KAWAMATA-VIEHWEG VANISHING VIA PURITY PROPERTIES OF THE ULTRA-FROBENIUS

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1. INTRODUCTION

Part 1 of this project is concerned with proving a relative version of the results in [1]. Consider the category of Lefschetz rings in which the objects are rings of equal characteristic zero which are realized as an ultraproduct of Noetherian rings of positive characteristic, and the morphisms between two such ultraproducts are given as ultraproducts of homomorphisms between the components. A Lefschetz ring $D$ is in general no longer Noetherian, but it has a very important feature which will play a pivotal role. Namely, let $D$ be the ultraproduct of the $D_w$, where $D_w$ has characteristic $p(w) > 0$. Note that in order for $D$ to have characteristic zero, $p(w)$ must grow unboundedly. On each $D_w$, choose an endomorphism $\varphi_w$ which is a power of the Frobenius, that is to say, $\varphi_w$ raises each element to its $p(w)^e(w)$-th power, where $e(w)$ is some positive integer. The ultraproduct of the $\varphi_w$ is denoted by $\varphi$ and is called an ultra-Frobenius on $D$. In the remainder of this proposal, $(R, m)$ will denote a Noetherian local ring containing $\mathbb{Q}$.

Conjecture 1. There exists a covariant functor $D_R$ from the category of finitely generated $R$-algebras to the category of Lefschetz rings and a natural transformation $\eta$ from the identity functor to $D_R$.

Let us assume that the functor $D_R$ and the natural transformation $\eta$ exist and let $A$ be a finitely generated $R$-algebra. We view $D_R(A)$ as an $A$-algebra via the homomorphism $\eta_A: A \to D_R(A)$. We say that $A$ is ultra-F-regular, if for every $c \in A$ not contained in a minimal prime of $A$, there exists an ultra-Frobenius $\varphi$ such that the morphism $A \to D_R(A)$ given by the rule $x \mapsto c\varphi(x)$, is pure.

The second part of this proposal is about certain vanishing theorems. For the remainder of this introduction, $X$ denotes a connected projective scheme over $\text{Spec}(R)$. Let $\mathcal{P}$ be an ample invertible sheaf on $X$. The section ring of $X$ with respect to $\mathcal{P}$ is by definition the ring of global sections $S(\mathcal{P}) := H^0(X, \oplus_n \mathcal{P}^n)$. Since $\mathcal{P}$ is ample, $S(\mathcal{P})$ is a finitely generated graded $R$-algebra (with $R$ lying in degree zero) and $X \cong \text{Proj}(S(\mathcal{P}))$.

Conjecture 2. Let $\mathcal{P}$ be an ample invertible sheaf on $X$.

(2.1) Let $\mathcal{L}$ be a numerically effective line bundle on $X$. If $S(\mathcal{P})$ is ultra-F-regular, then Kawamata-Viehweg vanishing holds for $\mathcal{L}$, that is to say, the higher sheaf cohomology $H^i(X, \mathcal{L})$ of $\mathcal{L}$ vanishes, and if in addition $\mathcal{L}$ is big, $H^i(X, \mathcal{L}^{-1})$ vanishes for $i < \dim X$.

(2.2) If $X$ is Fano, then we may choose $\mathcal{P}$ so that $S(\mathcal{P})$ is ultra-F-regular.

I gave a proof of Conjecture 1 in [18] in the geometric case, i.e., in the case that $R$ is an algebraically closed field of characteristic zero. For general $R$, I proved together with Aschenbrenner in [1] that there exists a faithfully flat homomorphism $R \to R_\infty$ with $R_\infty$ a Lefschetz ring (which is even functorial when one adds more structure to the category
of Noetherian local rings). This gives already the ‘base case’, by letting $D_R(R) := R_\infty$ and $\eta: R \to R_\infty$. A proof of the conjecture will now follow from a relativized version of the proof in [18] starting from the base case $\eta: R \to D_R(R)$. I also proved the geometric case of Conjecture 2 in [21]. The argument for (2.1) will rest on a calculation of the action of the ultra-Frobenius on certain cohomology groups, modeled after the proof in [21]. However, (2.2) might be harder to prove. Fano varieties are not the only varieties to which (2.1) applies: so does any globally ultra-F-regular variety, that is, a variety admitting an ample invertible sheaf $P$ whose section module $S(P)$ is ultra-F-regular. This class is expected to behave well under quotients:

**Conjecture 3.** If $X$ is globally ultra-F-regular and $G$ is a reduced group acting algebraically on $X$, then any GIT quotient $X//G$ of $X$ is also globally ultra-F-regular.

In particular, I postulate that any GIT quotient by a reduced group of a projective space over a regular local ring, and also any toric variety, is globally ultra-F-regular, whence admits Kawamata-Viehweg vanishing.

2. Background

2.1. Vanishing theorems. For $X$ a scheme and $F$ a sheaf on $X$, it is very useful to know whether some sheaf cohomology $H^i(X, F)$ is zero. Any general theorem that forces such vanishing is called a vanishing theorem. Examples that lie at the foundations of modern algebraic geometry are Grothendieck vanishing for $i$ larger than the dimension of $X$ [5], and Serre vanishing for affine schemes [24] and for high powers of ample sheaves [23]. Another important vanishing theorem is the so-called Kodaira vanishing [11], which asserts that if $X$ is a non-singular projective variety over a field of characteristic zero and $P$ is an ample invertible $O_X$-module, then $H^i(X, P^{-1}) = 0$ for all $i < \dim(X)$. Many more improvements exist and play a crucial role in birational geometry. Typically, however, they are only known when $X$ is of finite type over a field. All known algebraic proofs of Kodaira vanishing ([3, 4, 10, 26]) use characteristic $p$ methods to prove the result in characteristic zero (interestingly enough, Kodaira vanishing fails in positive characteristic!). The most recent of these use tight closure, which is a theory developed by Hochster and Huneke [7] to formalize many arguments using the Frobenius map. In [27], Smith proves Kawamata-Viehweg vanishing (that is to say, the analogue of Conjecture 2) for globally $F$-regular varieties over a field: in characteristic $p$ the definition is as above, replacing ultra-Frobenius simply by Frobenius; in characteristic zero the definition is by reduction modulo $p$. However, this latter reduction technique forms an obstruction in proving Conjecture 3 in characteristic zero. Using the notion of globally ultra-F-regular instead, I was able to circumvent this problem [21] when $R$ is a field. As a corollary, I obtained a proof of Smith’s conjecture that GIT quotients of Fano varieties admit Kawamata-Viehweg vanishing.

2.2. Non-standard tight closure. As mentioned above, tight closure has proven to be a very powerful tool to encode several characteristic $p$ methods. For instance, Conjectures 2 and 3 are proven by Smith in positive characteristic using tight closure ideas. There is also a tight closure theory in equal characteristic zero [9], which, as it is again given by reduction modulo $p$, has the same deficiency: it does not behave well under group quotients, or more generally, under pure descent. To rectify this, I introduced in [18] an alternative tight closure operation, called non-standard tight closure, for algebras of finite type over an algebraically closed field of characteristic zero, and in [1] for arbitrary Noetherian local rings of equal characteristic zero. The idea behind all these constructions goes back to my earlier work on transferring properties from positive characteristic to zero characteristic.
using first-order definability [13, 14], or using ultraproducts [16, 19]. I also used this technique to obtain transfer between mixed and equal characteristic [15, 17, 22].

Closely related to non-standard tight closure (and conjecturally equal to it) is generic tight closure. Since we will refer to it in Conjecture 5 below, I’ll briefly give the definition. Let \( R_\infty \) be the Lefschetz extension of \( R \) from [1], and let \( R_w \) be rings of positive characteristic whose ultraproduct is \( R_\infty \). An element \( z \in R \) belongs to the generic tight closure of an ideal \( I \subseteq R \), if \( z_w \) lies in the tight closure of \( I_w \), where \( z_w \) and \( I_w \) are chosen in \( R_w \) so that their ultraproduct equals the respective image of \( z \) and \( I \) in \( R_\infty \).

2.3. Current research–PSC-CUNY grant #60097-34. Having overcome the difficulties inherent in the reduction technique, I was able in [20] to reprove Boutot’s theorem [2] which states that quotient singularities (of finite type over a field) are rational. In fact, in [21] I drew the stronger conclusion that they are log-terminal whenever they are \( \mathbb{Q} \)-Gorenstein. In view of the extension of non-standard tight closure to all Noetherian local rings, we conjectured in [1] that any pure subring of an excellent regular local ring has rational singularities, which are log-terminal if the subring is \( \mathbb{Q} \)-Gorenstein. The proof of this Conjecture is the subject of my current PSC-CUNY sponsored research. In fact, to prove this conjecture, I already needed a partial version of Conjecture 1, namely for \( R \)-algebras \( A \) of the form \( R_f \). This part of the project has already been completed successfully, lending additional credibilility to the validity of Conjecture 1.

3. Project design

Recall that \( R \) denotes a Noetherian local ring containing \( \mathbb{Q} \), that \( R_\infty \) is a Lefschetz extension of \( R \) (as in [1]) and that \( X \) is a connected projective scheme over \( \text{Spec}(R) \).

3.1. Relative Lefschetz hulls. I already indicated the strategy for proving Conjecture 1. We start from the map \( \eta_R : R \to R_\infty \) given by [1], and then relativize the construction from [18]. Although not mentioned in the statement of the conjecture, I expect this functor to have additional properties. Observe that if \( I \subseteq A \), then \( \mathcal{D}_R(A)/\eta_R(I)\mathcal{D}_R(A) \) is a Lefschetz \( A/I \)-algebra. Hence, we could define \( \mathcal{D}_R(A/I) \) to be this Lefschetz ring. So, it appears that without loss of generality, we may impose on \( \mathcal{D}_R \) that it commutes with homomorphic images. We would thus have reduced the problem to the case that \( A \) is a polynomial ring \( R[X] \).

3.2. Section rings of globally ultra-F-regular varieties. As to Conjecture 3, the equivalence of (4.1) with (4.3) below is needed for its proof. Without it, we would only be able to show that some GIT quotient is globally ultra-F-regular. The remainder of the argument for Conjecture 3 is a formal consequence of the fact that ultra-F-regularity descends under pure maps and this has been established already as part of my present PSC-CUNY project.

Conjecture 4. The following are equivalent:

(4.1) \( X \) is globally ultra-F-regular;
(4.2) the localization of some section ring of \( X \) at its irrelevant ideal is ultra-F-regular;
(4.3) every section ring of \( X \) is ultra-F-regular.

The analogue of this claim in positive characteristic is true by [27, Theorem 3.10], and so is the geometric form (where \( R \) is a field) by [21]. I expect that the same argument can be ‘relativized’. Granting the validity of this claim, we can also give a formulation of Conjecture 2 which does not rely on the validity of Conjecture 1, by using (4.2) as the
definition of global ultra-F-regularity. However, we will still need Conjecture 1 for the proof, since we need the action of ultra-Frobenius on various cohomology modules.

3.3. Kawamata-Viehweg vanishing. Once we can prove Claim 1 below, (2.1) in Conjecture 2 will follow formally from Serre duality by the same arguments as in [21, 27].

Claim 1. Suppose $X$ is globally ultra-F-regular and let $\mathcal{F}$ be an invertible $\mathcal{O}_X$-module. If for some $i > 0$ and some effective Cartier divisor $D$, all $H^i(X, \mathcal{F}^n(D))$ vanish for $n \gg 0$, then $H^i(X, \mathcal{F})$ vanishes.

To explain the strategy for proving this claim, let me just deal with the case that $D = 0$. Let $S := S(\mathcal{P})$ be a section ring of $X$ which is ultra-F-regular and let $F := H^0(X, \bigoplus_n \mathcal{P}^n \otimes \mathcal{F})$ be the corresponding section module of $\mathcal{F}$. It follows that $F$ is a finitely generated graded $S$-module. Let $\varphi$ be a pure ultra-Frobenius (i.e., we just take $c = 1$ in the definition of ultra-F-regularity; we will need arbitrary $c$ in case $D \neq 0$) and let $\varphi^* F$ be the tensor product $F \otimes_\varphi \mathcal{D}_R(S)$ where we view $\mathcal{D}_R(S)$ as an $S$-algebra via the restriction of $\varphi$ to $S$. The idea now is that $\varphi^* F$ corresponds to an infinite power of $F$, which therefore has zero sheaf cohomology by overspill. Now, the map $F \rightarrow \varphi^* F$ is pure by base change, and remains so after taking cohomology, showing that the sheaf cohomology of $F$ is also zero. Let me discuss the overspill argument in more detail. We can descend the data to the components of $\mathcal{D}_R(S)$ and therefore assume that we are in (sufficiently large) positive characteristic $p$ and $\varphi$ is the $e$-th power of the Frobenius. In that case, $\varphi^* F$ is the section module associated to $\varphi^* F \cong F^e$, where the last isomorphism follows since $F$ is invertible. We then apply the assumption that $\mathcal{F}^n$ has no cohomology for large $n := p^e$. The main obstruction in carrying out this argument is that we do not have a space corresponding to $\mathcal{D}_R(S)$, and therefore have no sheaf cohomology which can be compared to the sheaf cohomology of its components and to the sheaf cohomology of $X$. In [21], this problem was circumvented by using Čech cohomology and working on the level of section modules rather than on their associated sheaves, and the same technique should work in general.

3.4. Globally ultra-F-regular varieties. In this last section, I propose some amendments to (2.2), which of all claims made in this proposal is perhaps the hardest to prove. Recall that $X$ is called Fano if it is smooth and its anti-canonical sheaf $\omega_X^{-1}$ is ample. It follows that the anti-canonical section ring of $S(\omega_X^{-1})$ is Gorenstein with log-terminal singularities. In my current PSC-CUNY project, I postulated that ultra-F-regular local rings are log-terminal (this claim is supported by evidence in positive characteristic [25, 6], and in the geometric case [21]). What we need now is its converse. This holds again in the geometric case, but all known proofs [6, 12, 21, 25] require a strong form of Kodaira vanishing, which is not available in the general case. Note that for Gorenstein local rings, rational singularities are automatically log-terminal. In view of our previous observations, I therefore postulate:

Conjecture 5. For an excellent local Gorenstein ring $R$ containing $\mathbb{Q}$, the following are equivalent:

(5.1) $R$ is ultra-F-regular;
(5.2) $R$ has rational (whence log-terminal) singularities;
(5.3) some parameter ideal (=an ideal generated by a (full) system of parameters) of $R$ is equal to its generic tight closure.

The positive characteristic analogue of this conjecture holds by [8, 25], and so does the geometric case by [20, 21]. Implication (5.1) $\Rightarrow$ (5.3) follows from the comparison of non-standard tight closure and generic tight closure in [1]. Implication (5.3) $\Rightarrow$ (5.2), without
any Gorenstein assumption, is under investigation in my current PSC-CUNY research. Of all implications, \((5.2) \Rightarrow (5.1)\) will be the hardest to prove.

REFERENCES