An Efficient Solution for Cauchy-Like Systems of Linear Equations

Z. Chen†
Mathematics Department
New York City College of Technology
300 Jay Street, Brooklyn, NY 11201, U.S.A.
zchen@citytech.cuny.edu

V. Pan‡
Mathematics and Computer Science Department
Lehman College, CUNY, Bronx, NY 10468, U.S.A.
vpan@alphalehman.cuny.edu

(Received and accepted September 2003)

Abstract—Using the displacement approach and transformation of vectors, the complexity of solving strongly nonsingular Cauchy-like systems of linear equations $C\vec{x} = \vec{b}$ will be improved off a factor of $\log n$ from the known algorithms of the recursive triangular factorization, where $F$ is a field and $\vec{b} \in F^{n \times 1}$ is a vector and $C \in F^{n \times n}$ is a strongly nonsingular Cauchy-like matrix. The efficient algorithms were presented by Morf, Bitmead and Anderson to solve strongly nonsingular Toeplitz-like equations of linear systems by using the recursive triangular factorization in 1980. Recently, this approach of the recursive triangular factorization was extended to solve Cauchy-like systems with the complexity of $O(n \log^2 n)$ operations by Pan et al. This is the best known complexity bound by using the direct approach of recursive triangular factorization in Cauchy-like cases. However, these algorithms are still slower than the well-known algorithms with the asymptotic bound of $O(n \log^2 n)$ operations, which have been proposed by the means of reducing Cauchy-like matrices into Toeplitz-like matrices. In our present paper, the algorithms of the recursive triangular factorization are modified so that the complexity bound of the direct recursive approach can be decreased to $O(n \log^2 n)$ operations. This matches the asymptotic bound without transforming to Toeplitz-like matrices. Our improvement of the direct recursive approach is by a factor of $\log n$ due to changing the original vectors expressed in the given Cauchy-like matrix into the special vectors, where the entries are unit roots. The applications of structured matrices include Nevanlinna-Pick tangential interpolation problems. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Cauchy-like matrices, Recursive factorization, Displacement and scaling generators, Fast algorithms, Linear systems equations.

1. INTRODUCTION

Solving systems of linear equations is extremely important in computer science, applied mathematics, and engineering. New techniques are always needed to develop the improvement of the
efficiency of algorithms. One way to speed up computations is to apply displacement approach. Originally, Kailath, Kung and Morf [1] have successfully exploited the displacement structure in fast computations. The displacement structure has been investigated thoroughly by many active researchers. It has been implemented in the efficient computations of many structured matrices such as Toeplitz-like. Computations with displacement structured matrices such as Cauchy-like, Vandermonde-like, and Toeplitz-like have a variety of applications in many areas such as conformal mapping, tangential Nevanlinna-Pick interpolation, solution of integral equations, rational interpolation, polynomial interpolation, and evaluation (cf. [2–12]).

The running time and memory spaces can be saved dramatically in solving Cauchy-like systems of linear equations by exploiting the displacement structure. Instead of working on the entries of a matrix itself, we use its generators in our computations. Many matrices can be reconstructed through their generators with the displacement operators [13–16]. The displacement representation provides many opportunities to accelerate our computations.

Let $\mathbf{F}$ be a field and $\mathbf{v}$ denotes a vector. We denote $A_{M\mathbf{v}}(n)$ as the complexity of the product of a matrix $M \in \mathbf{F}^{n \times n}$ by a vector $\mathbf{v} \in \mathbf{F}^{n \times 1}$. Moreover, let us work over a field $\mathbf{F}$ which supports fast Fourier transforms of length $n$.

To solve $n \times n$ strongly nonsingular Toeplitz-like equations of linear systems $T\mathbf{z} = \mathbf{v}$, where $T \in \mathbf{F}^{n \times n}$ is a Toeplitz-like matrix, Morf, Bitmead and Anderson (see, e.g., [17,18]) applied the displacement structure in the recursive triangular factorization to design fast algorithms with complexity of $O(n \log^2 n)$ arithmetic operations (hereafter, we refer to arithmetic operations as $ao$). These algorithms are faster than the Gaussian elimination algorithms, which run $O(n^3)$ times.

These efficient techniques in the recursive triangular factorization have been extended recently to $n \times n$ Cauchy-like matrices by Pan and Zheng [19] with complexity of

$$A_{C_{\mathbf{v}}^{-1} \mathbf{v}}(n) = O(A_{C\mathbf{v}}(n) \log n) = O(n \log^3 n),$$

(1.1)

where $C_{\mathbf{v}}^{-1}$ denotes as the inverse of a Cauchy-like matrix and $A_{C\mathbf{v}}(n) = O(n \log^2 n)$ denotes as the complexity of multiplication of an $n \times n$ Cauchy matrix by a vector. $O(n \log^3 n)$ is the best known bound by using direct recursive approach to solve Cauchy-like systems. However, these algorithms are still slower than the well-known algorithms with the asymptotic bound of $O(n \log^2 n)$ operations, which have been proposed by the means of reducing Cauchy-like matrices into Toeplitz-like matrices. Due to the difference structure between a Cauchy-like matrix and a Toeplitz-like matrix, the transition from a Cauchy-like matrix to a Toeplitz-like matrix is not extending the recursive triangular factorization from a Toeplitz-like matrix to a Cauchy-like. Moreover, the transition using the Vandermonde matrices may produce errors in numerical computations because of the ill-condition of the Vandermonde matrices [20] (except the Fourier matrices). In our present paper, we will modify the recursive triangular factorization so that the complexity bound of the direct recursive approach can be decreased to $O(n \log^2 n)$ operations. This matches the asymptotic bound without transforming to Toeplitz-like matrices.

We propose the new Cauchy-like extension of the recursive triangular factorization, which is inferior to the best known direct recursive approach.

By observing from (1.1), the complexity of the recursive triangular factorization depends or the complexity of the products of a Cauchy matrix by a vector. Let $$(1/(e_i - f_j))_{i,j=0}^{n-1}$$ be a Cauchy matrix. It is well known that a Cauchy matrix times a vector can be computed in $O(n \log^2 n)$ $ao$ (see [21]). If the entries of the vectors $\mathbf{e} = e_0, \ldots, e_{n-1}$ and $\mathbf{f} = f_0, \ldots, f_{n-1}$ are $n^{th}$ roots of unity it costs only $O(n \log n)$ $ao$ to compute their product. This property motivates us to change the original vectors which are expressed in a generalized Cauchy-like matrix into the special vectors where the entries of the vectors are unit roots.

Due to the fact that the new matrix preserves the same Cauchy-like structure, the computation of finding the inverse of the new matrix are faster than the computations of finding the invers
An Efficient Solution

of the original given matrix by using the recursive triangular factorization. This new proof of modifying the recursive triangular factorization can be used to solve strongly nonsinglu

Cauchy-like systems of linear equations in the complexity of

\[ A_{C,\tilde{v}}(n) = O(A_{C,v}(n) \log n) = O(n \log^2 n), \]

where \( A_{C,\tilde{v}}(n) = O(n \log n) \) denotes the complexity of multiplication of an \( n \times n \) special Cauchy matrix by a vector. (We refer to a Cauchy matrix \( (1/(e_i - f_j))_{i,j=0}^{n-1} \) as a special Cauchy mat if the entries of the vectors \( \tilde{e}, \tilde{f} \) are roots of unity.) Our improvement for the direct recurs approach is by a factor of \( \log n \) versus running time of the best known algorithms for the direct approach. In fact, Heinig [15] has proposed to change the vectors into unit roots applying the Vandermonde linear solver. Computations with the Vandermonde matrices create the numerical instabilities due to the ill-conditions of the Vandermonde matrices (cf. [20]). Our algorithms avoid the computations with a Vandermonde matrix. The complexity bound our algorithms reaches to the asymptotic bound.

We organize our paper as follows. In Section 2, we present our new representation of the inverse of Cauchy-like matrices. In Section 3, we recall the basic algorithms of the recursive triangular factorization. In our final section, we modify the recursive triangular factorization to solve strongly Cauchy-like linear systems of equations. Let us work over a field \( \mathbb{F} \) which suppo FFT and follow some lines of [19] and [22].

2. NEW EXPRESSION OF THE INVERSE OF A CAUCHY-LIKE MATRIX

It is easy for us to work with the Cauchy-like matrices with special vectors because of lower computational complexity. We will show that the vectors in Cauchy-like matrices can reduced to special vectors. The new representation of Cauchy-like matrices will be given in this section. Let us review some well-known definitions.

**Definition 2.1.** Let \( \alpha = e^{-2\pi i \sqrt{-1}/n} \), \( \beta = e^{2\pi i \sqrt{-1}/n} \). The elements \( \alpha_i = \alpha^i \), \( \beta_j = \beta^j \), \( i, j = 0, \ldots, n-1 \), are \( n \)-roots of 1 and \(-1\), respectively.

**Definition 2.2.** (Cf. [23].) We denote \( \mathbb{F} \) as a field and write \( W^T \) as the transpose of a matrix or a vector \( W \). The inverse of a matrix \( M \) is denoted as \( M^{-1} \). Let \( D(\tilde{z}) \in \mathbb{F}^{n \times n} \) be a diagonal matrix with the diagonal entries \( \tilde{z} = x_0, \ldots, x_{n-1} \). Let \( c(\tilde{z}, \tilde{v}) = (1/(t_i - v_j))_{i,j=0}^{n-1} \in \mathbb{F}^{n \times n} \) denote a Cauchy matrix with the distinct values of \( t_i \) and \( v_j \) over a field \( \mathbb{F} \).

Let us recall some well-known results as follows (cf. [21]).

**Lemma 2.1.** Let \( \tilde{u}, \tilde{v}, \) and \( \tilde{w} \) be triple vectors, where the elements \( u_i \in \tilde{u} \) and \( w_j \in \tilde{w} \) are equal in a field \( \mathbb{F} \), for \( i, j = 0, 1, \ldots, n-1 \). Let \( C(\tilde{u}, \tilde{w}) \) be an \( n \times n \) Cauchy matrix. Then computations of the product \( \tilde{z} = C(\tilde{u}, \tilde{w})\tilde{v} \) cost

\[ A_{C_0}(n) = O(n \log^2 n) \]  

(2)

If the elements of the vectors \( \tilde{u} \) and \( \tilde{w} \) are roots of unity, then the complexity can be reduced to

\[ A_{C,\tilde{v}}(n) = O(n \log n) \]  

(2)

By following [24], and [16], we have the following definitions.

**Definition 2.3.** A linear operator \( \Delta^{[E,F]}(\cdot) : \mathbb{F}^{n \times n} \rightarrow \mathbb{F}^{n \times n} \) is defined to map each matrix \( C \in \mathbb{F}^{n \times n} \) to its displacement \( EC - CF \), where \( E \in \mathbb{F}^{n \times n} \) and \( F \in \mathbb{F}^{n \times n} \) are given matrices. The operator \( \Delta^{[E,F]}(\cdot) \) is called the symmetric Sylvester operator. The rank of the image, \( \tau \)
rank$(EC - CF)$, is called $[E, F]$-displacement rank of the matrix $C$ for $r \ll n$. Let $E = D(\tilde{a})$ \( F = D(\tilde{b}) \in \mathbb{F}^{n \times n} \), and $G_r, H_r \in \mathbb{F}^{n \times r}$ be matrices. A matrix $C \in \mathbb{F}^{n \times n}$ is called Cauchy-like matrix if it satisfies

\[
\Delta^{[D(\tilde{a}), D(\tilde{b})]}(C) = D(\tilde{a}) \mathcal{C} - CD(\tilde{b}) = G_r H_r^T,
\]

where the pair of the matrices $G_r, H_r \in \mathbb{F}^{n \times r}$ is called the $\Delta^{[D(\tilde{a}), D(\tilde{b})]}$-generators of $C$ with length $r$. We denote this matrix as $C_{G_r, H_r}(\tilde{a}, \tilde{b})$ or $C_{\text{like}}$.

It is easy to realize the following well-known relationships of the Cauchy and Cauchy-like matrices (cf. [16]). We observe that $\Delta^{[D(\tilde{a}), D(\tilde{b})]}(C(\tilde{s}, \tilde{u})) = D(\tilde{s})C(\tilde{s}, \tilde{u}) - C(\tilde{s}, \tilde{u})D(\tilde{u}) = \mathbb{I}$ where $\mathbb{I} = (1, \ldots, 1)$. Thus, it follows.

**Lemma 2.2.** Let $c(\tilde{s}, \tilde{u})$ be an $n \times n$ Cauchy matrix with the vectors $\tilde{s}$ and $\tilde{u}$ having $2n$ distinct values in $\mathbb{F}$, then it has the $[D(\tilde{s}), D(\tilde{u})]$-displacement rank equal to 1.

Let the Sylvester operator, $\Delta^{[E, F]}(\cdot) : \mathbb{F}^{n \times n} \rightarrow \mathbb{F}^{n \times n}$, act on the linear space $\mathbb{F}^{n \times n}$ with the empty kernel. A Cauchy-like matrix can be completely reconstructed as follows.

**Proposition 2.1.** (See [25].) Let $C_{G_r, H_r}(\tilde{u}, \tilde{w}) \in \mathbb{F}^{n \times n}$ be a Cauchy-like matrix associated with the Sylvester operator $\Delta^{[D(\tilde{a}), D(\tilde{b})]}(\cdot) : \mathbb{F}^{n \times n} \rightarrow \mathbb{F}^{n \times n}$ and generators $G_k = [g_1, \ldots, g_k] H_k = [h_1, \ldots, h_k]$ as defined in Definition 2.3. Then the Cauchy-like matrix $C_{G_r, H_r}(\tilde{u}, \tilde{w})$ can be expressed as

\[
C_{G_r, H_r}(\tilde{u}, \tilde{w}) = \sum_{i=1}^{k} D(\tilde{g}_i) C(\tilde{v}_i, \tilde{w}) D(\tilde{h}_i) = \left( \begin{array}{cc}
\tilde{g}_i^T & \\
\tilde{v}_i - \tilde{w}_j
\end{array} \right)_{i,j=0}^{n-1},
\]

where $C(\tilde{v}, \tilde{w})$ is a Cauchy matrix and $k \ll n$.

By Combining Lemma 2.1 and Proposition 2.1, we obtain the following results.

**Corollary 2.1.** Let $\tilde{v} \in \mathbb{F}^{n \times 1}$ be a vector and $C_{G_r, H_r}(\tilde{d}, \tilde{p}) \in \mathbb{F}^{n \times n}$ be a Cauchy-like matrix. Then the product $\tilde{y} = C_{G_r, H_r}(\tilde{d}, \tilde{p})\tilde{u}$ can be computed in $\mathbb{A}_{\text{like}, e}(n) = O(rn \log^2 n)$.

Furthermore, if the elements of the vectors $\tilde{d}$ and $\tilde{p}$ are roots of unity, then the complexity can be accelerated to $\mathbb{A}_{\text{like}, e}(n) = O(rn \log n)$.

**Theorem 2.1.** Let $\Delta^{[E, F]}(\cdot) : \mathbb{F}^{n \times n} \rightarrow \mathbb{F}^{n \times n}$ be the Sylvester operator. Let $C_{G_r, H_r}(\tilde{z}, \tilde{y})$ and $C_{G_r H_r}(\tilde{y}, \tilde{w})$ denote the Cauchy-like matrices according to Definition 2.3 with $\Delta^{[D(\tilde{a}), D(\tilde{b})]}(C_{G_r H_r}(\tilde{z}, \tilde{y})) = G_r^a H_r^a^T$, $\Delta^{[D(\tilde{a}), D(\tilde{b})]}(C_{G_r H_r}(\tilde{z}, \tilde{y})) = G_r^b H_r^b^T$, and $\Delta^{[D(\tilde{s}), D(\tilde{u})]}(C_{G_r H_r}(\tilde{y}, \tilde{u})) = G_r^d H_r^d^T$, respectively, where the matrices $G_r^a, H_r^a \in \mathbb{F}^{n \times a}$, $G_r^b, H_r^b \in \mathbb{F}^{n \times b}$, $G_r^d, H_r^d \in \mathbb{F}^{n \times d}$ are generators, respectively, for $a, b, d \ll n$, and all values of $\tilde{x}_i, \tilde{y}_j, \tilde{z}_k, u_m$ are all distinct in a field $\mathbb{F}$. Then the product matrix $C_{G_r H_r}(\tilde{z}, \tilde{y}) = C_{G_r H_r}(\tilde{z}, \tilde{y}) C_{G_r H_r}(\tilde{z}, \tilde{y}) C_{G_r H_r}(\tilde{y}, \tilde{w})$ is a Cauchy-like matrix with the $[D(\tilde{z}), D(\tilde{y})]$-displacement rank $s$ ($s = a + b + d$), such that $\Delta^{[D(\tilde{z}), D(\tilde{y})]}(C_{G_r H_r}(\tilde{z}, \tilde{y})) = G_r^e H_r^e^T$, where the pair of matrices $G_r^e, H_r^e$ are $[D(\tilde{z}), D(\tilde{y})]$-generators of the size $n \times s$,

\[
G_r^e = \begin{bmatrix}
G_r^a, C_{G_r H_r}(\tilde{z}, \tilde{y}) G_r^b, C_{G_r H_r}(\tilde{z}, \tilde{y}) C_{G_r H_r}(\tilde{z}, \tilde{y}) G_r^d
\end{bmatrix},
\]

\[
H_r^e = \begin{bmatrix}
C_{G_r H_r}(\tilde{y}, \tilde{u})^T C_{G_r H_r}(\tilde{z}, \tilde{y})^T H_r^a, C_{G_r H_r}(\tilde{y}, \tilde{u})^T H_r^b, H_r^d
\end{bmatrix}.
\]
An Efficient Solution

PROOF. We have
\[
\Delta^{[D(\bar{\varepsilon}), D(\bar{\eta})]}(C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta})) = G^a H^a C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta}) C_{G_z H_z}(\bar{\eta}, \bar{\varepsilon}) + C_{G_z H_z}(\bar{\eta}, \bar{\varepsilon})
\]
for \(a = \alpha + \beta + \gamma\), \(\beta = \beta_1 + \beta_2 + \beta_3\) and \(\beta = \beta_4\).

THEOREM 2.2. Let \(C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta})\) be an \(n \times n\) nonsingular generalized Cauchy-like matrix associated with the symmetric Sylvester operator \(\Delta^{[D(\bar{\varepsilon}), D(\bar{\eta})]}(\cdot) : \mathbb{F}^{n \times n} \to \mathbb{F}^{n \times n}\) of (2.3), such that \(\Delta^{[D(\bar{\varepsilon}), D(\bar{\eta})]}(C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta})) = G_r H_z^T\). Let \((\bar{\eta}, \bar{\varepsilon})\), \((\bar{\eta}, \bar{\varepsilon})\) be a pair of Cauchy matrices, where \(f_j, p_k, q_l\) are all distinct. Then we have the following matrix equations:
\[
C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta})^{-1} = C(\bar{\eta}, \bar{\varepsilon}) C_{S_{r+2} Y_{r+2}}(\bar{\eta}, \bar{\varepsilon})^{-1} C(\bar{\eta}, \bar{\varepsilon})
\]
where \(C_{S_{r+2} Y_{r+2}}(\bar{\eta}, \bar{\varepsilon})\) is a Cauchy-like matrix associated with \([D(\bar{\eta}), D(\bar{\eta})]\)-displacement generators,
\[
S_{r+2} = \begin{bmatrix} 1, C(\bar{\eta}, \bar{\varepsilon}) C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta}) & C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta}) \end{bmatrix},
\]
\[
Y_{r+2} = \begin{bmatrix} C(\bar{\eta}, \bar{\varepsilon})^T C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta})^T 1, C(\bar{\eta}, \bar{\varepsilon})^T 1, C(\bar{\eta}, \bar{\varepsilon})^T 1\end{bmatrix},
\]
for \(\bar{1}^T = (1, \ldots, 1)\). Furthermore, the complexity of the computations of the generators and \(Y_{r+2}\) is \(O(n^2 \log^2 n)\) arithmetic operations.

PROOF. From Lemma 2.1, we have \(\Delta^{[D(\bar{\eta}), D(\bar{\eta})]}(C(\bar{\eta}, \bar{\eta})) = 1 1^T\) and \(\Delta^{[D(\bar{\eta}), D(\bar{\eta})]}(C(\bar{\eta}, \bar{\eta})) = 1 1^T\) where \(\bar{1} = (1, \ldots, 1)\). From Theorem 2.1, it follows that \(C(\bar{\eta}, \bar{\eta}) C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta}) C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta}) = C_{S_{r+2}}(\bar{\eta}, \bar{\varepsilon})\), where \(S_{r+2}\) and \(Y_{r+2}\) are as \((2.10)\) and \((2.11)\). Substitute this matrix equation into the trivial matrix identity \(C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta}) = C(\bar{\eta}, \bar{\eta})^{-1} C(\bar{\eta}, \bar{\eta}) C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta}) C(\bar{\eta}, \bar{\eta})^{-1}\) and obtain equations \(C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta}) = C(\bar{\eta}, \bar{\eta})^{-1} C_{S_{r+2} Y_{r+2}}(\bar{\eta}, \bar{\varepsilon}) C(\bar{\eta}, \bar{\eta})^{-1}\). By inverting both sides of this equation, we immediately obtain equation \((2.9)\). The complexity of computing the generators and \(Y_{r+2}\) is immediately followed by Corollary 2.1.

THEOREM 2.3. (Cf. [14].) Given the \([D(\bar{\eta}), D(\bar{\eta})]\)-displacement generators \(G_u, H_u\) with placement rank \(u\) of a Cauchy-like matrix \(C_{G_r H_z}(\bar{\eta}, \bar{\eta})\), \(x < u < n\), it requires \(O(u^2 n)\) operations to compute the \([D(\bar{\eta}), (\bar{\eta})]\)-displacement generators \(G_z, H_z\) of the Cauchy matrix \(C_{G_r H_z}(\bar{\eta})\).

COROLLARY 2.2. Let the matrices \(\bar{G}_r H_z(\bar{\varepsilon}, \bar{\varepsilon})\), \(\bar{G}_{S_{r+2} Y_{r+2}}(\bar{\eta}, \bar{\eta})\), \(\bar{C}(\bar{\varepsilon}, \bar{\eta})\), \(\bar{C}(\bar{\varepsilon}, \bar{\eta})\) and \(\bar{C}(\bar{\varepsilon}, \bar{\eta})\) be Cauchy-like matrices and Cauchy matrices as in Theorem 2.2. Then we use \(O(r^2 n)\) to pote the new \(D(\bar{\eta}, (\bar{\eta})\)-generator \(\bar{G}_r H_z\) for the Cauchy-like matrix \(C_{G_r H_z}(\bar{\varepsilon}, \bar{\eta})\), such that
\[
\bar{C}_{G r H z}(\bar{\varepsilon}, \bar{\eta})^{-1} = C(\bar{\varepsilon}, \bar{\eta}) C_{G r H z}(\bar{\varepsilon}, \bar{\eta})^{-1} C(\bar{\varepsilon}, \bar{\eta})
\]
where \(r < r + 2 < n\).

PROOF. The corollary is followed by the Theorem 2.2. The running time of computing displacement generators \(\bar{G}_r H_z\) is immediately followed by Theorem 2.3.

3. RECURSIVE TRIANGULAR FACTORIZATION OF A MATRIX

In this section, we will recall the known method of the recursive triangular factorization of a strongly nonsingular matrix [19,22].
Definition 3.1. A matrix $M$ is strongly nonsingular if all its leading principal submatrices nonsingular.

Let us denote $I$ identity matrix, $O$ denotes a null matrix. Given an $n \times n$ nonsingular matrix we partition the matrix $M$ into four $n/2 \times n/2$ blocks, e.g.,

$$\begin{pmatrix} A & B \\ D & F \end{pmatrix}.$$ (1)

The matrix $S$ is called the Schur complement of $A$ in $M$ and can be computed as

$$S = F - DA^{-1}B.$$ (1)

We may write its inverse as

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}DA^{-1} & -A^{-1}BS^{-1}DA^{-1} \\ -S^{-1}DA^{-1} \\ S^{-1} \end{pmatrix}.$$ (1)

Let matrices $A$ and $S$ be invertible. We can continue this process recursively for the submatrices $A$ and $S$. Based on the following proposition, Pan and Zheng [19] and Pan et al. [22] do factorizations to all leading principal submatrices, $1 \times 1, 2 \times 2, 4 \times 4, \ldots, n \times n$. The total number of the steps for the recursive triangular factorization is $\lceil \log_2 n \rceil$. Each step of the computation costs matrices multiplications and subtractions.

Proposition 3.1. (See [19,22].) If $M \in F^{n \times n}$ is matrix with leading principal submatrices being nonsingular, so are $A$ and $S$.

Proposition 3.2. (See [19,22].) Suppose $M$ is a matrix with leading principal submatrices being nonsingular and write $S$ as (3.2). Let $M_1$ be a leading principal submatrix of $S$ and let denote the Schur complement of $M_1$ in $S$. Then $S^{-1}$ and $S^{-1}_1$ are located in the southeast blocks of $M^{-1}$.

Here is the algorithm of the recursive triangular factorization for inversion.

Algorithm 3.1. Recursive Triangular Factorization for Inversion [19,22].

Input. A matrix $M \in F^{n \times n}$.

Output. The inverse matrix $M^{-1}$.

Computations.

1. Use Algorithm 3.1. Compute $A^{-1}$ as (3.1).
2. Compute the Schur complement $S = F - DA^{-1}B$.
3. Use Algorithm 3.1 to the inverse matrix $S^{-1}$.
4. Compute $M^{-1}$ from (3.3).

4. MODIFIED RECURSIVE TRIANGULAR FACTORIZATION TO A CAUCHY-LIKE MATRIX

In this section, we will describe the techniques of the modified recursive triangular factorization to a Cauchy-like matrix. As we can see from equation (1.2), the complexity of the recursive factorization depends on the complexity of a Cauchy matrix by a vector. It needs only $O(n \log n)$ to compute a special Cauchy matrix times a vector. It motivates us to modify the process of recursive factorization so that the complexity bound of [19,22] can be improved by a factor $\log n$.

Let us recall some well-known results first.
LEMMA 4.1. (See [19,22].) Let \( \tilde{b}_j \in F^{n \times 1}, j = 1, 2, 3, \) be three distinct vectors and \( M_i \in F^{n \times n} \) Cauchy-like matrices, such that \( \Delta^{[D(\tilde{b}),D(\tilde{b}+1)]}(M_i) = G_i H_i^T, \) where \( G_i, H_i \in F^{n\times r_i}, r_i < n, \) \( i = 1, 2. \) The matrix \( M = M_1 M_2 \) is a Cauchy-like matrix with \( \Delta^{[D(\tilde{b}),D(\tilde{b})]}(M) = G_r H_r^T, \) \( G = [G_1, M_1 M_2], H_r = [M_2^T H_1, H_2], G_r, H_r \in F^{n \times r}, r = r_1 + r_2. \) Furthermore, \( O(r_1 r_2 A C(n)) \) suffice to compute \( G_r \) and \( H_r. \)

By multiplying \( C_{M_i, W_i}(\tilde{x}, \tilde{b})^{-1} \) on both sides of the symmetric Sylvester equation for \( C_{M_i, W_i}(\tilde{x}, \tilde{b}) \), we get the following.

LEMMA 4.2. Given an \( n \times n \) Cauchy-like matrix \( C_{M_i, W_i}(\tilde{x}, \tilde{b}) \) as defined in Definition 2.3, so that (2.3) holds for \( t \ll n, \) then matrix \( C_{M_i, W_i}(\tilde{x}, \tilde{b})^{-1} \) satisfies \( \Delta^{[D(\tilde{b}),D(\tilde{b})]}(C_{M_i, W_i}(\tilde{x}, \tilde{b})^{-1} G_r H_r^T, \) where \( G_r = [C_{M_i, W_i}(\tilde{x}, \tilde{b})^{-1} M_2], H_r = [(C_{M_i, W_i}(\tilde{x}, \tilde{b})^{-1})^\top W_i]. \)

FACT 4.1. By Lemma 4.2, the rank of \( \Delta^{[D(\tilde{b}),D(\tilde{b})]}(C_{M_i, W_i}(\tilde{x}, \tilde{b})^{-1} \ll t. \)

LEMMA 4.3. (See [19,22].) Let \( M \) and \( N \) be \( n \times n \) matrices which satisfy \( \Delta^{[D(\tilde{b}),D(\tilde{b})]}(M = G_1 H_1^T, \) and \( \Delta^{[D(\tilde{b}),D(\tilde{b})]}(N) = G_2 H_2^T, \) respectively. Then the matrices \( M + N \) and \( M - N \) Cauchy-like matrices associated with a \([D(\tilde{b}),D(\tilde{b})]\)-generator of length at most \( r_2 + r_1. \)

LEMMA 4.4. (See [19,22].) Let \( C_{G_1; H_1} = (\tilde{p}, \tilde{q}) \) be an \( n \times n \) generalized Cauchy-like matrix associated with the symmetric Sylvester operator \( \Delta^{[D(\tilde{b}),D(\tilde{b})]}(\cdot) : F^{n \times n} \to F^{n \times n}, \) such that \( \Delta^{[D(\tilde{b}),D(\tilde{b})]}(C_{G_1; H_1} = G_1 H_1^T + (\tilde{p}, \tilde{q}) G_2 H_2^T. \) Let

\[
C_{G_1; H_1} = \begin{pmatrix} A & B \\ D & F \end{pmatrix}, \quad S = F - DA^{-1}B,
\]

where \( A, B, D, F, \) and \( S \) as (3.1) and (3.2). Then the displacement ranks of \( A, B, D, F, \) are all less than \( r. \)

LEMMA 4.5. (See [19,22,26–28].) Let \( p(x) = p(x_1, x_2, \ldots, x_m) \) be a nonzero \( m \)-variate polynomial of a total degree \( k \) with a finite set \( S \) in its domain. Let \( \vec{x} = (x_1^*, x_2^*, \ldots, x_m^*) \) be a point in \( S \) where the random values \( x_1^*, \ldots, x_m^* \) are chosen in \( S, \) independently of each other and under uniform probability distribution on \( S. \) Then

\[
\text{probability} (p(\vec{x}^*) = 0) \leq \frac{k}{|S|},
\]

where \( |S| \) is the cardinality of \( S. \)

Now let us describe the complexity of the modified recursive triangular factorization to Cauchy-like matrices.

THEOREM 4.1. Let \( \tilde{b} \) be a given vector and \( C_{G_1; H_1} = (\tilde{p}, \tilde{q}) \) be a given strongly nonsingular generalized Cauchy-like matrix which satisfies \( \Delta^{[D(\tilde{b}),D(\tilde{b})]}(C_{G_1; H_1} = G_1 H_1^T. \) Let the ma \( C_{G_2; H_2} = (\tilde{p}, \tilde{q}) \) as in Corollary 2.2 be a strongly nonsingular Cauchy-like matrix. Then it requ \( O(n^2 \log^2 n) \) arithmetic operations to solve a Cauchy-like equation linear system of equa \( C_{G_1; H_1}(\tilde{e}, \tilde{f}) = \tilde{v}. \)

Proof. If the vectors \( \tilde{e}, \tilde{f} \) in the matrix \( C_{G_1; H_1} = (\tilde{e}, \tilde{f}) \) are \( n^\text{th} \) roots of unity, \( \tilde{p} = \tilde{a}, \tilde{q} = \tilde{b}, \) wher and \( \beta_0 \) of Definition 2.1, then we can apply the recursive triangular factorization Algorithm 3. compute the generators of the inverse of the matrix \( C_{G_1; H_1}(\tilde{a}, \tilde{b}). \) If the vectors \( \tilde{e}, \tilde{f} \) in the ma \( C_{G_1; H_1} = (\tilde{e}, \tilde{f}) \) are not \( n^\text{th} \) roots of unity, then we apply Theorem 2.2 to compute the generators \( \tilde{e} \) and \( \gamma_{\tilde{e}+1} \) as (2.10) and (2.11). Using Corollary 2.2, we compute the generators \( G_r, H_r \) for matrix \( C_{G_2; H_2} = (\tilde{p}, \tilde{q}). \) We choose the new vectors \( \tilde{p}, \tilde{q}, \) such that \( \tilde{p} = \tilde{a}, \tilde{q} = \tilde{b}, \) where \( a_1 \) and \( b_1 \) of Definition 2.1. Then, we can apply the recursive triangular factorization Algorithm 3. compute the generators of the inverse of the matrix \( C_{G_2; H_2}(\tilde{a}, \tilde{b}). \) In both cases, instead working directly with the entries of the matrix itself, we compute the generators of \( A, B, D \)
and $S$, where $A$, $B$, $D$, $F$, and $S$ as (4.1). The algorithm will continue to the matrices $A$ and $S^{-1}$ until we have the $D((\tilde{\beta}), (\tilde{\alpha}))$-generators $W_r$ and $V_r$ of the inverse matrix of $C_{G, H}(\tilde{\alpha})$, or the $D((\tilde{\beta}), (\tilde{\alpha}))$-generators $W_r^*$ and $V_r^*$ of the inverse matrix of $C_{G, H}^*(\tilde{\alpha}, \tilde{\beta})$. The length of the displacement generators can be decreased applying Theorem 2.3. The computations of the generators in each step involve the multiplications of a special Cauchy matrix times a vector. From Proposition 2.1, we use the generators $W_r$ and $V_r$, or $W_r^*$ and $V_r^*$ to compute the solution vector $\tilde{x}$, where $\tilde{x} = C_{W_rV_r^*}(\tilde{\beta}, \tilde{\alpha})\tilde{u}$ or $\tilde{x} = C(\tilde{f}, \tilde{g})C_{W_rV_r^*}(\tilde{\beta}, \tilde{\alpha})C(\tilde{\alpha}, \tilde{\beta})\tilde{u}$. The complexity $O(n^2\log^2 n)$ operations is followed by Corollary 2.1, Lemma 2.1, and Theorem 2.2.

**Theorem 4.2.** (See [19,22].) Let the nonsingular Cauchy-like matrix $C_{G, H}(\tilde{\beta}, \tilde{\alpha})$ defined in Theorem 4.1 and $C(\tilde{p}, \tilde{d})$ be a Cauchy matrix as defined as Definition 2.2 for any dist vectors $\tilde{p}, \tilde{q}, \tilde{d}$. Let the vectors $\tilde{u}_i^*, \tilde{u}_j^*$ for $i, j = 1, \ldots, r$ be random values selected from a finite set $S$. Let $Z$ be a matrix satisfying

$$Z = UC(\tilde{p}, \tilde{d}),$$

where $U = \sum_{i=1}^r \text{diag}(\tilde{u}_i^*)C(\tilde{q}, \tilde{p})\text{diag}(\tilde{u}_i^*)$. Then, $C_{G, H}(\tilde{\beta}, \tilde{\alpha})Z$ is a strongly nonsingular Cauchy-like matrix with a probability at least $1 - n(n + 1)/|S|$.

**Corollary 4.1.** Let $C_{G, H}(\tilde{\beta}, \tilde{f}) \in F^{n \times n}$ be a given nonsingular generalized Cauchy-like matrix satisfying $\Delta(D(\tilde{\alpha}), D(\tilde{\beta})) = G_{H}H^T$ and $C(\tilde{f}, \tilde{f}) = \tilde{v}$ be a given vector. Then the Cauchy equation linear system of equation $C_{G, H}(\tilde{\beta}, \tilde{f})\tilde{x} = \tilde{v}$ can be solved by means of a random algorithm using $2nr$ random parameters from a set $S$ with the cardinality $|S|$ and $O(nr^2\log^2 n)$ arithmetic operations and failing with a probability at most $n(n + 1)/|S|$.

**Proof.** Let $Z$ be a matrix-defined equation as (4.2) and $B = \sum_{i=1}^r \text{diag}(\tilde{u}_i^*)C(\tilde{f}, \tilde{d})\text{diag}(\tilde{C}(\tilde{d}, \tilde{d}))$, where the vectors $\tilde{u}_i^*, \tilde{u}_j^*$ for $i, j = 1, \ldots, r$ are random values selected from a fixed set $S$ and $C(\tilde{d}, \tilde{d})$ is a Cauchy matrix as defined in Definition 2.2. Let $C_{G, H}(\tilde{\beta}, \tilde{q})$ be a nonsingular Cauchy-like matrix as in Theorem 4.1. By Theorem 4.2, the Cauchy-like matrices $C_{G, H}(\tilde{\beta}, \tilde{f})B$ are strongly nonsingular with a probability at least $1 - n(n + 1)/|S|$. We can compute the generators of the inverse of the matrix $(C_{G, H}(\tilde{\beta}, \tilde{f})B)$ or $(C_{G, H}^*(\tilde{\beta}, \tilde{\alpha})B)$ and $G_{H}H^T$. The length of the displacement generators can be decreased applying Theorem 2.3. From Proposition 2.1, we use the generators to compute $\tilde{x} = C(\tilde{f}, \tilde{q})Z(C_{G, H}^*(\tilde{\beta}, \tilde{\alpha})Z)^{-1}C(\tilde{\alpha}, \tilde{\beta})\tilde{u} = B(C_{G, H}^*(\tilde{\beta}, \tilde{\alpha})B)^{-1}\tilde{v}$ by using a total of $O(nr^2\log^2 n)$ arithmetic operations by Theorem 2.3. From Theorem 4.1, the bound on the number of random parameters used and on the failure probability is followed immediately.

**5. Conclusion**

A novel algorithm of solving Cauchy-like linear systems of equations is developed. The complexity of the algorithms matches the well-known asymptotic bound of $O(n\log^2 n)$. Instead of transforming Cauchy-like matrices into Toeplitz-like matrices, we change the vectors for a representation of a Cauchy-like matrix. By modifying the recursive of triangular factorization improve the known complexity by a factor of $\log n$.

**References**
