RECURSIVE SEQUENCES AND FAITHFULLY FLAT EXTENSIONS

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INTRODUCTION

This note arises from an attempt to give some model-theoretic interpretation of the concept of flatness. It is well-known that a necessary condition for a ring morphism \( A \to B \) to be faithfully flat, is that any linear system of equations with coefficients from \( A \) which has a solution over \( B \), must have already a solution over \( A \). In fact, if we strenghten this condition to any solution over \( B \) comes from solutions over \( A \) by base change, then this becomes also a sufficient condition for being faithfully flat. However, whereas the first (necessary) condition is reminiscent of the model-theoretic notion of existentially closedness, the second seems to have no model-theoretic counterpart. Recall that a subring \( A \) of a ring \( B \) is said to be existentially closed in \( B \), if any (not necessarily linear) system of equations with coefficients from \( A \) which is solvable over \( B \) is already solvable over \( A \). This is the relative version of this concept, the absolute version reads: a ring \( A \) in a class of rings \( K \) is called existentially closed or generic for that class, if for any overring \( B \in K \) we have that \( A \) is existentially closed in \( B \).\(^1\) So, paraphrasing this notion, one could say that if \( A \subset B \) is faithfully flat, then \( A \) is existentially closed in \( B \) with respect to linear equations. But as already observed, this is not a sufficient condition to guarantee faithful flatness.

I will present a property of rings which is a consequence of faithfully flatness, but presumably stronger than existentially closedness for linear equations. The key definition is that of a (linear) recursive sequence \((x_n)\) over a ring \( A \), as a sequence satisfying some fixed linear relation over \( A \) among \( t \) consecutive terms. We will show that if \( A \to B \) is faithfully flat and \((x_n)\) is a sequence of elements in \( A \) satisfying a linear recursion relation with coefficients in \( B \), then it already satisfies such a recursion relation (of the same length) with coefficients in \( A \). As there is a strong connection between recursive sequences and rational power series, we obtain the following corollary. Assume, moreover, that \( A \) and \( B \) are normal domains, then any power series over \( A \) which is rational (meaning that it can be written as a quotient of two polynomials) over \( B \), is already rational over \( A \). Any direct attempt, however, to prove this corollary just using faithfully flatness seems to fail as far as I can tell.

1. Definition. Let \( A \) be a Noetherian ring and let \( \bar{x} = (x_n)_{n<\omega} \) be a (countable) sequence of elements of \( A \). We say that \( \bar{x} \) is recursive over \( A \) (of length \( t \)), if there

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\(^1\)The typical example here is an algebraically closed field within the class of all fields.
exists a linear form \( L(X) = r_0X_0 + \cdots + r_{t-1}X_{t-1} \in A[X] \), where \( t \in \mathbb{N} \) and \( X = (X_i)_{i \leq t} \), such that for all \( n \gg 0 \), we have that

\[
L(x_n, \ldots, x_{n+t-1}) = x_{n+t}.
\]

Here we have used the notation \( n \gg 0 \) as an abbreviation for there exists \( n_0 \), such that for all \( n \geq n_0 \).

2. Theorem. Let \( A \to B \) be a morphism of Noetherian rings and let \( \bar{\mathbf{x}} \) be a sequence in \( A \). If \( B \) is faithfully flat over \( A \) and \( \bar{\mathbf{x}} \) is recursive over \( B \), then \( \bar{\mathbf{x}} \) is already recursive over \( A \) (of the same length).

Proof. Let \( L_t(\bar{x}) \) be the collection of all linear forms \( L(X) \in A[X] \), where \( X = (X_i)_{i \leq t} \) and \( t \in \mathbb{N} \), such that, for all \( n \gg 0 \), we have that \( L(x_n, \ldots, x_{n+t}) = 0 \).

Evidently, \( L_t(\bar{x}) \) carries the structure of an \( A \)-module. Let \( e \) be the \((t + 1)\)-tuple \((0, \ldots, 0, -1)\) and let

\[
a_t(\bar{x}) = \{ L(e) \mid L \in L_t(\bar{x}) \}.
\]

As the latter is the image of \( L_t(\bar{x}) \) under the morphism \( A[X] \to A \) defined by substituting \( e \) for \( X \), it is an ideal of \( A \).

We claim that \( \bar{\mathbf{x}} \) is recursive over \( A \) of length \( t \), if and only if \( a_t(\bar{x}) = A \).

Indeed, if \( \bar{\mathbf{x}} \) is recursive over \( A \) of length \( t \), then there is a linear form \( L(X) \in A[X] \), with \( X = (X_i)_{i \leq t} \), such that for all \( n \gg 0 \), we have that \( L(x_n, \ldots, x_{n+t}) = x_{n+t} \).

Let \( L'(X, X_t) = L(X) - X_t \), so that \( L' \in L_t(\bar{x}) \). Since \( 1 = L'(e) \), we proved one direction and the converse follows along the same lines.

Now, by assumption \( \bar{\mathbf{x}} \) is recursive over \( B \) of length \( t \), so that by the criterion we just established, \( a_t^B(\bar{x}) = B \). Hence let \( L(X) = b_0X_0 + \cdots + b_tX_t \in B[X] \) be a witness to this, where \( X = (X_i)_{i \leq t} \), i.e., such that for some \( n_0 \in \mathbb{N} \), we have, for all \( n \geq n_0 \), that \( L(x_n, \ldots, x_{t+n}) = 0 \) and \( L(e) = 1 \). Therefore \( b_t = -1 \). Put \( b = (b_i)_{i \leq t} \in B^t \). For every \( n \geq n_0 \), let \( M_n(Y) \in A[Y] \), where \( Y = (Y_i)_{i \leq t} \), be defined as

\[
M_n(Y) = x_nY_0 + \cdots + x_{n+t}Y_t.
\]

By Noetherianity, there exists some \( n_1 \geq n_0 \), such that each \( M_n \), for \( n \geq n_0 \), lies in the ideal of \( A[Y] \) generated by \( M_{n_0}, \ldots, M_{n_1} \). In other words, there exist \( p_{n,k}(Y) \in A[Y] \), such that, for all \( n \geq n_0 \) and all \( k \) with \( n_0 \leq k \leq n_1 \), we have that

\[
(1) \quad M_n(Y) = \sum_{k=n_0}^{n_1} p_{n,k}(Y)M_k(Y).
\]

By construction, \( M_k(b, -1) = 0 \), for \( n_0 \leq k \leq n_1 \). By flatness, we can find finitely many \( a^{(j)} \in A^t \), \( e^{(j)} \in A \) and \( \beta^{(j)} \in B \), such that

\[
(2) \quad M_k(a^{(j)}, e^{(j)}) = 0,
\]

for all \( j < s \) and \( n_0 \leq k \leq n_1 \), and

\[
(3) \quad b = \sum_{j<s} \beta^{(j)}a^{(j)},
\]

\[
-1 = \sum_{j<s} \beta^{(j)}e^{(j)}.
\]
However, from (1) and (2), it then follows that $M_n(a^{(j)}, e^{(j)}) = 0$, for all $j < s$ and all $n \geq n_0$. This means that $L^{(j)}(X) = a^{(j)}_{0}X_0 + \cdots + a^{(j)}_{t-1} + e^{(j)}X_t$ lies in $L_{t}(\vec{x})$, for each $j < s$, where $a^{(j)} = (a^{(j)}_{i})_{i < t}$. Hence $-e^{(j)} \in A^{t}(\vec{x})$. Together with (3), we therefore conclude that $A^{t}(\vec{x})B = B$. But faithfully flatness then implies that $A^{t}(\vec{x}) = A$, which by the above criterion means that $\vec{x}$ is recursive over $A$ of length $t$. \[\blacksquare\]

1.3. Proposition. Let $A$ be a Noetherian ring and $\vec{x}$ a sequence in $A$. Let $\xi_\vec{x}(T) \in A[[T]]$ be the generating series of $\vec{x}$, i.e.,

$$\xi_\vec{x}(T) = \sum_{n=0}^{\infty} x_n T^n.$$ 

Then $\vec{x}$ is recursive over $A$, if and only if, $\xi_\vec{x}(T)$ lies in $A(T)$, where the latter ring is the localization of $A[T]$ to the multiplicative set $1 + (T)A[T]$.

Proof. Suppose that $\vec{x}$ is recursive and let $\xi(T) = \xi_\vec{x}(T)$. There is some $n_0$ and some $a_k \in A$, for $k < t$, such that

(1) $x_n = a_0x_{n-t} + \cdots + a_{t-1}x_{n-1}$

for all $n \geq n_0$. We want to show that

(2) $\xi(T) \equiv Q(T)\xi(T) \mod A[T]$ 

for some polynomial $Q(T) \in (T)A[T]$. Indeed, if (2) holds, then

$$\xi(T) = Q(T)\xi(T) + P(T)$$

for some $P(T) \in A[T]$ and hence $\xi(T) = P(T)/(1 - Q(T))$ as required.

We work in the $A$-module $A[[T]]/A[T]$. Clearly

$$\xi(T) \equiv T^{n_0} \cdot \xi_{\vec{x}'}(T) \mod A[T],$$

where $\vec{x}'$ is the sequence obtained from $\vec{x}$ by deleting the first $n_0$ elements. Hence in order to prove (2) we may work with this new recursive sequence and hence assume from the start that $n_0 = 0$. We have, using (1), that

$$\xi(T) = \sum_{n < t} x_n T^n + \sum_{n \geq t} x_n T^n$$

$\equiv \sum_{n \geq t} (a_0x_{n-t} + \cdots + a_{t-1}x_{n-1})T^n \mod A[T]$ 

$\equiv a_0T^t\xi(T) + \cdots + a_{t-1}T\xi(T) \mod A[T]$.

This proves (2).

The converse is an easy exercise and left to the reader. \[\blacksquare\]
1.4. Corollary. Let \( \varphi \colon A \to B \) be a morphism of Noetherian rings. If \( \varphi \) is faithfully flat, then

\[
A(T) = A[[T]] \cap B(T),
\]

for \( T \) a single variable.

Proof. The \( \subset \)-inclusion is immediate hence take \( F \) in the right hand side of (1). Taking the coefficients of \( F \) as the members of a sequence \( \bar{x} \) over \( A \), we have that \( F(T) = \xi_{\bar{x}}(T) \). Since \( F(T) \in B(T) \), the sequence \( \bar{x} \) is recursive over \( B \) by (1.3). Therefore, by faithfully flatness and (1.2), \( \bar{x} \) is also recursive over \( A \) which by (1.3) again means that \( F \in A(T) \).

\[\blacksquare\]

1.5. Corollary. Let \( A \to B \) be a faithfully flat morphism of Noetherian normal domains. Then any power series with coefficients in \( A \) which is rational over \( B \), is already rational over \( A \).

Proof. A Noetherian normal domain \( A \) has the Fatou property by CHABERT [1, §3], meaning that \( A(T) \) is equal to the intersection of the fraction field of \( A[[T]] \) with \( A[[T]] \), i.e., \( A(T) \) is the ring of rational power series and we can apply (1.4).

References