $M$, and the canonical homomorphism $\mathbf{Gr}_0(\mathcal{M}) \to \mathbf{Gr}(\mathcal{M})$ is surjective. Inspecting the above proof, we see that all isomorphisms involved are in fact homeomorphisms, and so the result also holds in the strict Grothendieck ring $\mathbf{Gr}^{\mathrm{s}}(\mathcal{M})$. Since any function with discrete domain is continuous, we showed:

11.13. COROLLARY. *For a eukaryote, pseudo-o-minimal expansion of an ordered field, its Grothendieck ring and its strict Grothendieck ring coincide.* $\dashv$

**11.14. The partial order on $\mathfrak{D}(\mathcal{M})$.** Let $\mathfrak{D}(\mathcal{M})$ denote the collection of isomorphism classes of definable, discrete subsets in a pseudo-o-minimal structure $\mathcal{M}$. Recall that $X \preceq Y$ if there exists a definable injection $X \to Y$. We call a definable subset $X$ *univalent*, if $X \preceq M$. By Theorem 11.3, every definable curve is univalent in an o-minimal structure. In this section, we study $\preceq$ on definable, discrete subsets.

11.15. LEMMA. *In an expansion of an ordered field, every definable, discrete subset is univalent.*

PROOF. By induction, it suffices to show that if $D \subseteq M^{n+1}$ is discrete and definable, then there is a definable, injective map $g\colon D \to M^n$. The set of lines connecting two points of $D$ is again a discrete set (in the corresponding projective space) and hence we can find a hyperplane which is non-orthogonal to any of these lines. But then the restriction to $D$ of the projection onto this hyperplane is injective. $\dashv$

Assume $D$ and $E$ are discrete, definable subsets with $D \preceq E$ and $E \preceq D$. Hence there are definable injections $D \to E$ and $E \to D$. By Proposition 11.6, both compositions are bijections, showing that $D$ and $E$ are definably isomorphic. Since transitivity is trivial, we showed that we get a partial order on $\mathfrak{D}(\mathcal{M})$. To obtain a partial order on the zero-dimensional Grothendieck ring $\mathbf{Gr}_0(\mathcal{M})$, we define $[D] \preceq [E]$, if there exists a definable, discrete subset $A$ such that $D \sqcup A \preceq E \sqcup A$. To show that this well-defined, assume $[D] = [D']$ and $[E] = [E']$. By Lemma 11.1, there exist definable, discrete subsets $F$ and $G$ such that $D \sqcup F \cong D' \sqcup F$ and $E \sqcup G \cong E' \sqcup G$. Therefore,

$$D' \sqcup F \sqcup G \sqcup A \cong D \sqcup F \sqcup G \sqcup A \preceq E \sqcup F \sqcup G \sqcup A \cong E \sqcup F \sqcup G \sqcup A$$

since $D \sqcup A \preceq E \sqcup A$. We then extend this to a partial ordering on $\mathbf{Gr}_0(\mathcal{M})$ by linearity. In the o-minimal case, $\mathbf{Gr}_0(\mathcal{M})$ is just $\mathbb{Z}$ in its natural ordering.

In an expansion of an ordered group, let us call a definable, discrete set $D$ *equidistant*, if the map $a \mapsto \sigma_D(a) - a$ is constant on all non-maximal elements of $D$, where $\sigma_D$ is the successor function.

11.16. PROPOSITION. *In a pseudo-o-minimal expansion $\mathcal{M}$ of an ordered field, any two definable equidistant subsets of $M$ are comparable.*

PROOF. Let $D, E \subseteq M$ be definable equidistant subsets. Since they are bounded by Corollary 4.2, we may assume after a translation that both have minimum equal to $0$, and then after taking a scaling, that the distance between consecutive points in both is $1$. Let $m$ be the maximum of all $a \in D \cap E$ for which $D_{\leq a} = E_{\leq a}$. If $m$ is non-maximal in either set, then $m+1$ lies both in $D$ and in $E$ by assumption, contradiction. Hence $m$ is the maximum, say, of $D$, and therefore $D \subseteq E$, whence $D \preceq E$. $\dashv$

More generally, given a definable, discrete subset $D \subseteq M$ in a pseudo-o-minimal expansion $\mathcal{M}$ of an ordered field, define the *derivative* $D'$ of $D$ as the set of all differences $\sigma_D(a) - a$, where $a$ runs over all non-maximal elements of $D$. Hence an equidistant set is one whose derivative is a singleton. Since we have a surjective map $D \setminus \{\max D\} \to D'\colon a \mapsto \sigma_D(a) - a$, it follows from the next lemma that $D' \preceq D$.

11.17. LEMMA. *In a pseudo-o-minimal structure $\mathcal{M}$, if $g\colon X \to M^k$ is a definable map, then $g(D) \preceq D$, for every discrete, definable subset $D \subseteq X$.*

PROOF. This follows by considering the injective map $g(D) \to D$ sending $a$ to the minimum of $g^{-1}(a)$. ⊣

I do not expect $\preceq$ to be always total (although it can be made total by extending the class of isomorphisms as we shall see in Theorem 13.3 below). Since $D \preceq E$ implies $[D] \preceq [E]$, but not necessarily the converse, the former being total implies that the latter is too, but again, the converse is not clear. To construct potential counterexamples, let us introduce the following notation.

11.18. EXAMPLE (Discrete Overspill). Given a sequence $\mathbf{a} = (a_n)$ of real numbers, let $\mathbb{R}_\natural\langle\mathbf{a}\rangle$ be the ultraproduct of the $\mathcal{R}_n$, where each $\mathcal{R}_n$ is the expansion of the real field with a unary predicate $\mathtt{D}$ interpreting the first $n$ elements $a_1, \dots, a_n$ in the sequence. Since each $\mathcal{R}_n$ is o-minimal, $\mathbb{R}_\natural\langle\mathbf{a}\rangle$ is pseudo-o-minimal. Moreover, $\mathbf{a}$ is the "finite" part of the set $D_\mathbf{a} := \mathtt{D}(\mathbb{R}_\natural\langle\mathbf{a}\rangle)$ defined by $\mathtt{D}$, that is to say,

$$D_\mathbf{a} \cap \mathbb{R} = \{a_1, a_2, \dots, \}.$$

so that we refer to $\mathbb{R}_\natural\langle\mathbf{a}\rangle$ as the structure obtained from $\mathbf{a}$ by *discrete overspill* (for a related construction, see also §15 below).

In this notation, Example 2.2 is the discrete overspill $\mathbb{R}_\natural\langle\mathbb{N}\rangle$ of $\mathbb{N}$ listed in its natural order. I do not know whether $\preceq$ is total on it. Any countable subset can be enumerated, including $\mathbb{Q}$, although this enumeration might not be order preserving. Nonetheless, we get a structure $\mathbb{R}_\natural\langle\mathbf{q}\rangle$ with $D_\mathbf{q} \cap \mathbb{R} = \mathbb{Q}$ (the non-standard elements of $D_\mathbf{q}$ form a proper subset of $\mathbb{Q}_\natural$ and are harder to describe as they depend on the choice of enumeration). We can repeat this construction with more than one sequence, taking one unary predicate for each. Any structure obtained by discrete overspill is eukaryote by Remark 9.16.

11.19. EXAMPLE. Now, if we take two unary predicates, representing, say, the sequence of prime numbers $\mathbf{p}$ and the sequence of powers of two $\mathbf{t}$, then in $\mathbb{R}_\natural\langle\mathbf{p}, \mathbf{t}\rangle$, it seems very unlikely that the discrete sets $D_\mathbf{p}$ and $D_\mathbf{t}$ are comparable. For if they were, they would have to be definably isomorphic by Lemma 13.2 below, as they have the same ultra-Euler characteristic (equal to $\omega_\natural$, the ultraproduct of the diagonal sequence $(n)_n$). It is easy to combine these two unary sets into a single one, by letting $a_{2n} := p_n$ and $a_{2n-1} := -t_n$, so that then $D_\mathbf{a} \cap (\mathbb{R}_\natural)_{\leq 0} = D_\mathbf{t}$ and $D_\mathbf{a} \cap (\mathbb{R}_\natural)_{\geq 0} = D_\mathbf{p}$, giving an example of a single discrete overspill $\mathbb{R}_\natural\langle\mathbf{a}\rangle$ in which $\preceq$ is most likely not total.

11.20. THEOREM (Euler O-minimality Criterion). *A necessary and sufficient condition for an ultra-o-minimal structure $\mathcal{M}_\natural$, given as the ultraproduct of o-minimal structures $\mathcal{M}_i$, to be o-minimal is that, for each formula $\varphi$ without parameters, there exists an $N_\varphi \in \mathbb{N}$ such that $|\chi_{\mathcal{M}_i}(\varphi)| \leq N_\varphi$, for almost all $i$.*

PROOF. If $\mathcal{M}_\natural$ is o-minimal, then $\varphi(\mathcal{M}_\natural)$ is a disjoint union of $N$ cells, whence by Łos' Theorem, so are almost all $\varphi(\mathcal{M}_i)$. Since a cell has Euler characteristic $\pm 1$, additivity yields $|\varphi(\mathcal{M}_i)| \leq N$, for almost all $i$. Conversely, let $Y_\natural \subseteq M_\natural$ be definable, say, given as the fiber of a $\emptyset$-definable subset $X_\natural \subseteq M_\natural^{1+n}$ over a tuple $\mathbf{b}_\natural$. Let $X_i \subseteq M_i^{1+n}$ be the corresponding $\emptyset$-definable subset, and choose $\mathbf{b}_i$ in $M_i$ with ultraproduct $\mathbf{b}_\natural$, so that $Y_\natural$ is the ultraproduct of the $Y_i := X_i[\mathbf{b}_i]$. By the proof of Theorem 8.13

(which in the o-minimal case does yield a finite cell decomposition), we can decompose each $X_i$ as a disjoint union of $\emptyset$-definable subsets $X_i^{(e)}$ consisting of the union of all $e$-cells in a cell decomposition of $X_i$. In fact, this proof can be carried out in the theory DCTC, so that it holds uniformly in any $\mathcal{M} \models$ DCTC. For instance, if $X = \varphi(\mathcal{M})$ is planar, then $X^{(2)}$ consists exactly of all interior points that do not lie on a vertical fiber containing some node of $\partial X$, whereas $X^{(0)}$ consists of all nodes of $\partial X$ that belong to $X$, and $X^{(1)}$ of all remaining points. Let $\varphi^{(e)}$ define in each model $\mathcal{M} \models$ DCTC the set $X^{(e)}$, for $e \leq n+1$. Since each $X_i^{(e)}$ is a disjoint union of $e$-cells, its Euler characteristic is equal to $(-1)^e N_{i,e}$, where $N_{i,e}$ is the number of $e$-cells in the decomposition. By assumption (applied to the formula $\varphi^{(e)}$), this Euler characteristic is bounded in absolute value, whence so are the $N_{i,e}$, that is to say, there exist $N_e \in \mathbb{N}$ such that $N_{i,e} < N_e$ for all $i$. But then the fiber $X_i^{(e)}[\mathbf{b}_i]$ admits a decomposition in at most $N_e$ cells. Since the union of the latter for all $e$ is just $Y_i$, we showed that there is a uniform bound on the number of cells (whence intervals) in a decomposition of $Y_i$. Since this is now first-order expressible, $Y_\natural$ too is a finite union of intervals. $\dashv$

§**12. Expansions of pseudo-o-minimal structures.** In this section, $\mathcal{M}$ will always denote a pseudo-o-minimal structure. Since an expansion by definable sets does not alter the collection of definable sets, we immediately have:

12.1. LEMMA. *If $X$ is definable in $\mathcal{M}$, then $(\mathcal{M}, X)$ is again pseudo-o-minimal.* $\dashv$

So we ask in more generality, what properties does a subset of a pseudo-o-minimal structure need to have in order for the expansion by that subset to be again pseudo-o-minimal? Let us call such a subset *o-minimalistic* (or, more correctly, *$\mathcal{M}$-o-minimalistic* as this depends on the surrounding structure), where we just proved that definable subsets are.

12.2. COROLLARY. *The image of an o-minimalistic subset under a definable map is again o-minimalistic, and so is its complement, its closure, its boundary, and its interior. More generally, any set definable from an o-minimalistic set is again o-minimalistic.*

PROOF. It suffices to prove the last assertion. Let $X$ be o-minimalistic. Since $(\mathcal{M}, X)$ is pseudo-o-minimal, any set definable in $(\mathcal{M}, X)$ is o-minimalistic (in the expansion, whence also in the reduct) by Lemma 12.1. $\dashv$

To define a weaker isomorphism relation, we introduce the following notation. Let $X$ be a definable subset in a structure $\mathcal{N}$, say, defined by the formula (with parameters) $\varphi$, that is to say, $X = \varphi(\mathcal{N})$. If $\mathcal{N}'$ is an elementary extension of $\mathcal{N}$, then we set $X^{\mathcal{N}'} := \varphi(\mathcal{N}')$, and call it the *definitional extension* of $X$ in $\mathcal{N}'$.

Let us call two $\mathcal{M}$-definable subsets *o-minimalistically isomorphic*, denoted $X \equiv Y$, if their definitional extensions have the same ultra-Euler measure in every ultra-o-minimal elementary extension $\mathcal{M} \preceq \mathcal{N}$, that is to say, if $\mu_{\mathcal{N}}(X^{\mathcal{N}}) = \mu_{\mathcal{N}}(Y^{\mathcal{N}})$. It is easy to see that this constitutes an equivalence relation on definable subsets.

12.3. PROPOSITION. *In a pseudo-o-minimal expansion $\mathcal{M}$ of an ordered field, if two $\mathcal{M}$-definable subsets $X$ and $Y$ are o-minimalistically isomorphic, then there exists a pseudo-o-minimal expansion of $\mathcal{M}$ in which they become definably isomorphic.*

PROOF. Suppose $X$ and $Y$ are o-minimalistically definable, and let $\mathcal{N}$ be some ultra-o-minimal elementary extension of $\mathcal{M}$, given as the ultraproduct of o-minimal structures $\mathcal{N}_i$. Let $X_i$ and $Y_i$ be $\mathcal{N}_i$-definable subsets with respective ultraproducts $X^{\mathcal{N}}$ and $Y^{\mathcal{N}}$. Since by Proposition 10.3 dimension is definable, $X^{\mathcal{N}}$ and $Y^{\mathcal{N}}$ have the same dimension, whence so do almost each $X_i$ and $Y_i$ by Łos' Theorem. By assumption, they have also the same Euler characteristic for almost all $i$, so that they are definably isomorphic by Theorem 11.3. Hence, there exists for almost all $i$, a definable isomorphism $f_i\colon X_i \to Y_i$. Let $\Gamma_\natural$ be the ultraproduct of the graphs $\Gamma(f_i)$. Since almost all $(\mathcal{N}_i, \Gamma(f_i))$ are o-minimal, their ultraproduct $(\mathcal{N}, \Gamma_\natural)$ is pseudo-o-minimal, whence so is $(\mathcal{M}, \Gamma)$, where $\Gamma$ is the restriction of $\Gamma_\natural$ to $\mathcal{M}$. Moreover, by Łos' Theorem, $\Gamma_\natural$ is the graph of a bijection $X^{\mathcal{N}} \to Y^{\mathcal{N}}$, and hence its restriction $\Gamma$ is the graph of a bijection $X \to Y$, proving that $X$ and $Y$ are definably isomorphic in $(\mathcal{M}, \Gamma)$. $\dashv$

I do not know whether the converse is also true: if $X$ and $Y$ are definably isomorphic in some pseudo-o-minimal expansion $\mathcal{M}'$, are they o-minimalistically isomorphic? They will have the same Euler characteristic in any (reduct of an) ultra-o-minimal elementary extension of $\mathcal{M}'$ by essentially the same argument, but what about ultra-o-minimal elementary extensions of $\mathcal{M}$ that are not such reducts? A related question is in case $\mathcal{M}$ itself is already ultra-o-minimal, if two sets have the same Euler characteristic, do their definitional extensions also have the same Euler characteristic in an ultra-o-minimal elementary extension? This would follow if Euler characteristic was definable, but at the moment, we can only prove a weaker version (see Theorem 14.7). Before we address these issues, we prove a result yielding non-trivial examples of o-minimalistically isomorphic sets that need not be definably isomorphic.

12.4. COROLLARY. *In a pseudo-o-minimal expansion $\mathcal{M}$ of an ordered field, if two definable subsets $X$ and $Y$ have the same dimension and the same class in $\mathbf{Gr}(\mathcal{M})$, then they are o-minimalistically isomorphic.*

PROOF. By Lemma 11.1, there exists a definable subset $Z$ such that $X \sqcup Z$ and $Y \sqcup Z$ are definably isomorphic. Let $\mathcal{M} \preceq \mathcal{N}$ be an ultra-o-minimal elementary extension. Hence $X^{\mathcal{N}} \sqcup Z^{\mathcal{N}}$ and $Y^{\mathcal{N}} \sqcup Z^{\mathcal{N}}$ are definably isomorphic, and therefore

$$\chi_{\mathcal{N}}(X^{\mathcal{N}}) + \chi_{\mathcal{N}}(Z^{\mathcal{N}}) = \chi_{\mathcal{N}}(X^{\mathcal{N}} \sqcup Z^{\mathcal{N}}) = \chi_{\mathcal{N}}(Y^{\mathcal{N}} \sqcup Z^{\mathcal{N}}) = \chi_{\mathcal{N}}(Y^{\mathcal{N}}) + \chi_{\mathcal{N}}(Z^{\mathcal{N}})$$

showing that $X^{\mathcal{N}}$ and $Y^{\mathcal{N}}$ have the same ultra-Euler characteristic, as we needed to show. $\dashv$

**12.5. Contexts and virtual isomorphisms.** To overcome the difficulties alluded to above, we must make our definitions context-dependable in the following sense. Given a pseudo-o-minimal structure $\mathcal{M}$, by a *context* for $\mathcal{M}$, we mean an ultra-o-minimal structure $\mathcal{N}$ that contains $\mathcal{M}$ as an elementary substructure (which always exists by Corollary 10.2). An expansion $\mathcal{M}'$ of $\mathcal{M}$ is then called *permissible* (with respect to the context $\mathcal{N}$), if $\mathcal{N}$ can be expanded to a context $\mathcal{N}'$, that is to say, $\mathcal{M}' \preceq \mathcal{N}'$ and $\mathcal{N}'$ is again ultra-o-minimal. If $\mathcal{M}$ itself is ultra-o-minimal, then we may take it as its own context, but even in this case, not every expansion will be permissible, as it may fail to be an ultraproduct.

From now on, we fix a pseudo-o-minimal structure $\mathcal{M}$ and a context $\mathcal{N}$. We define a (context-dependable) *Euler characteristic* $\chi_{\mathcal{M}}(\cdot)$ (or, simply $\chi$) by restricting the ultra-Euler characteristic of $\mathcal{N}$, that is to say, by setting $\chi(X) := \chi_{\mathcal{N}}(X^{\mathcal{N}})$, for any $\mathcal{M}$-definable subset $X$, and we define similarly its *Euler measure* $\mu(X) :=$

$(\dim(X), \chi(X))$. We say that two definable subsets are *virtually isomorphic*, if there exists a permissible expansion of $\mathcal{M}$ in which they become definably isomorphic. In particular, two definable subsets that are o-minimalistically isomorphic are also virtually isomorphic, but the converse is unclear. We can now prove an o-minimalistic analogue of Theorem 11.3.

12.6. THEOREM. *In a pseudo-o-minimal expansion $\mathcal{M}$ of an ordered field, two definable subsets are virtually isomorphic if and only if they have the same Euler measure.*

PROOF. One direction is proven in the same way as Proposition 12.3, so assume $X$ and $Y$ are virtually isomorphic definable subsets. By assumption, $\mathcal{M} \preceq \mathcal{N}$ expands into pseudo-o-minimal structures $\mathcal{M}' \preceq \mathcal{N}'$, with $\mathcal{N}'$ again ultra-o-minimal, such that $X$ and $Y$ are $\mathcal{M}'$-definably isomorphic. Let $\mathcal{N}'$ be the ultraproduct of o-minimal structures $\mathcal{N}'_i$. Since $X^{\mathcal{N}'}$ and $Y^{\mathcal{N}'}$ are definably isomorphic, so are almost all $X_i$ and $Y_i$, where $X_i$ and $Y_i$ are $\mathcal{N}'_i$-definable subsets with respective ultraproducts $X^{\mathcal{N}'}$ and $Y^{\mathcal{N}'}$. In particular, $X_i$ and $Y_i$ have the same Euler measure for almost all $i$, by Theorem 11.3. Hence $X^{\mathcal{N}'}$ and $Y^{\mathcal{N}'}$ have the same ultra-Euler measure, by Proposition 10.3. Since both invariants remain the same in the reduct $\mathcal{N}$, elementarity then yields $\mu(X) = \mu(Y)$. ⊣

**12.7. O-finitism.** As we already mentioned in the introduction, in the o-minimalistic context, discrete sets play the role of finite sets, and so we briefly discuss the first-order aspects of this assertion. Given a (non-empty) collection of $L$-structures $\mathfrak{K}$, and a subset $X \subseteq N^p$ in some $L$-structure $\mathcal{N}$, we say that $X$ is *pseudo-$\mathfrak{K}$-finite*, if $(\mathcal{N}, X)$ satisfies every $L(\mathtt{U})$-sentence $\sigma$ which holds in every expansion $(\mathcal{K}, F)$ of a structure $\mathcal{K} \in \mathfrak{K}$ by a finite set $F \subseteq K^p$. In case $\mathfrak{K}$ is the collection of o-minimal structures, we call $X$ *pseudo-o-finite*. Applying the definition just to $L$-sentences $\sigma$ (not containing the predicate $\mathtt{U}$, so that $(\mathcal{K}, F) \models \sigma$ if and only if $\mathcal{K} \models \sigma$), we see that $\mathcal{N}$ is then necessarily pseudo-o-minimal. Put differently, a pseudo-o-finite set in a pseudo-o-minimal structure is a model of *o-finitism*, that is to say, of the theory of a finite set in an o-minimal structure. By Proposition 11.6, Proposition 2.6 and Proposition 8.7, we have:

12.8. COROLLARY. *A pseudo-o-finite set is discrete, closed, bounded, and locally definable, every non-empty intersection with an open interval has a maximum and a minimum, and every injective, definable self-map on it is an isomorphism.* ⊣

It seems unlikely that these properties characterize fully o-finitism. A complete axiomatization of o-finitism would be of interest in view of the following results.

12.9. THEOREM. *A subset $X \subseteq M^p$ is pseudo-o-finite if and only if it is discrete and o-minimalistic. In particular, any definable, discrete subset in a pseudo-o-minimal structure is pseudo-o-finite.*

PROOF. Assume first that $X$ is pseudo-o-finite, whence discrete by Corollary 12.8. We have to show that given an $L(\mathtt{U})$-sentence $\sigma$ holding true in every o-minimal $L(\mathtt{U})$-structure, then $(\mathcal{M}, X) \models \sigma$. Let $\mathcal{K}$ be an o-minimal structure and let $F \subseteq K^p$ be a finite subset. Hence $(\mathcal{K}, F)$ is also o-minimal and therefore satisfies $\sigma$. Since this holds for all such expansions, $\sigma$ is true in $(\mathcal{M}, X)$ by o-finitism, as we needed to show.

Conversely, suppose $X \subseteq M^p$ is discrete and o-minimalistic, that is to say, $(\mathcal{M}, X)$ is pseudo-o-minimal. To show that $X$ is pseudo-o-finite, let $\sigma$ be a sentence true in every expansion $(\mathcal{K}, F)$ of an o-minimal structure $\mathcal{K}$ by a finite subset $F \subseteq K^p$. Consider the disjunction $\sigma'$ of $\sigma$ with the sentence expressing that the set defined by $\mathtt{U}$ is not discrete. Hence $\sigma'$ is true in any o-minimal expansion $(\mathcal{K}, F)$. Since $X$ is o-minimalistic, this means that $(\mathcal{M}, X) \models \sigma'$, and since $X$ is discrete, this in turn implies that $\sigma$ is true in $(\mathcal{M}, X)$, as we needed to show. The last assertion then follows from Lemma 12.1. $\dashv$

Let us call a subset of an ultra-o-minimal structure *ultra-finite*, if it is the ultraproduct of finite subsets (such a set may fail to be definable, since the definition in each component may not be uniform).

12.10. THEOREM. *A subset $X \subseteq M^p$ is pseudo-o-finite if and only if there exists an elementary extension $\mathcal{M} \preceq \mathcal{N}$ with $\mathcal{N}$ ultra-o-minimal and an ultra-finite subset $Y \subseteq N^p$, such that $X = Y \cap M^p$.*

PROOF. Suppose $\mathcal{N}$ and $Y$ have the stated properties, and let $\mathcal{N}_i$ be o-minimal structures and $Y_i \subseteq N_i^p$ finite subsets, so that $\mathcal{N}$ and $Y$ are their respective ultraproducts. Since $(\mathcal{N}_i, Y_i)$ is again o-minimal, their ultraproduct $(\mathcal{N}, Y)$ is pseudo-o-minimal. Since $(\mathcal{M}, X)$ is then an elementary substructure, the latter is also pseudo-o-minimal. Moreover, since $Y$ is discrete, so must $X$ be, and hence $X$ is pseudo-o-finite by Theorem 12.9. Conversely, by the same theorem, if $X$ is pseudo-o-finite, then $(\mathcal{M}, X)$ is pseudo-o-minimal. Hence there exists an elementary extension $(\mathcal{N}, Y)$ which is ultra-o-minimal as an $L(\mathtt{U})$-structure by Corollary 10.2. Write $(\mathcal{N}, Y)$ as an ultraproduct of o-minimal structures $(\mathcal{N}_i, Y_i)$. Since $X$ is discrete, so must $Y$ be by elementarity, whence so are almost all $Y_i$ by Łos' Theorem. The latter means that almost all are in fact finite, showing that $Y$ is ultra-finite, and the assertion follows since $X = Y \cap M^p$. $\dashv$

Next, we give a criterion for a subset $Y \subseteq M$ to be o-minimalistic. By Theorem 2.10, its boundary $\partial Y$ should be discrete, and $Y^\circ = Y \setminus \partial Y$ should be a disjoint union of open intervals. Given an arbitrary set $Y \subseteq M$, define its *enhanced boundary* $\Delta Y$ as the set consisting of the pairs $(y, \varepsilon)$ with $y \in \partial Y$ and $\varepsilon$ equal to 0, 1, or $-1$, depending on whether respectively $y$, $y^+$, and/or $y^-$ belongs to $Y$. Recall from §2 that $a^+$ (respectively, $a^-$) *belongs to* $Y$ if there exists an open interval with left (respectively, right) endpoint $a$ contained in $Y$. No fiber of an enhanced boundary can have more than two points and its projection is the ordinary boundary $\partial Y$. If $Y$ is o-minimalistic, then $\Delta Y$ must satisfy some extra conditions: it must be bounded, discrete and closed, and, by type completeness, if $(y, 1)$ belongs to it, then so must $(y', -1)$, where $y'$ is the immediate successor of $y$ in $\partial Y$.

12.11. THEOREM. *A subset $Y \subseteq M$ is o-minimalistic if and only if its enhanced boundary $\Delta Y$ is pseudo-o-finite and its interior is a disjoint union of open intervals.*

PROOF. Suppose $Y$ is o-minimalistic, so that $Y^\circ$ is a disjoint union of open intervals. Since $\Delta Y$ is definable from $Y$, it too is o-minimalistic by Corollary 12.2, whence pseudo-o-finite by Theorem 12.9. To prove the converse, let $D := \partial Y = \pi(\Delta Y)$, a bounded, closed, discrete set, and let $l$ be its minimum. Define $X \subseteq M$ as the set of all $x \in M$ such that one of the following three conditions holds

12.11.i. $(x, 0) \in \Delta Y$;
12.11.ii. $x > l$ and $(d, 1) \in \Delta Y$, where $d = \max D_{<x}$;
12.11.iii. $x < l$ and $(l, -1) \in \Delta Y$.

Since $X$ is definable from $\Delta Y$, it is o-minimalistic by Corollary 12.2. Remains to show that $X = Y$. It follows from (12.11.i) that $X \cap D = Y \cap D$, so that it suffices to show that $X^\circ = Y^\circ$. Therefore, we may as well assume from the start that $Y$ is open. Write $Y = \sqcup_n I_n$ as a disjoint union of open intervals, and let $]a,b[$ one of the $I_n$ (we leave the unbounded case to the reader, for which one needs (12.11.iii)). In particular, $a \in D$ and $a^+$ belongs to $Y$, so that $(a,1) \in \Delta Y$. By (12.11.ii), the entire interval $]a,b[$ lies in $X$, whence so does the whole of $Y$. Conversely, if $x \in X$, let $d := \max D_{<x}$, so that $(d,1) \in \Delta Y$. Hence $d^+$ belongs to $Y$, and so $d$ must be an endpoint of one of the $I_n$. The other endpoint must be bigger than $d$, and hence bigger than $x$, showing that $x \in I_n \subseteq Y$. ⊣

§**13. The virtual Grothendieck ring.** We fix again a pseudo-o-minimal structure $\mathcal{M}$ and a context $\mathcal{N}$. We can use virtual isomorphisms instead of definable isomorphisms in the definition of the zero-dimensional or the full Grothendieck ring, that is to say, the quotient modulo the scissor relations of the free Abelian group on virtual isomorphism classes of respectively all discrete, definable subsets, and of all definable subsets, yielding the *virtual Grothendieck rings* $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$ and $\mathbf{Gr}^{\text{virt}}(\mathcal{M})$ respectively. We have surjective homomorphisms $\mathbf{Gr}_0(\mathcal{M}) \to \mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$ and $\mathbf{Gr}(\mathcal{M}) \to \mathbf{Gr}^{\text{virt}}(\mathcal{M})$.

13.1. COROLLARY. *Given a pseudo-o-minimal expansion $\mathcal{M}$ of an ordered field, there exist embeddings $\mathbf{Gr}_0^{virt}(\mathcal{M}) \subseteq \mathbf{Gr}^{virt}(\mathcal{M}) \hookrightarrow \mathbb{Z}_\natural$, where $\mathbb{Z}_\natural$ is the ring of nonstandard integers in the given context.*

PROOF. Since the Euler characteristic vanishes on any scissor relation, it induces by Theorem 12.6 a homomorphism $\chi\colon \mathbf{Gr}^{\text{virt}}(\mathcal{M}) \to \mathbb{Z}_\natural$. By the same result, its restriction to $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$ is injective. To see that $\chi$ is everywhere injective, assume $\chi(X) = \chi(Y)$ for some definable subsets $X$ and $Y$. If they have the same dimension, then they are virtually isomorphic, again by Theorem 12.6. So assume $X$ has dimension $d \geq 1$ and $Y$ has lesser dimension. Let $U$ be the difference of a $d$-dimensional box minus a $(d-1)$-dimensional sub-box, so that in particular $[U]$ vanishes, whence also $\chi(U)$. As $X$ and $Y \sqcup U$ now have the same Euler measure, they are virtually isomorphic by Theorem 12.6, and hence $[X] = [Y] + [U] = [Y]$ in $\mathbf{Gr}^{\text{virt}}(\mathcal{M})$, as we needed to show. The injectivity of $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M}) \to \mathbf{Gr}^{\text{virt}}(\mathcal{M})$ is then also clear. ⊣

In particular, if $\mathcal{M}$ is moreover eukaryote, then we have an equality of virtual Grothendieck rings $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M}) = \mathbf{Gr}^{\text{virt}}(\mathcal{M})$ by Corollary 11.11.

13.2. LEMMA. *If two discrete, $\mathcal{M}$-definable subsets with the same ultra-Euler characteristic are comparable, then they are definably isomorphic.*

PROOF. Suppose $D$ and $E$ are discrete, definable subsets with $D \preceq E$ and $\chi(D) = \chi(E)$. Upon replacing $D$ by a definable copy, we may assume $D \subseteq E$. Taking ultra-Euler characteristics, we get $\chi(E \setminus D) = \chi(E) - \chi(D) = 0$. By Łos' Theorem, the definitional expansion of $E \setminus D$ is empty, whence so is then $E \setminus D$ itself. ⊣

To obtain a 'virtual' generalization, we extend the partial order on $\mathfrak{D}(\mathcal{M})$ to a total order on $\mathfrak{D}^{\text{viso}}(\mathcal{M})$, the set of virtual isomorphism classes of definable, discrete subsets. First, given definable subsets $X$ and $Y$, we say that $X \leq Y$, if $X \preceq_{\mathcal{M}'} Y$ in some permissible pseudo-o-minimal expansion $\mathcal{M}'$ of $\mathcal{M}$. Clearly, if $X \preceq Y$, then $X \leq Y$. The following two results are the o-minimalistic analogues of Theorem 11.3.

13.3. THEOREM. *Given two $\mathcal{M}$-definable, discrete subsets $F$ and $G$, we have $F \leq G$ if and only if $\chi(F) \leq \chi(G)$. In particular, $\leq$ is a total order on $\mathfrak{D}^{viso}(\mathcal{M})$.*

PROOF. Suppose first that $\chi(F) \leq \chi(G)$. Write $\mathcal{N}$ as the ultraproduct of o-minimal structures $\mathcal{N}_i$, and let $F_i$ and $G_i$ be finite sets with respective ultraproducts the definitional extensions $F^{\mathcal{N}}$ and $G^{\mathcal{N}}$ of $F$ and $G$ respectively. Since $\chi_{\mathcal{N}}(F^{\mathcal{N}}) \leq \chi_{\mathcal{N}}(G^{\mathcal{N}})$, the cardinality of $F_i$ is at most that of $G_i$, for almost all $i$. In particular, there exists an injective map $F_i \to G_i$ for almost all $i$. Let $\Gamma_\natural$ be the ultraproduct of the graphs of these maps $F_i \to G_i$. Hence $\Gamma_\natural$ is ultra-finite and therefore its restriction $\Gamma$ to $\mathcal{M}$ is pseudo-o-finite by Theorem 12.10, whence o-minimalistic by Theorem 12.9. By Łos' Theorem and elementarity, $\Gamma$ is the graph of an injective map $F \to G$, showing that $F \preceq_{(\mathcal{M},\Gamma)} G$. Since $(\mathcal{M},\Gamma)$ is permissible, $F \leq G$. The converse goes along the same lines: suppose $F \preceq_{\mathcal{M}'} G$, for some permissible pseudo-o-minimal expansion $\mathcal{M}'$ of $\mathcal{M}$. By definition, there is an ultra-o-minimal expansion $\mathcal{N}'$ of $\mathcal{N}$ with $\mathcal{M}' \preceq \mathcal{N}'$. Since $F^{\mathcal{N}} \preceq_{\mathcal{N}'} G^{\mathcal{N}}$, we have $\chi(F) = \chi_{\mathcal{N}'}(F^{\mathcal{N}}) \leq \chi_{\mathcal{N}'}(G^{\mathcal{N}}) = \chi(G)$. ⊣

13.4. PROPOSITION. *If $\mathcal{M}$ expands an ordered field, then $X \leq Y$ if and only if $\dim(X) \leq \dim(Y)$, for $X$ and $Y$ definable subsets with $\dim(Y) > 0$.*

PROOF. The direct implication is clear. For the converse, by definability of dimension, we may pass to the context of $\mathcal{M}$ and therefore already assume $\mathcal{M}$ is ultra-o-minimal, given as the ultraproduct of o-minimal structures $\mathcal{M}_i$. Let $X_i$ and $Y_i$ be definable subsets in $\mathcal{M}_i$ with respective ultraproducts $X$ and $Y$. By Łos' Theorem, $\dim(X_i) \leq \dim(Y_i)$, and hence $X_i \preceq Y_i$, by Theorem 11.3, for almost all $i$. Let $f_i\colon X_i \to Y_i$ be a definable injection and let $\Gamma_\natural$ be the ultraproduct of the graphs $\Gamma(f_i)$. Since each $(\mathcal{M}_i, \Gamma(f_i))$ is again o-minimal, $(\mathcal{M}, \Gamma_\natural)$ is ultra-o-minimal and hence in particular a permissible expansion. Since $\Gamma_\natural$ is the graph of an injective map by Łos' Theorem, $X \preceq_{(\mathcal{M},\Gamma_\natural)} Y$, as we needed to show. ⊣

In particular, any definable, discrete subset $D$ is virtually univalent, meaning that $D \leq M$.

13.5. COROLLARY. *[Virtual Pigeonhole Principle] Two $\mathcal{M}$-definable, discrete subsets $D$ and $E$ are virtually isomorphic if and only if, for some definable subset $X$, the sets $D \sqcup X$ and $E \sqcup X$ are virtually isomorphic, if and only if $[D] = [E]$ in $\mathbf{Gr}^{virt}(\mathcal{M})$.*

PROOF. One direction in the first equivalence is immediate, so assume $D \sqcup X$ and $E \sqcup X$ are virtually isomorphic. Passing to a permissible pseudo-o-minimal expansion, we may assume that they are already definably isomorphic, say, by an isomorphism $f\colon D \sqcup X \to E \sqcup X$. By totality (Theorem 13.3), we may assume that $E \leq D$, and hence after taking another permissible pseudo-o-minimal expansion, and replacing $E$ with an isomorphic image, we may even assume that $E \subseteq D$. Therefore, the composition of $f$ and the inclusion $E \sqcup X \subseteq D \sqcup X$ is a map with co-discrete image, and hence is surjective by (DPP). However, this can only be the case if $E = D$, as we needed to show. The last equivalence is now just Lemma 11.1. ⊣

13.6. COROLLARY. *The zero-dimensional, virtual Grothendieck ring $\mathbf{Gr}_0^{virt}(\mathcal{M})$ is an ordered ring with respect to $\leq$.*

PROOF. Every element in $\mathbf{Gr}_0^{\mathrm{virt}}(\mathcal{M})$ is of the form $[A] - [B]$, for some definable, discrete subsets $A$ and $B$ in the pseudo-o-minimal structure $\mathcal{M}$. Therefore, for definable, discrete subsets $A_i$ and $B_i$, with $i = 1, 2$, we set $[A_1] - [B_1] \leq [A_2] - [B_2]$ if and only if

$$A_1 \sqcup B_2 \leq A_2 \sqcup B_1. \tag{3}$$

To see that this is well-defined, suppose $[A_i] - [B_i] = [A'_i] - [B'_i]$, for $i = 1, 2$ and definable, discrete subsets $A'_i$ and $B'_i$. Therefore, $[A_i \sqcup B'_i] = [A'_i \sqcup B_i]$, whence $A_i \sqcup B'_i$ and $A'_i \sqcup B_i$ are virtually isomorphic by Corollary 13.5. We have to show that assuming (3), the same inequality holds for the accented sets. Taking the disjoint union with $B'_1 \sqcup B'_2$ on both sides of (3), yields inequalities

$$(A_1 \sqcup B'_1) \sqcup B_2 \sqcup B'_2 \leq (A_2 \sqcup B'_2) \sqcup B_1 \sqcup B'_1$$
$$(A'_1 \sqcup B_1) \sqcup B_2 \sqcup B'_2 \leq (A'_2 \sqcup B_2) \sqcup B_1 \sqcup B'_1$$
$$(A'_1 \sqcup B'_2) \sqcup (B_1 \sqcup B_2) \leq (A'_2 \sqcup B'_1) \sqcup (B_1 \sqcup B_2)$$

which by another application of Corollary 13.5 then gives $A'_1 \sqcup B'_2 \leq A'_2 \sqcup B'_1$, as we needed to show. It is now easy to check that $\leq$ makes $\mathbf{Gr}_0^{\mathrm{virt}}(\mathcal{M})$ into a totally ordered ring. $\dashv$

13.7. COROLLARY. *Every pseudo-o-finite subset defines a cut in $\mathfrak{D}^{viso}(\mathcal{M})$. In particular, we can put a total pre-order on the collection of pseudo-o-finite subsets.*

PROOF. Let $F$ be a pseudo-o-finite subset of a pseudo-o-minimal structure $\mathcal{M}$ and let $D \in \mathfrak{D}^{\mathrm{viso}}(\mathcal{M})$ be arbitrary. Since $(\mathcal{M}, F)$ is pseudo-o-minimal by Theorem 12.9, we can compare $D$ and $F$ in $\mathfrak{D}(\mathcal{M}, F)$ by Theorem 13.3. If $G$ is another pseudo-o-finite subset, then we set $F \leq G$ if and only if the lower cut in $\mathfrak{D}(\mathcal{M})$ determined by $F$ is contained in the lower cut of $G$. $\dashv$

A note of caution: even if $F \leq G$ and $G \leq F$, for $F$ and $G$ pseudo-o-finite subsets, they need not be virtually isomorphic. For instance, taking $D$ as in Example 2.2, it is a pseudo-o-finite subset of $\mathbb{R}_\natural$, and since $\mathfrak{D}(\mathbb{R}_\natural)$ is just $\mathbb{N}$ by o-minimality, its cut is $\infty$. However, $D \setminus \{\omega_\natural\}$ determines the same cut, whence $D \leq D \setminus \{\omega_\natural\} \leq D$, but we know that they cannot be definably isomorphic in any pseudo-o-minimal expansion by (DPP). In fact, it is not clear whether two given pseudo-o-finite subsets live in a common pseudo-o-minimal expansion, and therefore can be compared directly. This is also why we cannot (yet?) define a Grothendieck ring on pseudo-o-finite subsets.

§**14. Discretely valued Euler characteristics.** In order to calculate the zero-dimensional virtual Grothendieck ring, we introduce a new type of Euler characteristic. Fix a pseudo-o-minimal structure $\mathcal{M}$ and a context $\mathcal{N}$, and let $D$ be a definable, infinite, discrete subset. In this section, we will always view $D$ in its lexicographical order $\leq_{\mathrm{lex}}$ (or, when there is no risk for confusion, simply denoted $\leq$).

14.1. COROLLARY. *Any definable subset of an $\mathcal{M}$-definable, discrete subset $D$ is virtually isomorphic to an initial segment $D_{\leq a}$.*

PROOF. The set of initial segments is a maximal chain in $\mathfrak{D}^{\mathrm{viso}}(\mathcal{M})$, since any two consecutive subsets in this chain differ by a single point. Hence, any definable subset $E \subseteq D$ must be a member of this chain up to virtual isomorphism. $\dashv$

Clearly, such an $a$ must be unique, and so, given a non-empty definable subset $E \subseteq D$, we let $\chi_D(E)$ be the unique $a$ such that $E$ is virtually isomorphic with $D_{\leq a}$. We add a new symbol $\varnothing$ to $D$ and set $\chi_D(\emptyset) := \varnothing$. For definable subsets $E_1, E_2 \subseteq D$, we have $E_1 \leq E_2$ if and only if $\chi_D(E_1) \leq \chi_D(E_2)$. Given a definable map $g$ with domain $D$, we can define by Lemma 11.17 its *rank* as $\mathrm{rk}(g) := \chi_D(g(D))$. A map is constant if and only if its rank is minimal (that is to say, equal to the minimum of its domain). By (DPP), we immediately have:

14.2. COROLLARY. *An $\mathcal{M}$-definable map with discrete domain is injective if and only if its rank is maximal (that is to say, equal to the maximum of its domain).* $\dashv$

Assume now that $\mathcal{M}$ is a pseudo-o-minimal expansion of an ordered field, so that in particular all definable discrete subsets are univalent (see Lemma 11.15). Let $D \subseteq M$ be definable, infinite, and discrete, with minimal element $l$ and maximal element $h$. For each $n$, we view the Cartesian power $D^n$ as a definable subset of $D^{n+1}$ via the map $\mathbf{a} \mapsto (l, \mathbf{a})$. We also need to take into consideration the empty set, and so we define $\varnothing$ to be lower than any element in any $D^n$, and we let $D^\infty$ be the direct limit of the ordered sets $D^n \cup \{\varnothing\}$. Under this identification, the elements of $D^n \cup \{\varnothing\}$ form an initial segment in $D^{n+1} \cup \{\varnothing\}$ with respect to the lexicographical ordering. In particular, if $E \subseteq D^n$ is a non-empty definable subset, then $\chi_{D^n}(E) = \chi_{D^{n+1}}(E')$, where $E'$ is the image of $E$ in $D^{n+1}$. After identification therefore, we will view $\chi_{D^n}(E)$ simply as an element of $D^\infty$, and we just denote it $\chi_D(E)$. More generally, given an arbitrary definable subset $X \subseteq M^n$, we define its *$D$-valued Euler characteristic* (or, simply *Euler characteristic*) $\chi_D(X) := \chi_{D^n}(X \cap D^n)$.

We define an addition and a multiplication on $D^\infty$ as follows. First, let us define the *disjoint union* $A \sqcup B$ of two definable subsets $A, B \subseteq M^n$ as the definable subset in $M^{n+1}$ consisting of all $(a, l)$ and $(b, h)$ with $a \in A$ and $b \in B$. For $a \in D^\infty$, we set $a \oplus \varnothing = \varnothing \oplus a = a$ and $a \otimes \varnothing = \varnothing \otimes a = \varnothing$. For the general case, assume $a, b \in D^n$, and let $a \oplus b$ be the Euler characteristic of the disjoint union $(D^n)_{\leq a} \sqcup (D^n)_{\leq b} \subseteq D^{n+1}$, and let $a \otimes b$ be the Euler characteristic of the Cartesian product $(D^n)_{\leq a} \times (D^n)_{\leq b} \subseteq D^{2n}$. One verifies that both operations are independent of the choice of $n$, making $D^\infty$ into a commutative semi-ring, where the zero for $\oplus$ is $\varnothing$, and where the unit for $\otimes$ is $l$, the minimum of $D$. We even can define a subtraction: if $a \leq b$ in $D^\infty$, then we define $b \ominus a$ as the Euler characteristic of $D^n_{>a} \cap D^n_{\leq b}$, where $n$ is sufficiently large so that $a, b \in D^n$. This allows us to define the (Grothendieck) group generated by $(D^\infty, \oplus)$, defined as all pairs $(x, y)$ with $x, y \in D^\infty$ up to the equivalence $(x, y) \sim (x', y')$ if and only if $x \oplus y' = x' \oplus y$; the induced commutative ring will be denoted $\mathfrak{Z}(D)$, and called the ring of *$D$-integers*.

To turn this into a genuine Euler characteristic, recall the construction of the *induced structure* $\mathcal{D}_{\text{ind}}$ on a subset $D \subseteq M$ of a first-order structure: for each definable subset $X \subseteq M^n$, we have a predicate defining in $\mathcal{D}_{\text{ind}}$ the subset $M \cap D^n$. If $\mathcal{M}$ is an ordered structure, then so is $\mathcal{D}_{\text{ind}}$. If $D$ is definable, then we have an induced homomorphism of Grothendieck rings $\mathbf{Gr}(\mathcal{D}_{\text{ind}}) \to \mathbf{Gr}_0(\mathcal{M})$. If instead of definable isomorphism, we take virtual isomorphism, we get the virtual variant $\mathbf{Gr}^{\text{virt}}(\mathcal{D}_{\text{ind}}) \to \mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$. By the Virtual Pigeonhole Principle (Corollary 13.5), this latter homomorphism is injective. To discuss when they are isomorphic, let us call *$D$ power dominant*, if for every definable, discrete subset $A$, there is some $n$ such that $A \leq D^n$.

14.3. PROPOSITION. *A definable, discrete subset $D \subseteq M$ is power dominant if and only if $\mathbf{Gr}^{virt}(\mathcal{D}_{ind}) \cong \mathbf{Gr}_0^{virt}(\mathcal{M})$.*

PROOF. Suppose first that $D$ is power dominant and let $A$ be an arbitrary definable, discrete subset. By assumption, there exists an $n$ and a definable subset $B \subseteq D^n$, such that $A$ is virtually isomorphic with $B$. Hence $[A] = [B]$ in $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$, proving that it lies in the image of $\mathbf{Gr}^{\text{virt}}(\mathcal{D}_{\text{ind}}) \to \mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$.

Conversely, assume that the latter map is surjective, and let $A$ be an arbitrary definable, discrete subset. Hence, there exists an $n$ and definable subsets $E, F \subseteq D^n$ such that $[A] = [E] - [F]$ in $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$. By the Virtual Pigeonhole Principle (Corollary 13.5), this means that there is a virtual isomorphism $A \sqcup F \to E$. Hence the composition $A \subseteq A \sqcup F \to E \subseteq D^n$, shows that $A \leq D^n$. ⊣

To study the existence of power dominant sets, let us say, for $D$ and $E$ discrete, definable subsets, that $D \lll E$, if $D^n \leq E$ for all $n$. If neither $D \lll E$ nor $E \lll D$, then $D$ and $E$ are mutually power bounded, that is to say, there exist $m$ and $n$ such that $D \leq E^m$ and $E \leq D^m$, and we write $D \approx E$. Hence $\lll$ induces a total order relation on the set $\text{Arch}^{\text{pow}}(\mathcal{M})$ of $\approx$-classes of definable, discrete subsets of $\mathcal{M}$. The class of the empty set is the minimal element of $\text{Arch}^{\text{pow}}(\mathcal{M})$, the class of a singleton is the next smallest element, and the class of a two-element set is the next (and consists of all finite sets). For an o-minimal structure, these are the only three classes, whereas for a proper pseudo-o-minimal structure, there must be at least one more class, of some infinite set. I do not know whether $\text{Arch}^{\text{pow}}(\mathcal{M})$ is always discretely ordered or even finite. In any case, it follows easily from the definitions that a class is maximal in $\text{Arch}^{\text{pow}}(\mathcal{M})$ if and only if it is the class of a power dominant set. Thus, the existence of a power dominant set corresponds to $\text{Arch}^{\text{pow}}(\mathcal{M})$ having a maximal element, which is especially interesting in view of Proposition 14.3 and its applications below. I conjecture that $D$ as in Example 2.2 is power dominant (and a similar property for any set obtained by discrete overspill). This would follow from the following growth conjecture for an o-minimal $L$-expansion $\mathcal{R}$ of $\mathbb{R}$:

14.4. CONJECTURE. *There exists, for every formula $\varphi$ in the language $L(\mathtt{U})$, some $n \in \mathbb{N}$, such that for any finite subset $F$, if $\varphi(\mathcal{R}, F)$ is finite, then it has cardinality at most $|F|^n$.*

Recall that $\varphi(\mathcal{R}, F)$ is the set defined by $\varphi$ in the structure $(\mathcal{R}, F)$ in which we interpret the unary predicate $\mathtt{U}$ by $F$. Likewise, I conjecture that the following always produces a power dominant set: let $\mathcal{M}$ be o-minimal and let $D$ be an infinite pseudo-o-finite subset, then $D$ is power dominant in the (pseudo-o-minimal) expansion $(\mathcal{M}, D)$.

14.5. THEOREM. *Every definable, discrete subset $D \subseteq M$ induces a ring isomorphism $\mathbf{Gr}^{virt}(\mathcal{D}_{ind}) \cong \mathfrak{Z}(D)$ by sending the class of a definable subset to its $D$-valued Euler characteristic.*

PROOF. We already observed that the ring operations on $\mathfrak{Z}(D)$ are invariant under virtual isomorphism. It is now easy to see that they also respect the scissor relations (sciss) in the Grothendieck ring of $\mathcal{D}_{\text{ind}}$. Surjectivity follows since every element in $\mathfrak{Z}(D)$ is of the form $a \ominus b$ for some $n$ and some $a, b \in D^n$, and hence is the image of $[(D^n)_{\leq a}] - [(D^n)_{\leq b}]$. To calculate the kernel, we can write a general element as $[E] - [F]$, with $E, F$ definable subsets in $\mathcal{D}_{\text{ind}}$. Such an element lies in the kernel if $\chi_D(E) = \chi_D(F)$, which means that $E$ and $F$ are virtually isomorphic, whence $[E] = [F]$ in $\mathbf{Gr}^{\text{virt}}(\mathcal{D}_{\text{ind}})$. ⊣

Summarizing, we have the following diagram of homomorphisms among the various Grothendieck rings, for $\mathcal{M}$ a pseudo-o-minimal expansion of an ordered field:

$$\begin{tikzcd} \mathbf{Gr}(\mathcal{D}_{\text{ind}}) \arrow[r, two heads] \arrow[d] & \mathbf{Gr}^{\text{virt}}(\mathcal{D}_{\text{ind}}) \arrow[r, "\sim"] \arrow[d, hook, "i"] & \mathfrak{Z}(D) \\ \mathbf{Gr}_0(\mathcal{M}) \arrow[r, two heads] \arrow[d] & \mathbf{Gr}_0^{\text{virt}}(\mathcal{M}) \arrow[d, hook, "j"] & \\ \mathbf{Gr}(\mathcal{M}) \arrow[r, two heads] & \mathbf{Gr}^{\text{virt}}(\mathcal{M}) & \end{tikzcd} \tag{4}$$

with $i$ an isomorphism if $D$ is power dominant by Proposition 14.3, and with $j$ an isomorphism if $\mathcal{M}$ is eukaryote, by Corollary 11.11, that is to say, we proved:

14.6. COROLLARY. *If $\mathcal{M}$ is a eukaryote, pseudo-o-minimal expansion of an ordered field admitting a definable, power dominant subset $D$, then its virtual Grothendieck ring $\mathbf{Gr}^{virt}(\mathcal{M})$ is isomorphic to the ring of $D$-integers $\mathfrak{Z}(D)$.* ⊣

If we would allow classes of pseudo-o-finite subsets in $\text{Arch}^{\text{pow}}(\mathcal{M})$, then there never is a maximal element: let $D$ be any definable, discrete subset (or even any pseudo-o-finite subset). Take an ultra-o-minimal elementary extension $\mathcal{N}$, and choose $D_i \subseteq N_i$ such that their ultraproduct is $D^{\mathcal{N}}$. By the observation following Proposition 13.4, we can choose $A_i \subseteq N_i$ to be virtually isomorphic with $D_i^i$ and let $A_\natural \subseteq N$ be their ultraproduct. By Theorem 12.10, the restriction $A_\natural \cap M$ is pseudo-o-finite and satisfies by Łos' Theorem $D^n \leq A$ for all $n$, that is to say, $D \lll A$.

14.7. THEOREM (O-minimalism of Euler characteristics). *Let $D \subseteq M$ be a definable, discrete subset of a pseudo-o-minimal structure $\mathcal{M}$, and let $X \subseteq M^{n+k}$ be any definable subset. For each $e \in D^n$, the set of parameters $\mathbf{a} \in M^n$ such that $\chi_D(X[\mathbf{a}]) = e$ is o-minimalistic.*

PROOF. If $\mathbf{a}$ does not belong to $D^n$, then $D^n \cap X[\mathbf{a}]$ is empty, whence the $D$-valued Euler characteristic of the fiber $X[\mathbf{a}]$ is $\varnothing$. As these $\mathbf{a}$ form a definable subset, we may therefore replace $X$ by $X \cap D^{n+k}$ and assume already that $X$ is a definable subset of $D^n$. Let $\mathcal{N}$ be the context and write it as the ultraproduct of o-minimal structures $\mathcal{N}_i$. Choose $D_i \subseteq N_i$, $e_i \in D_i^n$ and $X_i \subseteq D_i^{n+k}$ with respective ultraproducts $D^{\mathcal{N}}$, $e$, and $X^{\mathcal{N}}$. For each $i$, let $F_i \subseteq N_i^n$ be the (finite) set of parameters for which the fiber has the same cardinality as $(D_i^n)_{\leq e_i}$. Hence, for each $\mathbf{a} \in F_i$, there exists a bijection $f_{\mathbf{a}} \colon X_i[\mathbf{a}] \to (D_i^n)_{\leq e_i}$. Let $H_i \subseteq N_i^{3n}$ be the union of all $\{\mathbf{a}\} \times \Gamma(f_{\mathbf{a}})$, where $\mathbf{a}$ runs over all tuples in $F_i$. Let $F_\natural \subseteq N^n$ and $H_\natural \subseteq N^{3n}$ be their ultraproduct, so that both sets are ultra-finite. By Łos' Theorem, for each $\mathbf{a} \in F_\natural$, the fiber $H_\natural[\mathbf{a}]$ is the graph of a bijection $X^{\mathcal{N}}[\mathbf{a}] \to \big((D^{\mathcal{N}})^n\big)_{\leq e}$. Therefore, $F := F_\natural \cap M^n$ consists precisely of those $\mathbf{a} \in M^n$ for which the fiber $X[\mathbf{a}]$ has $D$-valued Euler characteristic $e$ in the expansion $(\mathcal{M}, H_\natural \cap M^{3n})$ whence in $\mathcal{M}$, as the former is pseudo-o-minimal by Theorem 12.10. For the same reason, $F$ is pseudo-o-finite, whence o-minimalistic by Theorem 12.9, so that we are done. ⊣

14.8. *Remark.* In everything in this section on Euler characteristics, we may, by passing to a suitable permissible expansion, even assume that $D$ is only pseudo-o-finite.

**14.9. Archimedean reducts.** As before, let $D$ be definable and discrete with respective minimum $l$ and maximum $h$. By (4.1.iii), we have a successor function $\sigma := \sigma_D$, defined on $D \setminus \{h\}$, with inverse $\sigma^{-1}$ defined on $D \setminus \{l\}$. Let us write $e \ll d$, if $\sigma^n(e) < d$, for all $n \in \mathbb{N}$. If neither $d \ll e$ nor $e \ll d$, then $\sigma^n(d) = e$ for some $n \in \mathbb{Z}$, and we write $d \sim_D e$. The set of $\sim_D$-equivalence classes is totally ordered by $\ll$, and is called the *Archimedean reduct* $\operatorname{Arch}(D)$ of $D$.

14.10. THEOREM. *The Archimedean reduct* $\operatorname{Arch}(D)$ *of a definable, discrete subset* $D$ *in a pseudo-o-minimal structure* $\mathcal{M}$ *is dense.*

PROOF. This is clear if $D$ is finite, since then there is only one Archimedean class, so assume it is infinite. If $\operatorname{Arch}(D)$ is not dense, there would exist $l \ll h$ in $D$ so that for no $d \in D$ we have $l \ll d \ll h$. Therefore, upon replacing $D$ with $D \cap [l, h]$, we may assume that $\operatorname{Arch}(D)$ consists of exactly two classes, those of $l$ and $h$. By Corollary 10.2 (or, Theorem 12.10), we can embed $\mathcal{M}$ elementary in an ultra-o-minimal structure $\mathcal{N}$ so that $D$ is the restriction of a (definable) ultra-finite set $F$ in $\mathcal{N}$. Let $\mathcal{N}_i$ and $F_i$ be respectively o-minimal structures and finite subsets in these with ultraproduct equal to $\mathcal{N}$ and $F$ respectively. For each $i$, let $f_i \colon F_i \to F_i$ be the map reversing the (lexicographical) order and let $\Gamma_\natural$ be the ultraproduct of the graphs of the $f_i$. Since this is an ultra-finite set, its restriction $\Gamma$ to $\mathcal{M}$ is an pseudo-o-finite set by Theorem 12.10. By Łos' Theorem, $\Gamma$ is the graph of the order reversing permutation $f \colon D \to D$. In particular, $f$ is definable in the pseudo-o-minimal expansion $(\mathcal{M}, \Gamma)$ and maps any element in the class of $l$ to an element in the class of $h$ and vice versa. By definability, there is a maximal $a \in D$ such that $f(a) \geq a$. In particular, $f(a') < a'$, where $a'$ is the successor of $a$ in $D$. A moment's reflection then shows that then either $f(a) = a$ or $f(a) = a'$, which contradicts that no element is $\sim_D$-equivalent with its image. $\dashv$

14.11. *Remark.* Similarly, given $D, E \in \mathfrak{D}^{\text{viso}}(\mathcal{M})$, we can define $D \ll E$ if for every finite subset $F$, we have $D \cup F \leq E$. If neither $D \ll E$ nor $E \ll D$, then we say that $D$ and $E$ have the same *virtual Archimedean class*, and write $D \sim E$. This is equivalent with the existence of finite subsets $F$ and $G$ such that $D \cup F$ and $E \cup G$ are virtually isomorphic. The induced order $\ll$ on virtual Archimedean classes is dense: indeed, suppose $D \ll E$ and let $d := \chi_E(D)$ and $h := \chi_E(E)$ (i.e., the maximum of $E$). By Theorem 14.10, since $d \ll h$, there is some $a \in D$ with $d \ll a \ll h$. It follows that $D \ll E_{\leq a} \ll E$.

§**15. Taylor sets.** In this section, we work in an expansion of $\mathbb{R}$ and its ultrapower $\mathbb{R}_\natural$, and we introduce some notation and terminology tailored to this situation. Recall that an element in $\mathbb{R}_\natural$ is called *infinitesimal* if its norm is smaller than $1/n$, for all positive $n$. The *standard part* of $\alpha \in \mathbb{R}_\natural$, denoted $\operatorname{st}(\alpha)$, is the supremum of all $r \in \mathbb{R}$ with $r \leq \alpha$; we say that $\alpha$ is *finite*, denoted $\alpha \in \mathbb{R}_\natural^{\text{fin}}$, if $\operatorname{st}(\alpha) < \infty$. In that case, $\operatorname{st}(\alpha) - \alpha$ is infinitesimal and $\operatorname{st}(\alpha)$ is the unique real number with this property. If $\alpha$ is a tuple $(\alpha_1, \dots, \alpha_k)$, then we define $\operatorname{st}(\alpha)$ coordinate-wise as $(\operatorname{st}(\alpha_1), \dots, \operatorname{st}(\alpha_k))$. For a subset $X \subseteq \mathbb{R}^k$, we write $X_\sharp$ for the set of all $\operatorname{st}(\alpha)$ where $\alpha$ runs over all elements in $X_\natural^{\text{fin}} := X_\natural \cap (\mathbb{R}_\natural^{\text{fin}})^k$, and, following the ideology from [18, §8], we call $X_\sharp$ the *catapower* of $X$. We note the following simple result from non-standard analysis, the proof of which we leave to the reader:

15.1. LEMMA. *The catapower of a subset* $X \subseteq \mathbb{R}^k$ *is equal to its closure* $\bar{X}$. $\dashv$

For $X \subseteq \mathbb{R}^k$, define, for each $n$, its *truncation* $X_{\downharpoonright n}$ as the set of points in $X$ whose coordinates have norm at most $n$, where the norm of a point is defined as the maximum of the absolute values of its coordinates. In particular, the ultraproduct of the $\mathbb{R}_{\downharpoonright n}$ is equal to the interval $\mathbb{R}_\flat := \,]\omega_\natural, \omega_\natural[$. We extend this to any subset $X \subseteq \mathbb{R}^k$, by denoting the ultraproduct of the truncations $X_{\downharpoonright n}$ by $X_\flat$ and call it the *protopower* of $X$.[2] In other words, $X_\flat = X_\natural \cap \mathbb{R}^k_\flat$, where $X_\natural$ is the ultrapower of $X$.

By the *trace* of a subset $\Xi \subseteq \mathbb{R}^k_\natural$, denoted $\operatorname{tr}(\Xi)$, we mean the set of its real points, that is to say, $\operatorname{tr}(\Xi) = \Xi \cap \mathbb{R}^k$. If $\Xi$ is definable by a formula $\varphi$ in some expansion of $\mathbb{R}_\natural$, we may use the slightly ambiguous notation $\varphi(\mathbb{R})$ for its trace as well. For any subset $X \subseteq \mathbb{R}^k$, we have $X = \operatorname{tr}(X_\flat)$: indeed, $\mathbf{a} \in \mathbb{R}^k$ satisfies $\mathbf{a} \in X_\flat$, if and only if $\mathbf{a} \in X_{\downharpoonright n}$ for almost all $n$ (by Łos' Theorem), if and only if $\mathbf{a} \in X$. For given $n \in \mathbb{N}$ and a $k$-ary function $f$, let us write $f_{\downharpoonright n}$ for the *truncated* function defined by sending a point $\mathbf{a}$ to $f(\mathbf{a})$ if $|\mathbf{a}| \leq n$ and to zero otherwise (note that this is not the same as taking the truncation of the graph of $f$, since we allow values of arbitrary high norm).

Let $L^{\mathrm{an}}$ be the language of ordered fields together with a function symbol for each everywhere convergent power series (also referred to as a *globally analytic* function). Clearly, we may view $\mathbb{R}$ as an $L^{\mathrm{an}}$-structure, but this is not very useful, since $\mathbb{Z}$ is definable in it (as the zero set of $\sin(\pi x)$), and therefore neither eukaryote nor pseudo-o-minimal. Denef and van den Dries therefore considered in [3] instead restricted analytic functions, that is to say, the structure $\mathcal{R}^{\mathrm{an}}$, where each function symbol $f$ from $L^{\mathrm{an}}$ is interpreted as the truncated analytic function $f_{\downharpoonright 1}$. They then showed that $\mathcal{R}^{\mathrm{an}}$ is o-minimal. Let us write $\mathcal{R}^{\mathrm{an}}_n$ for the $L^{\mathrm{an}}$-structure on $\mathbb{R}$ where each function symbol corresponding to a convergent power series $f$ is interpreted as its truncation $f_{\downharpoonright n}$. It follows that each $\mathcal{R}^{\mathrm{an}}_n$ is also o-minimal, and hence their ultraproduct $\mathcal{R}^{\mathrm{an}}_\natural$ is pseudo-o-minimal. Moreover, $\mathcal{R}^{\mathrm{an}}_\natural$ is eukaryote by Corollary 9.15. While not part of the signature, power series with a smaller radius of convergence can also be encoded, at least in one variable: using a combination of linear transformations $x \mapsto ax + b$, and the (inverse) trig functions $\tan x$ and $\arctan x$, any two open intervals (bounded or unbounded) are isomorphic via a globally analytic map. For instance, if $f$ is defined on the open interval $]-1,1[$, then $g(x) := f(\frac{2}{\pi}\arctan x)$ is globally analytic, and hence $f$ is definable in $L^{\mathrm{an}}$.

15.2. DEFINITION (Taylor sets). We call $X \subseteq \mathbb{R}^k$ a *Taylor* set, if there exists an $L^{\mathrm{an}}$-formula $\varphi(\mathbf{x}, \mathbf{y})$ (without parameters), such that for each sufficiently large $n$, there exists a tuple of parameters $\mathbf{b}_n$ so that $X_{\downharpoonright n} = \varphi(\mathcal{R}^{\mathrm{an}}_n, \mathbf{b}_n)$.

Modifying $\varphi$ if necessary, we may even assume that this holds for all $n$, and that $|\mathbf{x}| \leq n$ is a conjunct in $\varphi$. If $\mathbf{b}_\natural$ is the ultraproduct of the $\mathbf{b}_n$, then the protopower $X_\flat$ is equal to $\varphi(\mathcal{R}^{\mathrm{an}}_\natural, \mathbf{b}_\natural)$ by Łos' Theorem, and hence $X = \operatorname{tr}(X_\flat)$. Any set realized as a *protopower* of a Taylor set will be called an *analytic protopower*, giving a one-one correspondence between Taylor sets and analytic protopowers. We refer to the defining formula $\varphi(\mathbf{x}, \mathbf{b}_\natural)$ of $X_\flat$ as the *analytic* formula for $X$, and we express this by writing $X = \varphi(\mathbb{R})$ (this does not mean that $X$ is definable, since the parameters might be non-standard; in the terminology of §8.6, a Taylor set is in general only locally $L^{\mathrm{an}}$-definable). Not every definable subset is an analytic protopower (equivalently, not every $L^{\mathrm{an}}(\mathbb{R}_\natural)$-formula is analytic): let $\Theta$ be defined by $(\exists y)\, xy = 1 \wedge \sin(\pi y) =$

[2] This is slight variant of the notion of *protoproduct* studied in [18, Chapter 9].

0. Its trace $\mathrm{tr}(\Theta)$ is equal to the set of reciprocals of positive natural numbers and cannot be a Taylor set by Lemma 15.3 below. Any quantifier free $L^{\mathrm{an}}(\mathbb{R}_\natural)$-formula is analytic, so that in particular, any globally real analytic variety is Taylor. Taylor sets are closed under (finite) Boolean combinations, but not under definable (analytic) images, nor under projections. In particular, the Taylor sets do not form a first-order structure.

15.3. LEMMA. *A real discrete subset is Taylor if and only if it is closed. Moreover, a discrete Taylor set intersects any bounded set in finitely many points.*

PROOF. If $X$ is discrete, then $X_{\downharpoonright n}$ must be finite by o-minimality, and hence $X$ cannot have an accumulation point whence is closed. Conversely, if $X$ is discrete and closed, then it is the zero set of some analytic function $f$ (taking sums of squares allows us to reduce to a single equation), and hence $X_{\downharpoonright n}$ is defined in $\mathcal{R}_n^{\mathrm{an}}$ by $f_{\downharpoonright n}(\mathbf{x}) = 0$, and $|\mathbf{x}| \leq n$. ⊣

15.4. LEMMA. *A subset $X \subseteq \mathbb{R}^k$ is Taylor if and only if its protopower $X_\flat$ is $\mathcal{R}_\natural^{an}$-definable.*

PROOF. Recall that $X_\flat$ is the ultraproduct of the truncations $X_{\downharpoonright n}$. One direction has already been observed. Assume $X_\flat$ is $\mathcal{R}_\natural^{\mathrm{an}}$-definable, say $X_\flat = \varphi(\mathcal{R}_\natural^{\mathrm{an}}, \mathbf{b})$, for some $L^{\mathrm{an}}$-formula $\varphi$ and some tuple of parameters $\mathbf{b}$. Writing $\mathbf{b}$ as the ultraproduct of tuples $\mathbf{b}_n$, it follows from Łos' Theorem that $X_{\downharpoonright n} = \varphi(\mathcal{R}_n^{\mathrm{an}}, \mathbf{b}_n)$ for almost all $n$. Enlarging the tuple of parameters if necessary, we may assume that $n$ is one of the entries of $\mathbf{b}_n$. Choosing for each $n$ some $m > n$ such that $X_{\downharpoonright m} = \varphi(\mathcal{R}_m^{\mathrm{an}}, \mathbf{b}_m)$, we get $X_{\downharpoonright n} = \varphi(\mathcal{R}_n^{\mathrm{an}}, \mathbf{b}_m) \wedge |\mathbf{x}| \leq n$, showing that $X$ is Taylor. ⊣

We can rephrase this as a criterion for analytic protopowers:

15.5. COROLLARY. *A protopower is analytic if and only if it is $\mathcal{R}_\natural^{an}$-definable if and only if its trace is Taylor.* ⊣

In terms of formulae, we might paraphrase this as: an $L^{\mathrm{an}}(\mathbb{R}_\natural)$-formula $\varphi$ is analytic if and only if $\varphi(\mathbb{R}_\natural)$ is the ultraproduct of the $\varphi(\mathbb{R}_{\downharpoonright n})$. Thus, an open interval in $\mathbb{R}_\natural$ is an (analytic) protopower if and only if its endpoints are either real or equal to $\pm\omega_\natural$: indeed, suppose $]\alpha, \beta[$ is a protopower, and let $\mathrm{st}(\alpha)$ and $\mathrm{st}(\beta)$ be the respective standard parts of $\alpha$ and $\beta$. Hence $I := \,]\alpha, \beta[\, \cap \mathbb{R}$ is a (not necessarily open) interval with endpoints $\mathrm{st}(\alpha)$ and $\mathrm{st}(\beta)$. If $\mathrm{st}(\alpha)$ is finite, then $I_{\downharpoonright n}$ is an interval with left endpoint $\mathrm{st}(\alpha)$ for $n$ sufficiently large, and hence the same is true for the ultraproduct of these truncations. By Corollary 15.5, this forces $\mathrm{st}(\alpha) = \alpha$. In the other case, the left endpoint of $I_{\downharpoonright n}$ is $-n$, and hence their ultraproduct has left endpoint $-\omega_\natural$, showing that $\alpha = -\omega_\natural$. The same argument applies to $\beta$, proving the claim.

15.6. EXAMPLE. By Lemma 15.3, every closed, discrete subset, whence in particular any subset of $\mathbb{Z}$, is Taylor. To give a non-discrete example, consider the spiral $C \subseteq \mathbb{R}^2$ with parametric equations $x = \exp\tau \sin\tau$ and $y = \exp\tau\cos\tau$, for $\tau \in \mathbb{R}$. If $(x, y) \in C_{\downharpoonright n}$, then $\exp\tau = \sqrt{x^2 + y^2} \leq n\sqrt{2}$ and hence $\tau \leq \log(n\sqrt{2}) \leq n$. In particular, the negative values of $\tau$ can be larger in absolute value than $n$. Hence $C$ is not Taylor. However, if $C^+$ is the 'positive' part, given by the same equations but only for $\tau \geq 0$, then $C^+_{\downharpoonright n}$ is defined in $\mathcal{R}_n^{\mathrm{an}}$ by $x = \exp_{\downharpoonright n}(\tau)\sin_{\downharpoonright n}(\tau)$, $y = \exp_{\downharpoonright n}(\tau)\cos_{\downharpoonright n}(\tau)$, and $\tau \leq \log_{\downharpoonright n}(n\sqrt{2})$, showing that $C^+$ is Taylor (see Corollary 15.10 below).

15.7. PROPOSITION. *The closure, interior, frontier, and boundary of a Taylor set is again Taylor.*

PROOF. Since all concepts are obtained by either taking closures or Boolean combinations, it suffices to show that the closure $\bar{X}$ of a Taylor set $X$ is again Taylor. Let $\varphi(x,z)$ be an analytic formula for $X$, so that $X_{\downharpoonright n} = \varphi(\mathcal{R}_n^{\text{an}}, \mathbf{b}_n)$, for some parameters $\mathbf{b}_n$ and all $n$. If $\psi(x,z)$ is the formula $(\forall a > 0)(\exists y)\,|x-y| < a \wedge \varphi(y,z)$, then $\psi(\mathcal{R}_n^{\text{an}}, \mathbf{b}_n)$ defines the closure of $X_{\downharpoonright n}$. It is now easy to check that the latter is equal to $\bar{X}_{\downharpoonright n}$, showing that $\bar{X}$ is Taylor. ⊣

15.8. *Remark.* From the proof it is also clear that if $X_\flat$ is the protopower of $X$, then the closure $\overline{X_\flat}$ of $X_\flat$ is the protopower of $\bar{X}$, and the analogous properties for the other topological operations. Inspecting the above proofs and examples, we can single out the following geometric feature of Taylor sets.[3]

15.9. PROPOSITION. *Let $X \subseteq \mathbb{R}^{k+1}$ be a Taylor set and let $Y \subseteq \mathbb{R}^k$ be its projection onto the first $k$ coordinates. If there exists $l \in \mathbb{N}$ such that $Y_{\downharpoonright n}$ is contained in the projection of $X_{\downharpoonright ln}$, for all sufficiently large $n$, then $Y$ is again Taylor.*

PROOF. Let $\varphi(\mathbf{x}, y, \mathbf{c}_\natural)$ be the analytic formula defining $X$, and choose tuples $\mathbf{c}_n$ with ultraproduct equal to $\mathbf{c}_\natural$. By definition, $X_{\downharpoonright n}$ is defined in $\mathcal{R}_n^{\text{an}}$ by $\varphi(\mathbf{x}, y, \mathbf{c}_n)$. Let $\tilde{\varphi}(\mathbf{x}, y, \mathbf{c}_\natural)$ be the formula obtained from $\varphi$ by replacing every power series $f(\mathbf{x}, y)$ occurring in it by the power series$f(\mathbf{x}, ly)$, and put $\psi(\mathbf{x}, \mathbf{c}_\natural) := (\exists y)\tilde{\varphi}(\mathbf{x}, y, \mathbf{c}_\natural)$. I claim that $\psi$ is an analytic formula with $\psi(\mathbb{R}) = Y$. To this end, we have to show that $Y_{\downharpoonright n} = \psi(\mathcal{R}_n^{\text{an}}, \mathbf{c}_n)$, for almost all $n$. One inclusion is clear, so assume $\mathbf{a} \in Y_{\downharpoonright n}$, for some $n$. Hence $|\mathbf{a}| \leq n$ and there exists $b \in \mathbb{R}$ such that $(\mathbf{a}, b) \in X$. By assumption, we can choose $|b| \leq ln$. Let $b' := b/l$, so that $|b'| \leq n$. Since then $\mathcal{R}_n^{\text{an}} \models \tilde{\varphi}(\mathbf{a}, b')$, as the point $(\mathbf{a}, b')$ has norm at most $n$, whence agrees on any power series with its $n$-th truncation, we get $\mathcal{R}_n^{\text{an}} \models \psi(\mathbf{a})$, as required. ⊣

Given a $\mathcal{C}_1$-function $f\colon \mathbb{R} \to \mathbb{R}$ on an open interval $]a,b[$, we say that $f$ is *increasing* at $b$ if $f'(b^-) > 0$, where $f'(b^-)$ denotes the left limit at $b$ of the derivative $f'$, with a similar definition for decreasing or at the left endpoint.

15.10. COROLLARY. *Let $f$ be a power series converging on a half-open interval $[a,b[$. If $f$ is increasing at $b$, then the curve $C \subseteq \mathbb{R}^2$ with polar equation $R = f(\theta)$, for $a \leq \theta < b$, is Taylor.*

PROOF. As discussed above, we may make an order-preserving, analytic change of variables so that $f$ becomes convergent on $\mathbb{R}_{\geq 0}$. In particular, $f$ is increasing at $\infty$, which by L'Hôpital's rule means that the limit of $f(x)/x$ for $x \to \infty$ exists and is positive. Hence, we may choose $l \in \mathbb{N}$ large enough so that $1/l < f(x)/x$ for all $x \geq l$. Let $X \subseteq \mathbb{R}^3$ be the semi-analytic set given by $x = f(z)\sin(z)$, $y = f(z)\cos(z)$, and $a \leq z < b$, so that $C$ is just the projection of $X$ onto the first two coordinates. By Proposition 15.9, it suffices to show that $C_{\downharpoonright n}$ is contained in the projection of $X_{\downharpoonright 2ln}$, for all $n$. To this end, let $(a,b) \in C_{\downharpoonright n}$, so that $a = f(\theta)\sin\theta$ and $b = f(\theta)\cos\theta$, for some $\theta \geq 0$. In particular, $f(\theta) = \sqrt{a^2+b^2} \leq n\sqrt{2}$. There is nothing to prove if $\theta \leq l$, so let $\theta > l$ and hence $1/l < f(\theta)/\theta$. The result now follows since $\theta < lf(\theta) \leq 2ln$. ⊣

[3] The corresponding syntactic characterization of analytic formulae is not yet clear to me.

Of course, a similar criterion exists if the domain is open at the left endpoint, where the function now has to be decreasing. We already observed that a Taylor set is of the form $\varphi(\mathbb{R})$ for some $L^{\mathrm{an}}(\mathbb{R}_\natural)$-formula $\varphi$, that is to say, is a trace of an $\mathcal{R}^{\mathrm{an}}_\natural$-definable subset. For each such trace $X := \varphi(\mathbb{R})$, we can define its *dimension* $\dim(X)$ to be the dimension of $\varphi(\mathcal{R}^{\mathrm{an}}_\natural)$. In general, this notion is not well behaved: the trace of the discrete, zero-dimensional set given by the formula $(\exists y > 0)\sin(\pi y) = 0 \wedge \sin(\pi xy) = 0$ is equal to $\mathbb{Q}$, a non-discrete set. Fortunately, Taylor sets behave tamely, as witnessed, for instance, by the following planar trichotomy (compare with Theorem 7.4):

15.11. THEOREM. *A non-empty Taylor subset $X \subseteq \mathbb{R}^2$ is either*

15.11.i. *zero-dimensional, discrete, and closed;*
15.11.ii. *one-dimensional, nowhere dense, but at least one projection has non-empty interior;*
15.11.iii. *two-dimensional with non-empty interior.*

PROOF. Let $X_\flat$ be the protopower of $X$ and $d$ its dimension. By Proposition 10.3, almost all truncations $X_{\rfloor n}$ have dimension $d$ . Hence, if $d = 0$, then almost all (whence all) $X_{\rfloor n}$ are finite and $X$ is closed and discrete. If $d = 2$, then almost all (whence all) $X_{\rfloor n}$ have non-empty interior, whence so does $X$. Finally, if $d = 1$, (almost) all $X_{\rfloor n}$ are nowhere dense, and some projection has interior. Therefore, $X$ itself has the same properties. $\dashv$

In view of Remark 15.8, the dimension of the frontier $\mathrm{fr}(X)$ of a Taylor set $X$ is strictly less than its dimension $\dim(X)$. Hence, by the same argument as for Corollary 7.11, we immediately get:

15.12. COROLLARY. *Any Taylor set is constructible.* $\dashv$

Next, we study maps in this context. For $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^l$, let us call a map $f\colon X \to Y$ *Taylor*, if its graph is a Taylor set. It is not hard to conclude that:

15.13. COROLLARY. *The domain and image of a Taylor map are Taylor, and so is any fiber. Likewise, if the graph of an $\mathcal{R}^{an}_\natural$-definable map $\gamma\colon \Xi \to \Theta$ is a protopower, then so are $\Xi$ and $\gamma(\Xi)$, as well as every fiber $\gamma^{-1}(\mathbf{b})$ with $\mathbf{b} \in Y$. Moreover, the trace of $\gamma$ induces a Taylor map $g\colon \mathrm{tr}(\Xi) \to \mathrm{tr}(\gamma(\Xi))$, and any Taylor map is obtained in this way.* $\dashv$

15.14. *Remark* (Taylor cell decomposition). In particular, a horizontal Taylor 1-cell in $\mathbb{R}^2$ must be the graph of a continuous, Taylor map, and similarly, a Taylor 2-cell in $\mathbb{R}^2$ is the region between two Taylor graphs. Let $X$ be a Taylor set with protopower $X_\flat$. Since $\mathcal{R}^{\mathrm{an}}_\natural$ is eukaryote, we can find a surjective, cellular map $\delta\colon X_\flat \to \Delta$ with $\Delta$ a discrete, closed set. I conjecture that we may take $\delta$ to be a protopower too. Assuming this, taking traces yields a Taylor map $d\colon X \to \mathrm{tr}(\Delta)$, whose fibers are all Taylor cells, and hence defined by means of continuous Taylor maps. Hypothetically, this yields a Taylor cell decomposition of $X$ which is finite on each compact subset by Lemma 15.3.

Using the o-minimalistic DPP (Proposition 11.6), one easily shows:

15.15. COROLLARY. *Any discrete Taylor set $D$ satisfies DPP in the sense that a Taylor map $D \to D$ is injective if and only if it is surjective.* $\dashv$

15.16. COROLLARY (Monotonicity for Taylor maps). *A Taylor map $g\colon X \to Y$ is continuous outside a set of dimension strictly less than the dimension of $X$. In particular, one-variable Taylor maps are monotone outside a discrete, closed (Taylor) subset.*

PROOF. We may assume, for the purposes of this proof that $g$ is surjective, so that, in particular, both $X$ and $Y$ are Taylor, by Corollary 15.13. By the same result, taking protopowers yields a definable map $g_\flat\colon X_\flat \to Y_\flat$ whose restriction to $X$ is equal to $g$. By the Monotonicity Theorem (Theorem 3.2), the set of discontinuities $\Delta$ of $g_\flat$ has dimension strictly less than $\dim(X_\flat) = \dim(X)$. Replacing $\Delta$ by its closure,[4] which does not change the dimension, we may assume $\Delta$ is closed. I claim that $g$ is continuous outside the trace $D := \mathrm{tr}(\Delta)$. Indeed, if $a \in X \setminus D$, then by the non-standard criterion for continuity, we have to show that for all $\alpha$ infinitesimally close to $a$, their images under $g_\natural$ remain infinitesimally close, where $g_\natural$ is the ultrapower of $g$. However, since $g_\flat$ is the ultrapower of the restrictions $g|_{X_{\downharpoonleft n}}$, both maps agree on bounded elements, and so we have to show that $g_\flat(a)$ and $g_\flat(\alpha)$ are infinitesimally close. This does hold indeed for $\alpha$ sufficiently close to $a$ since $a \notin \Delta$ and $\Delta$ is closed.

In the one-variable case, we may choose $\Delta$ so that $g_\flat$ is monotone on any interval with endpoints in $\Delta$, and clearly, $g$ is then monotone on $D$. It follows from Lemma 15.3 and Theorem 15.11 that $D$ is Taylor. $\dashv$

15.17. *Remark.* Using the discussion in Remark 3.6, we can choose $\Delta$ in the above statement also to be Taylor in higher dimensions.

If $f\colon X \to Y$ is Taylor and bijective, then its inverse is also Taylor, and we will say that $X$ and $Y$ are *analytically isomorphic*. In the definition of a Grothendieck ring, it was not necessary that the collection of subsets formed a first-order structure, only that they were preserved under Boolean combinations and products. Since this is true also of Taylor sets, we can define the *analytic* Grothendieck ring $\mathbf{Gr}^{\mathrm{an}}$ as the free Abelian group of analytic isomorphism classes of Taylor sets modulo the scissor relations.

15.18. PROPOSITION. *There is a natural homomorphism $\mathbf{Gr}^{an} \to \mathbf{Gr}(\mathcal{R}_\natural^{an})$ of Grothendieck rings sending the class of a Taylor set $X$ to the class of its protopower $X_\flat$.*

PROOF. To show that the map $[X] \to [X_\flat]$ is well-defined, suppose $f\colon X \to Y$ is an analytic isomorphism. The ultraproduct of the truncations $f_n\colon X_{\downharpoonleft n} \to Y_{\downharpoonleft n}$ induces then a definable map $f_\flat\colon X_\flat \to Y_\flat$, and by Łos' Theorem, this is again a bijection. $\dashv$

By Corollary 11.4, composition with the ultra-Euler-characteristic yields the *analytic Euler characteristic* $\chi^{\mathrm{an}}(X)$ of a Taylor set $X$; by definition, it is the ultraproduct of the $\chi(X_{\downharpoonleft n})$. In particular, for $D$ discrete and closed, $\chi^{\mathrm{an}}(D)$ is the ultraproduct of the cardinalities of its truncations. For which $D$ does there exist a *density* $d$ such that $\chi^{\mathrm{an}}(D)/\omega_\natural^d$ is a bounded element, and if so, what is its standard part?

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[4] In fact, this is not needed since one can show that $\Delta$ is already closed.

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