On the vanishing of Tor of the absolute integral closure

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Abstract

Let $R$ be an excellent local domain of positive characteristic with residue field $k$ and let $R^+$ be its absolute integral closure. If $\text{Tor}_1^R(R^+, k)$ vanishes, then $R$ is weakly F-regular. If $R$ has at most an isolated singularity or has dimension at most two, then $R$ is regular.

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1 Introduction

Recall that the absolute integral closure $A^+$ is defined for an arbitrary domain $A$ as the integral closure of $A$ inside an algebraic closure of the field of fractions of $A$. A key property of the absolute integral closure was discovered by Hochster and Huneke (1992): for $R$ an excellent local domain of positive characteristic, $R^+$ is a balanced big Cohen-Macaulay algebra, that is to say, any system of parameters on $R$ is an $R^+$-regular sequence. It is well-known that this implies that an excellent local domain $R$ of positive characteristic is regular if, and only if, $R \to R^+$ is flat. Indeed, the direct implication follows since $R^+$ is a balanced big Cohen-Macaulay algebra of finite projective dimension (use for instance (Schoutens, 2003b, Theorem IV.1)) and the converse follows since $R \to R^+$ and $R^{1/p} \to R^+$ are isomorphic whence both faithfully flat, implying that $R \to R^{1/p}$ is flat, and therefore,
by Kunz’s Theorem, that $R$ is regular (here $R^{1/p}$ denotes the extension of $R$ obtained by adding all $p$-th roots of element of $R$; for more details see (Huneke, 1996, Theorem 9.1 and Exercise 8.8)).

Huneke (1996, Exercise 8.8) points out that it is not known whether the weaker condition that all Betti numbers of $R^+$ vanish, that is to say, that all $\text{Tor}_n^R(R^+, k)$ vanish for $n \geq 1$, already implies that $R$ is regular. It is not hard to see, using that $R^+$ is a big Cohen-Macaulay algebra, that this is equivalent with requiring that only $\text{Tor}_1^R(R^+, k)$ vanishes. The main result of this paper is then the following positive solution for isolated singularities.

**Theorem 1.1** Let $(R, m)$ be an excellent local domain of positive characteristic with residue field $k$. Suppose $R$ has either an isolated singularity or has dimension at most two. If $\text{Tor}_1^R(R^+, k) = 0$, then $R$ is regular.

For arbitrary domains, we obtain at least the following.

**Theorem 1.2** Let $(R, m)$ be an excellent local domain of positive characteristic with residue field $k$. If $\text{Tor}_1^R(R^+, k) = 0$, then $R$ is weakly F-regular. In particular, $R$ is normal, Cohen-Macaulay, pseudo-rational and any finite extension of $R$ is split (i.e., $R$ is a splinter).

We have some more precise information on the vanishing of certain Tor’s in terms of the singular locus of $R$.

**Theorem 1.3** Let $(R, m)$ be an excellent local domain of positive characteristic and let $\alpha$ be an ideal defining the singular locus of $R$ (e.g., $\alpha$ is the Jacobian of $R$). If $\text{Tor}_1^R(R^+, k) = 0$, where $k$ is the residue field of $R$, then $\text{Tor}_n^R(R^+, M) = 0$ for all $n \geq 1$ and all finitely generated $R$-modules $M$ for which $M/\alpha M$ has finite length.

The key observation in obtaining all these results, is that, in general, the vanishing of $\text{Tor}_1^R(S, k)$ implies that $R \to S$ is cyclically pure (or ideal-pure), meaning that $IS \cap R = I$, for all ideals $I$ of $R$. This is explained in Section 2. To prove Theorem 1.1, we need a result from (Schoutens, 2003b): if the first Betti number of a module over an isolated singularity vanishes, then the module has finite projective dimension. Now, the argument which proofs that $R \to R^+$ is flat when $R$ is regular, yields the same conclusion under the weaker assumption that $R^+$ has finite projective dimension. This proves also the two-dimensional case, since we know already that $R$ is normal.

Balanced big Cohen-Macaulay algebras in characteristic zero exist by the work of Hochster-Huneke, basically by a lifting procedure due to Hochster. However, the balanced big Cohen-Macaulay algebras obtained in (Hochster and Huneke, 1992) are not canonically defined. In (Schoutens, 2003a), I give an alternative but canonical construction $B(R)$ of a balanced big Cohen-Macaulay algebra for a $\mathbb{C}$-affine
local domain \( R \) using ultraproducts and the absolute integral closure in positive characteristic. It follows from the present results that if \( \text{Tor}_1^R(B(R), k) = 0 \), where \( k \) is the residue field of \( R \), then \( R \) is regular provided \( R \) has an isolated singularity or has dimension at most two (moreover, without these additional assumptions, \( R \) has at most rational singularities). This is the more interesting because it is not clear whether in general flatness of \( R \to B(R) \) implies regularity of \( R \). For a further generalization to arbitrary excellent local domains, see the forthcoming Aschenbrenner and Schoutens (2003).

Acknowledgement

The original proof of Theorem 1.2 only established F-rationality; the argument yielding weak F-regularity is due to I. Aberbach, whom I would like to thank for letting me reproduce it here. I also want to thank the anonymous referee, for providing a simplified argument for Theorem 1.3.

2 Vanishing of Betti numbers and cyclic purity

We derive a simple criterion for a local ring homomorphism to be cyclically pure. We start with an easy lemma, the proof of which is included for sake of completeness.

Lemma 2.1 Let \( A \) be a ring, \( \mathfrak{a} \) an ideal in \( A \) and \( M \) and \( N \) two \( A \)-modules. If \( \mathfrak{a}N = 0 \) and \( \text{Tor}_1^A(M, N) = 0 \), then \( \text{Tor}_1^{A/\mathfrak{a}}(M/\mathfrak{a}M, N) = 0 \).

PROOF. One can derive this by aid of spectral sequences, but the following argument is more direct. Put \( \bar{A} := A/\mathfrak{a} \). Since \( N \) is an \( \bar{A} \)-module, we can choose an exact sequence of \( \bar{A} \)-modules

\[
0 \to \bar{H} \to \bar{F} \to N \to 0
\]

with \( \bar{F} \) a free \( \bar{A} \)-module. Tensoring with the \( \bar{A} \)-module \( \bar{M} := M/\mathfrak{a}M \), we get an exact sequence

\[
0 \to \text{Tor}_1^A(\bar{M}, N) \to \bar{M} \otimes_{\bar{A}} \bar{H} \to \bar{M} \otimes_{\bar{A}} \bar{F}.
\]

Since the last two modules are equal to \( M \otimes_A \bar{H} \) and \( M \otimes_A \bar{F} \) respectively and since \( \text{Tor}_1^A(M, N) = 0 \), the last morphism in this exact sequence is injective. Therefore, \( \text{Tor}_1^{A/\mathfrak{a}}(M/\mathfrak{a}M, N) = 0 \), as required. \( \square \)

Theorem 2.2 Let \((R, \mathfrak{m})\) be a Noetherian local ring with residue field \( k \) and let \( S \) be an arbitrary \( R \)-algebra. If \( \text{Tor}_1^R(S, k) = 0 \) and \( \mathfrak{m}S \neq S \), then \( R \to S \) is...
cyclically pure. Moreover, if \( n \) is an \( m \)-primary ideal, then

\[(n :_R I)S = (nS :_S IS)\]

for every ideal \( I \) in \( R \).

**PROOF.** Since \( \text{Tor}_1^R(S, k) \) vanishes, so does \( \text{Tor}_1^{R/n}(S/nS, k) \) by Lemma 2.1, for every \( m \)-primary ideal \( n \). By the Local Flatness Criterion (see (Matsumura, 1986, Theorem 22.3)) applied to the Artinian local ring \( R/n \), the base change \( R/n \to S/nS \) is flat, whence faithfully flat, since \( mS \neq S \). In particular, this base change is injective, showing that \( nS \cap R = n \). Since every ideal is the intersection of \( m \)-primary ideals by Krull’s Intersection Theorem, the assertion follows.

The final assertion follows from the flatness of \( R/n \to S/nS \) (use for instance (Matsumura, 1986, Theorem 7.4)). \( \square \)

**Remark 2.3** Note that with notation from the Theorem, we have that the induced map of affine schemes \( \text{Spec } S \to \text{Spec } R \) is surjective, since the fiber rings \( S_p/pS_p \) are non-zero.

The following lemma shows that for a local Cohen-Macaulay ring, the vanishing of some Betti number of a big Cohen-Macaulay algebra is equivalent with the vanishing of all of its Betti numbers.

**Lemma 2.4** If \( (R, m) \) is a local Cohen-Macaulay ring with residue field \( k \) and if \( S \) is a big Cohen-Macaulay \( R \)-algebra, such that \( \text{Tor}_j^R(S, k) = 0 \) for some \( j \geq 1 \), then \( \text{Tor}_n^R(S, k) = 0 \), for all \( n \geq 1 \).

**PROOF.** Let \( x \) be a maximal \( R \)-regular sequence which is also \( S \)-regular. Put \( I := xR \). Since \( \text{Tor}_j^R(S, k) \) vanishes, so does \( \text{Tor}_j^{R/I}(S/IS, k) \) by (Matsumura, 1986, Lemma 2, p.140), so that \( S/IS \) is flat over \( R/I \) by the Local Flatness Criterion. However, since the finitistic weak dimension is at most the dimension of a ring by (Auslander and Buchsbaum, 1958, Theorem 2.4), it follows that \( S/IS \) is flat over \( R/I \). Therefore, \( 0 = \text{Tor}_n^{R/I}(S/IS, k) = \text{Tor}_n^R(S, k) \), for all \( n \geq 1 \). \( \square \)

Therefore, below, we may replace everywhere the condition that \( \text{Tor}_1^R(S, k) = 0 \) by the weaker condition that some \( \text{Tor}_j^R(S, k) = 0 \), provided we also assume that \( R \) is Cohen-Macaulay. In fact, if \( j \) is either 1 or 2, we do not need to assume that \( R \) is Cohen-Macaulay, since this then holds automatically.
Proposition 2.5 If \((R, \mathfrak{m})\) is a Noetherian local ring with residue field \(k\) and if \(S\) is a big Cohen-Macaulay \(R\)-algebra, such that either \(\text{Tor}_1^R(S, k)\) or \(\text{Tor}_2^R(S, k)\) vanishes, then \(R\) is Cohen-Macaulay.

**PROOF.** I claim that \(\mathcal{I}S \cap R = I\), for some parameter ideal \(I\) of \(R\). By a standard argument, it then follows that \(\mathcal{R}\) is Cohen-Macaulay (see for instance the argument in (Schoutens, 2003a, Theorem 4.2)). For \(j = 1\), we can use Lemma 2.1 to conclude that \(\text{Tor}_1^{\mathcal{R}/I}(\mathcal{S}/\mathcal{I}\mathcal{S}, k) = 0\), so that by the argument above, \(\mathcal{R}/I \to \mathcal{S}/\mathcal{I}\mathcal{S}\) is faithfully flat. For \(j = 2\), we reason as follows. Let

\[
0 \to M \to F \to S \to 0
\]

be a short exact sequence with \(F\) free. It follows that \(\text{Tor}_1^R(M, k)\) is equal to \(\text{Tor}_2^R(S, k)\), whence is zero. Therefore, by the same argument as before, \(M/IM\) is flat over \(\mathcal{R}/I\). On the other hand, since we may choose \(I\) so that it is generated by an \(S\)-regular sequence, we get that \(\text{Tor}_1^R(S/\mathcal{R}/I) = 0\) (indeed, the canonical morphism \(I \otimes M \to IM\) is easily seen to be injective). Hence we get an exact sequence

\[
0 \to M/IM \to F/IF \to S/\mathcal{I}\mathcal{S} \to 0
\]

showing that \(S/\mathcal{I}\mathcal{S}\) has finite flat dimension, whence is flat, since \(\mathcal{R}/I\) is Artinian. \(\square\)

Is there a counterexample in which some \(\text{Tor}_j^R(S, k)\) vanishes for some big Cohen-Macaulay algebra \(S\) and some \(j > 2\), without \(\mathcal{R}\) being Cohen-Macaulay?

3 Proofs

Recall that an excellent local ring of positive characteristic is called \(F\)-rational, if some ideal generated by a system of parameters is tightly closed, and weakly \(F\)-regular, if every ideal is tightly closed. It is well-known that for excellent local rings, weakly \(F\)-regular implies splinter, and \(F\)-rational implies Cohen-Macaulay and normal ((Huneke, 1996, Theorem 4.2)). By (Smith, 1997, Theorem 3.1), an \(F\)-rational ring is pseudo-rational.

**Proof of Theorem 1.2**

Suppose \(\mathcal{R}\) is as in the statement of the theorem, so that in particular \(\text{Tor}_1^R(R^+, k)\) vanishes. By Theorem 2.2, the embedding \(R \to R^+\) is cyclically pure. In order to show that \(\mathcal{R}\) is weakly \(F\)-regular, it suffices to show by (Huneke, 1996, Theorem 1.5) that every \(\mathfrak{m}\)-primary ideal is tightly closed. Towards a contradiction, suppose
that \( n \) is an \( m \)-primary ideal which is not tightly closed. Therefore, we can find a \( u \) in the tight closure of \( n \) such that \((n :_R u) = m\). By Theorem 2.2, we have
\[
(nR^+ :_{R^+} u) = mR^+.
\] (1)

By definition, there is a \( c \in R \) not contained in any minimal prime of \( R \) such that \( cu^q \in n[q] \), for all powers \( q = p^e \) (as usual, \( I[q] \) denotes the ideal generated by the \( q \)-th powers of elements in an ideal \( I \)). Since therefore \( c^{1/q}u \in nR^+ \), we get from (1) that \( c^{1/q} \in mR^+ \) whence \( c \in m^qR^+ \). By cyclical purity, \( c \in m^q \) for all \( q \), contradiction.

In particular, \( R \) is F-rational whence pseudo-rational, normal and Cohen-Macaulay (in fact, \( R \) is Cohen-Macaulay, by Proposition 2.5, and normal, by the cyclic purity of \( R \to R^+ \)). Since \( R \) is normal, it follows from (Hochster, 1977) that \( R \to R^+ \) is pure. Let us give a direct argument for showing that \( R \) is a splinter. Let \( R \subset S \) be a finite extension. In order to show that this is split, we may factor out a minimal prime of \( S \) and hence assume that \( S \) is a domain. So \( R \subset S \) extends to the pure map \( R \to R^+ \) and hence is itself pure. Since a pure map with finitely generated cokernel is split ((Matsumura, 1986, Theorem 7.14)), we showed that any finite extension splits.  

**Proof of Theorem 1.1**

The vanishing of \( \text{Tor}^R_1(R^+, k) \) implies that \( R \) is Cohen-Macaulay by Theorem 1.2. Since \( R^+ \) is a balanced big Cohen-Macaulay algebra and since \( R \) has an isolated singularity, we get from (Schoutens, 2003b, Theorem IV.1) that \( R \to R^+ \) is flat. As already observed, this implies that \( R \) is regular. If \( R \) has dimension at most 2, then by Theorem 1.2, it is normal and therefore has an isolated singularity, so that the previous argument applies.

Recall that by the argument at the end of the previous section, the vanishing of a single \( \text{Tor}^R_1(R^+, k) \) implies already that \( R \) is regular, if apart from being an isolated singularity, we also assume that \( R \) is Cohen-Macaulay, when \( j \geq 3 \). In order to derive a regularity criterion from Theorem 1.1, we need a lemma on flatness over Artinian local Gorenstein rings of embedding dimension one.

**Lemma 3.1** Let \((A, m)\) be an Artinian local ring of embedding dimension one and let \( M \) be an arbitrary \( A \)-module. Then \( M \) is \( A \)-flat if, and only if, \( \text{Ann}_A(I) = mM \), where \( I \) denotes the socle of \( A \), that is to say, \( I = \text{Ann}_A(m) \).

**Proof.** By assumption \( m = xA \), for some \( x \in A \). It follows that the socle \( I \) of \( A \) is equal to \( x^{e-1}A \), where \( e \) is the smallest integer for which \( x^e = 0 \). I claim that \( \text{Ann}_M(x^{e-1}) = x^iM \), for all \( i \). We will induct on \( i \), where the case \( i = 1 \) is just our assumption. For \( i > 1 \), let \( \mu \in M \) be such that \( x^{e-1}\mu = 0 \). Therefore,
\[ xe^{-i+1} \mu = 0 \], so that by our induction hypothesis, \( \mu \in x^{-1}M \), say, \( \mu = x^{-1} \nu \).

Since \( 0 = xe^{-i} \mu = xe^{-1} \nu \), we get \( \nu \in xM \) whence \( \mu \in x^iM \), as required.

Flatness now follows by the Local Flatness Criterion (Matsumura, 1986, Theorem 22.3). Indeed, it suffices to show that \( A/xA \to M/xM \) is flat and \( xA \otimes M \cong xM \).

The first assertion is immediate since \( A/xA \) is a field. For the second assertion, observe that \( xA \cong A/x^{-1}A \) and by what we just proved \( xM \cong M/\text{Ann}_M(x) \cong M/x^{-1}M \). It follows that \( xA \otimes M \) is isomorphic with \( xM \), as required.

\[ \text{Corollary 3.2} \]

Let \((R, m)\) be a \( d \)-dimensional excellent local Cohen-Macaulay domain of positive characteristic. Suppose that there exists an ideal \( I \) in \( R \) generated by a regular sequence such that \( m/I \) is a cyclic module. Suppose also that \( R \) has either an isolated singularity or that \( d \leq 2 \). If for each finite extension domain \( R \subset S \), we can find a finite extension \( S \subset T \), such that

\[ (IS :_S (I :_R m)_S) \subset mT, \]

then \( R \) is regular.

**Proof.** Let \((x_1, \ldots, x_i)\) be the regular sequence generating \( I \) and write \( m = I + xR \). If \( i < d \) then necessary \( i = d - 1 \) and \( m \) is generated by \( d \) elements, so \( R \) is regular. Hence assume \( i = d \), that is to say, \( I \) is \( m \)-primary. It follows that \( \overline{R} := R/I \) is an Artinian local ring with maximal ideal \( x\overline{R} \). Let \( e \) be the smallest integer for which \( xe \in I \). Hence the socle of \( \overline{R} \) is \( x^{e-1}\overline{R} \). Let \( \overline{R}^+ := R^+/IR^+ \). I claim that

\[ \text{Ann}_{\overline{R}^+}(x^{e-1}) = x\overline{R}^+. \]

Assuming the claim, Lemma 3.1 yields that \( \overline{R}^+ \) is \( \overline{R} \)-flat. Therefore, if \( k \) is the residue field of \( R \), then \( \text{Tor}_1^R(\overline{R}^+, k) = 0 \). But \((x_1, \ldots, x_d)\) is both \( R \)-regular and \( R^+ \)-regular, so that \( \text{Tor}_1^R(R^+, k) = 0 \). Regularity of \( R \) then follows from Theorem 1.1.

To prove the claim, one inclusion is clear, so assume that \( a \in R^+ \) is such that \( ax^{e-1} \in IR^+ \). Choose a finite extension \( R \subset S \subset R^+ \) containing \( a \) and such that we already have a relation \( ax^{e-1} \in IS \). By assumption, we can find a finite extension \( T \) of \( S \), such that \((IS : xe^{-1}) \subset mT \). Hence \( a \in mT \). Since \( T \) maps to \( R^+ \), we get \( a \in mR^+ \), and hence \( a \in x\overline{R}^+ \), as we wanted to show. \( \square \)

The condition that \( m \) is cyclic modulo a regular sequence is in this case equivalent with \( R \) being Cohen-Macaulay with regularity defect at most one (recall that the regularity defect of \( R \) is by definition the difference between its embedding dimension and its Krull dimension). If \( R \) is regular, then (2) is true for any \( m \)-primary ideal \( I \) of \( R \) (use the fact that \( R \to R^+ \) is flat).
Proof of Theorem 1.3

Let \((R, \mathfrak{m})\) be as in the statement of Theorem 1.3. In particular, \(R\) is Cohen-Macaulay by Theorem 1.2. Let \(M\) be a finitely generated \(R\)-module such that \(M/\mathfrak{a}M\) has finite length. Let \(I\) be the annihilator of \(M\). By Nakayama’s Lemma, \(M/\mathfrak{a}M\) having finite length implies that \(I + \mathfrak{a}\) is \(\mathfrak{m}\)-primary. We will induct on the dimension \(e\) of \(M\). If \(e = 0\), so that \(M\) has finite length, the vanishing of \(\text{Tor}_n^R(R^+, M)\) follows from Lemma 2.4 and a well-known inductive argument on the length of \(M\) (see for instance (Schoutens, 2003b, Corollary II.6)). Hence assume \(e > 0\) and let \(H\) be the largest submodule of finite length in \(M\). The \(\text{Tor}\) long exact sequence obtained from

\[
0 \to H \to M \to M/H \to 0
\]

shows that it suffices to prove the result for \(M/H\) instead of \(M\). Therefore, after modding out \(H\), me may assume that \(M\) has positive depth. By prime avoidance and since \(I + \mathfrak{a}\) is \(\mathfrak{m}\)-primary, we can find an \(M\)-regular element \(x \in \mathfrak{a}\). The short exact sequence

\[
0 \to M \xrightarrow{x} M \to M/xM \to 0
\]

gives rise to a long exact sequence

\[
\text{Tor}_{n+1}^R(R^+, M/xM) \to \text{Tor}_n^R(R^+, M) \xrightarrow{x} \text{Tor}_n^R(R^+, M),
\]

for all \(n \geq 1\). Since the left most module is zero by induction on \(e\), multiplication with \(x\) on \(\text{Tor}_n^R(R^+, M)\) is injective, for all \(n \geq 1\). In particular, we have for each \(n\) an embedding

\[
\text{Tor}_n^R(R^+, M) \subset (\text{Tor}_n^R(R^+, M))_x = \text{Tor}_n^R((R^+_x, M_x)). \tag{3}
\]

Since \(x \in \mathfrak{a}\), the localization \(R_x\) is regular. Therefore, \(R_x \to (R^+_x)\) is flat. An easy calculation shows that \((R^+_x) = (R^+_x)\) (see (Hochster and Huneke, 1992, Lemma 6.5)). In particular, \(\text{Tor}_n^R((R^+_x, M_x)) = 0\), and hence \(\text{Tor}_n^R(R^+, M) = 0\) by (3). \(\square\)

If \(R\) has dimension three, then \(\text{Tor}_n^R(R^+, R/\mathfrak{p})\) vanishes for every \(n \geq 1\) and every prime ideal \(\mathfrak{p}\) of \(R\) not in the singular locus of \(R\), since \(R\) is normal by Theorem 1.2 and hence \(\mathfrak{a}\) has height at least two. On the other hand, we have the following non-vanishing result.

Corollary 3.3 Let \((R, \mathfrak{m})\) be an excellent local domain of positive characteristic. If \(\mathfrak{p}\) is a prime ideal defining an irreducible component of the singular locus of \(R\), then \(\text{Tor}_1^R(R^+, R/\mathfrak{p})\) is non-zero.

\textbf{PROOF.} Assume \(\text{Tor}_1^R(R^+, R/\mathfrak{p})\) vanishes. Hence so does \(\text{Tor}_1^R((R^+)_\mathfrak{p}, k(\mathfrak{p}))\), where \(k(\mathfrak{p})\) is the residue field of \(\mathfrak{p}\). Since \((R^+)_\mathfrak{p}\) is equal to \((R^+)_\mathfrak{p}\) by (Hochster and Huneke, 1992, Lemma 6.5), and since \(R^+\) has an isolated singularity, it follows from Theorem 1.1 that \(R^+\) is regular, contradicting the choice of \(\mathfrak{p}\). \(\square\)
In view of Lemma 2.4 we can generalize this even further: if $R$ is Cohen-Macaulay, then each $\text{Tor}_n^R(R^+, R/\mathfrak{p})$ is non-zero, for $n \geq 1$ and for $\mathfrak{p}$ defining an irreducible component of the singular locus of $R$.

References


